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INEQUALITIES FOR DECOMPOSABLE FORMS OF DEGREE $n+1$ IN n VARIABLES

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Abstract. We consider the number of integral solutions to the inequality $|F(\mathbf{x})| \leq m$, where $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ is a decomposable form of degree $n+1$ in n variables. We show that the number of such solutions is finite for all m only if the discriminant of F is not zero. We get estimates for the number of such solutions that display appropriate behavior in terms of the discriminant. These estimates sharpen recent results of the author for the general case of arbitrary degree.

Introduction

Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form in $n \geq 2$ variables of degree d with integral coefficients, i.e., $F(\mathbf{X}) = \prod_{i=1}^{d} L_i(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$, where each $L_i(\mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ is a homogeneous linear form. Let $m \geq 1$. In a previous paper [T1] we studied the integer solutions $\mathbf{x} \in \mathbb{Z}^n$ to the inequality

$$(1) \qquad |F(\mathbf{x})| \leq m.$$

Let $V(F)$ denote the n-dimensional volume of $\{\mathbf{x} \in \mathbb{R}^n \colon |F(\mathbf{x})| \leq 1\}$, so that $m^{n/d}V(F)$ is the volume of the set of all real solutions to (1), and let $N_F(m)$ denote the number of integral solutions to (1). Generally speaking, one would expect $N_F(m)$ to be approximately $m^{n/d}V(F)$. Of course, such an approximation cannot hold unless both these quantities are finite. We say F is of *finite type* if, for any n'-dimensional subspace $S \subseteq \mathbb{R}^n$ defined over $\mathbb{Q}$, the n'-dimensional volume $V(F|_S)$ obtained by restricting F to the subspace S is finite. In [T1] we showed that $N_F(m)$ is finite for all $m \geq 1$ if and only if F is of finite type. Further, when F is of finite type

$$(2) \qquad |N_F(m) - m^{n/d}V(F)| \ll m^{(n-1)/(d-a(F))}(1+\log m)^{n-2}\mathcal{H}(F)^{c(F)}.$$

Here the implicit constant depends only on n and d, $\mathcal{H}(F)$ is the height of F (defined below in section 1), and $a(F), c(F)$ are positive constants depending on n and d. (When the discriminant is not zero, $a(F) = 1$ and $c(F) = \binom{d-1}{n-1} - 1$.) This generalizes a similar result due to Mahler [M] in the case $n = 2$, and also a result due to Ramachandra [R] for a certain special class of norm forms.

One easily sees that, except for the (in this context trivial) case of a positive definite quadratic form in two variables, it is necessary that the degree d be greater

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than the number of variables n for the volume $V(F)$ to be finite. In this paper we look at the case where $d = n+1$, and derive a much better error term than in (2).

Theorem 1. *Let $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ be a decomposable form in $n \geq 2$ variables of degree $n+1$. If F is of finite type, then the discriminant $D(F) \neq 0$. Further,*

$$|N_F(m) - m^{n/(n+1)}V(F)| \ll \frac{m^{(n-1)/n}}{|D(F)|^{1/n(n+1)!}}(1+\log m)^{n-2} + m^{(n-1)/(n+1)}(1+\log m)^{n-1},$$

where the implicit constant depends only on n.

The $m^{(n-1)/(n+1)}$ in the error term here is needed, since one may have that many solutions in an $(n-1)$-dimensional subspace. The logarithmic terms are likely not necessary. Compare, for example, the following result for cubic forms in two variables.

[T2, Theorem]. *Let $F(X,Y) \in \mathbb{Z}[X,Y]$ be a cubic form in two variables that is irreducible over $\mathbb{Q}$, and let $m \geq 1$. Then*

$$|N_F(m) - m^{2/3}A(F)| < 9 + \frac{2008m^{1/2}}{|D(F)|^{1/12}} + 3156m^{1/3}.$$

We explain here how the estimate for $N_F(m)$ in Theorem 1 above is almost as good as one could hope, in a certain sense. In all instances of the $\ll$ notation that follow, the implicit constant depends only on n.

If F is a form as above and $T \in \mathrm{GL}_n(\mathbb{R})$, then we can compose F with T to get a new form $F \circ T(\mathbf{X})$ with $|D(F \circ T)| = |\det(T)|^{d!/(d-n)!}|D(F)|$ and $V(F \circ T) = |\det(T)|^{-1}V(F)$. Specifically,

$$V(F)|D(F)|^{(d-n)!/d!} = V(F \circ T)|D(F \circ T)|^{(d-n)!/d!}.$$

Suppose $d = n+1$ and F has r_1 real linear factors and r_2 pairs of complex conjugate linear factors; so $r_1 + 2r_2 = n+1$. Then for an appropriate T,

$$F \circ T(\mathbf{X}) = \prod_{i=1}^{r_1} X_i \cdot \prod_{i=r_1+1}^{r_1+r_2} (X_i^2 + X_{i+r_2}^2).$$

Since $0 \leq r_2 \leq 1 + [(n+1)/2]$, there are $1 + [(n+1)/2]$ possible values for the quantity $V(F)|D(F)|^{1/(n+1)!}$. In particular,

$$V(F)|D(F)|^{1/(n+1)!} \gg\ll 1.$$

Looking back on Theorem 1, we see that

$$V(F)m^{n/(n+1)} \gg m^{(n-1)/(n+1)} \iff |D(F)| \ll m^{n!}$$

and

$$V(F)m^{n/(n+1)} \gg m^{(n-1)/n}|D(F)|^{-1/n(n+1)!} \iff |D(F)| \ll m^{n!/(n-1)}.$$

Also, one expects roughly $m^{(n-1)/(n+1)}V(F|_S)$ solutions in $(n-1)$-dimensional rational subspaces S. Thus, with the exception of the logarithmic factors, the main term $m^{n/(n+1)}V(F)$ in the estimate for $N_F(m)$ in Theorem 1 is larger than the second error term exactly when one would reasonably expect the volume to estimate the number of integral points; when the error terms dominate, one expects a large proportion of solutions to come from lower-dimensional subspaces.

When the discriminant is so large that one *doesn't* expect the volume to estimate the number of integral solutions, we can do better than Theorem 1, getting rid of the logarithmic terms.

Theorem 2. *Let F be a form as in Theorem 1 of finite type. Suppose further that $|D(F)|^{1-\varepsilon} \geq m^{n!}$ for some $\varepsilon > 0$. Then*

$$N_F(m) \ll \varepsilon^{-(n-1)} m^{(n-1)/(n+1)},$$

where the implicit constant depends only on n.

A very important tool in our proofs is the notion of equivalent forms. Suppose that $T \in \mathrm{GL}_n(\mathbb{Z})$. Then composing with T leaves the absolute value of the discriminant and the associated volume both fixed. Further, the integral solutions to (1) are in one-to-one correspondence (via T^{-1}) with the integral solutions to $|F \circ T(\mathbf{x})| \leq m$. Because of this, we say two forms F and G are equivalent if there is a $T \in \mathrm{GL}_n(\mathbb{Z})$ with $G = F \circ T$. We prove Theorems 1 and 2 by first finding an equivalent form with desirable properties. This is made possible by the hypothesis that the degree is one more than the number of variables. Once this is accomplished, we then show that there are n linear factors of F which are relatively small at any solution $\mathbf{x} \in \mathbb{R}^n$ of (1) (see Lemma 3 below). The proofs are completed by specializing several "gap arguments" developed in [T1].

1. Finding an Equivalent Form

For the remainder of this paper, $F(\mathbf{X}) = \prod_{i=1}^{n+1} L_i(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ will be a fixed decomposable form in n variables as in the Introduction, and $m \geq 1$ will be a fixed positive parameter.

In this section we will see how to find an equivalent form with certain desirable properties. Exactly what those "desirable properties" are is stated below in Lemma 1. After that, we will assume that the given form F itself has these properties.

We first establish some notation and recall pertinent definitions from [T1]. We will denote the usual L_2 norm of $\mathbf{x} \in \mathbb{C}^n$ by $\|\mathbf{x}\|$. We will denote the coefficient vector of a linear form $L_i(\mathbf{X})$ by $\mathbf{L}_i \in \mathbb{C}^n$. Complex conjugation will be denoted by an overline: $\overline{\alpha}$. This notation will be extended to vectors as well, e.g., $\overline{\mathbf{L}}$.

We define the *height* of F to be

$$\mathcal{H}(F) = \prod_{i=1}^{n+1} \|\mathbf{L}_i\|.$$

Note that $\mathcal{H}(F)$ is actually independent of the particular factorization of F used, though it is not preserved under equivalence.

Given a factorization of F, the discriminant of F is given by

$$D(F) = \prod \det(\mathbf{L}_{i_1}^{tr}, \dots, \mathbf{L}_{i_n}^{tr}),$$

where the product is over all ordered n-tuples $(i_1, \dots, i_n)$ of distinct indices i_j. As with the height, the discriminant is independent of the particular factorization used. Let $I(F)$ denote the set of all ordered n-tuples $(\mathbf{L}_{i_1}, \dots, \mathbf{L}_{i_n})$ of linearly independent coefficient vectors. Let $J(F)$ be the subset of $I(F)$ consisting of n-tuples that satisfy the following restriction: if $j < n$, then either $\mathbf{L}_{i_{j+1}}$ is proportional to $\overline{\mathbf{L}_{i_j}}$ or $\overline{\mathbf{L}_{i_j}}$ is in the span of $\mathbf{L}_{i_1}, \dots, \mathbf{L}_{i_j}$. It was shown in [T1] that $J(F)$ is in fact empty only when $I(F)$ is. Let

$$a(F) = \max\left\{\frac{\text{the number of } \mathbf{L}_i \text{ in the span of } \mathbf{L}_{i_1}, \dots, \mathbf{L}_{i_j}}{j}\right\},$$

where the maximum is over all n-tuples in $J(F)$ and $j = 1, \dots, n-1$. Note that $a(F) \geq 1$ if it is defined (i.e., $J(F)$ is not empty), with equality if and only if the discriminant of F is not zero.

We now state a characterization of finite volume given in [T1], and apply it to our situation here.

[T1, Proposition]. *For a decomposable form $F(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ of degree n in d variables not a power of a positive definite quadratic form in two variables, $V(F)$ is finite if and only if $a(F)$ is defined and less than d/n.*

In our case, one easily sees that $a(F)$ is less than $(n+1)/n$ if and only if $a(F) = 1$. We thus have the following.

Corollary. *For a decomposable form F of degree $n+1$ in n variables, $V(F)$ is finite if and only if $D(F) \neq 0$.*

Without loss of generality, we assume that $\mathbf{L}_i \in \mathbb{R}^n$ for $i > 2r$ and $\mathbf{L}_i \in \mathbb{C}^n \setminus \mathbb{R}^n$ for $i \leq 2r$, with $\overline{\mathbf{L}_i} = \mathbf{L}_{i+r}$ for $i \leq r$. (Here r may be zero, in which case all the linear factors of F are real.) Of course, such a decomposition is not uniquely determined by F. We will just work with one such. Let

$$\Delta_i = \det(\mathbf{L}_1^{tr}, \dots, \widehat{\mathbf{L}_i^{tr}}, \dots, \mathbf{L}_{n+1}^{tr})$$

and

$$\Delta = \prod_{i=1}^{n+1} \Delta_i,$$

so that $|D(F)|^{1/n!} = |\Delta|$. Note that any equivalent form has a decomposition into linear forms in the same manner, with the same r and same $|\Delta_i|$'s.

One readily sees that $\overline{\Delta_i} = (-1)^r \Delta_i$ for $i > 2r$ and that $\overline{\Delta_i} = \Delta_{i+r}$ for $i \leq r$. In particular, $\overline{\Delta_i \mathbf{L}_i} = \Delta_{i+r}\mathbf{L}_{i+r}$ for $i \leq r$ and $\overline{\Delta_i \mathbf{L}_i} = (-1)^r \Delta_i \mathbf{L}_i$ for $i > 2r$. From linear algebra we have

$$\sum_{j=1}^{n+1} (-1)^j \Delta_j \mathbf{L}_j = \mathbf{0}, \tag{3}$$

so that

$$\mathbf{0} = \begin{cases} \sum_{j=1}^{r} (-1)^j 2\Re(\Delta_j \mathbf{L}_j) + \sum_{j=2r+1}^{n+1} (-1)^j \Delta_j \mathbf{L}_j & \text{if } r \text{ is even,} \\ \sum_{j=1}^{r} (-1)^j 2i\Im(\Delta_j \mathbf{L}_j) + \sum_{j=2r+1}^{n+1} (-1)^j \Delta_j \mathbf{L}_j & \text{if } r \text{ is odd.} \end{cases} \tag{4}$$

(As usual, empty sums are interpreted as 0.) For notational convenience, we will write

$$\mathbf{M}_1 = \Re(\Delta_1 \mathbf{L}_1),\ \mathbf{M}_2 = \Im(\Delta_1 \mathbf{L}_1), \dots, \mathbf{M}_{2r} = \Im(\Delta_r \mathbf{L}_r)$$

if r is even and

$$\mathbf{M}_1 = \Im(\Delta_1 \mathbf{L}_1),\ \mathbf{M}_2 = \Re(\Delta_1 \mathbf{L}_1), \dots, \mathbf{M}_{2r} = \Re(\Delta_r \mathbf{L}_r)$$

if r is odd. In either case we write $\mathbf{M}_i = \Delta_i \mathbf{L}_i$ for $2r+1 \le i \le n+1$, and also write $M_i(\mathbf{X})$, $1 \le i \le n+1$, for the corresponding linear forms.

We will work with the parallelepiped $\mathcal{P}$ defined by

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \colon |M_i(\mathbf{x})| \le 1 \text{ for } i > 1\}.$$

One easily verifies that, by (4),

$$\frac{2^n}{\mathrm{Vol}(\mathcal{P})} = |\det\big((\mathbf{M}_i)_{i>1}\big)| = 2^{-r}|\det\big((\Delta_i \mathbf{L}_i)_{i>1}\big)| = 2^{-r}|\Delta|. \tag{5}$$

Here $(\mathbf{M}_i)_{i>1}$ denotes the $n \times n$ matrix with rows $\mathbf{M}_2, \dots, \mathbf{M}_{n+1}$, and similarly for $(\Delta_i \mathbf{L}_i)_{i>1}$. Let $\lambda_1 \le \cdots \le \lambda_n$ be the successive minima of $\mathcal{P}$ with respect to the integer lattice $\mathbb{Z}^n$. By Minkowski's theorem and (5) we have

$$\prod_{j=1}^{n} \lambda_j \le 2^{-r}|\Delta|. \tag{6}$$

Choose a $\mathbf{z}_j \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ for $1 \le j \le n$ such that $\{\mathbf{z}_1, \dots, \mathbf{z}_n\}$ is a basis for $\mathbb{Z}^n$ and $\mathbf{z}_j \in j\lambda_j \mathcal{P}$. Let $\mu_j = j2^{n+j}\lambda_j$. We note for future use that by (6)

$$\prod_{i=1}^{n} \mu_i = c_2(n) \prod_{i=1}^{n} \lambda_i \le c_2(n) 2^{-r}|\Delta|, \tag{6'}$$

where $c_2(n) = n! 2^{3n^2/2 - n/2}$.

By Lemma 15 of [E] there are a basis $\mathbf{z}'_1, \dots, \mathbf{z}'_n$ of $\mathbb{Z}^n$ and a permutation σ of $\{1, \dots, n\}$ that satisfy

$$|M_{j+1}(\mathbf{z}'_l)| \le \min\{\mu_{\sigma(j)}, \mu_l\}, \qquad 1 \le j, l \le n.$$

Further, by (4) we get

$$|M_1(\mathbf{z}'_l)| \le n\mu_l, \qquad 1 \le l \le n.$$

Since $\{\mathbf{z}'_1, \dots, \mathbf{z}'_n\}$ is a basis for $\mathbb{Z}^n$, the matrix T taking $\mathbf{e}_i$ to $\mathbf{z}'_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the canonical basis, is in $\mathrm{GL}_n(\mathbb{Z})$. We have thus proven the following.

Lemma 1. *Let $\mathbf{M}_i = (M_{i,1}, \dots, M_{i,n})$ for $1 \le i \le n+1$ be as above. After possibly applying a suitable transformation $T \in \mathrm{GL}_n(\mathbb{Z})$ first, there is a permutation σ of $\{1, \dots, n\}$ such that*

$$|M_{i+1,j}| \le \min\{\mu_{\sigma(i)}, \mu_j\}, \qquad 1 \le i, j \le n,$$

and

$$|M_{1,j}| \le n\mu_j.$$

2. Some Consequences of Lemma 1

From now on we assume that our form F is of finite type and satisfies the inequalities of Lemma 1. One easily sees from Hadamard's inequality that

$$\mathcal{H}(F)^n \geq |\Delta|,$$

since each $\mathbf{L}_i$ occurs in n of the determinants Δ_j. Our first consequence is an upper bound for the height of F in terms of the discriminant.

Lemma 2. *We have*

$$\mathcal{H}(F) \leq c_3(n)|\Delta|^{4/(n+1)},$$

where $c_3(n) = 2^{4n-(n+1)^2} n^{(n+22)/2} c_2^2(n)$. *If* $r = 0$, *then*

$$\mathcal{H}(F) \leq c_3'(n)|\Delta|^{2/(n+1)},$$

where $c_3'(n) = n^{(n+9)/2} 2^{2n} c_2(n)$.

Proof. We need an upper bound for μ_n. To do this, we first find a lower bound for λ_1. Using Lemma 1 and the fact that F does not have a nontrivial integral zero (this is implied by the hypothesis that F is of finite type) gives

$$\begin{aligned}
|\Delta| \leq |\Delta F(\mathbf{e}_1)| &= \prod_{j=1}^{n+1} |\Delta_i L_i(\mathbf{e}_1)| \\
&= \left(|M_{1,1}|^2 + |M_{2,1}|^2\right) \cdots \left(|M_{2r-1,1}|^2 + |M_{2r,1}|^2\right) \\
&\qquad \times |M_{2r+1,1}| \cdots |M_{n+1,1}| \\
&\leq 2^r n^2 \lambda_1^{n+1};
\end{aligned}$$

in other words,

$$\lambda_1 \geq \left(2^{-r} n^{-2} |\Delta|\right)^{1/(n+1)}. \tag{7}$$

By (6) we have

$$2^{-r}|\Delta| \geq \prod_{j=1}^{n} \lambda_j \geq \lambda_n \lambda_1^{n-1};$$

thus

$$\lambda_n \leq \frac{2^{-r}|\Delta|}{\lambda_1^{n-1}} < 2^{-r} 2^r n^2 |\Delta|^{1-(n-1)/(n+1)} = n^2 |\Delta|^{2/(n+1)}.$$

Combining this with Lemma 1, (6′) and (7) gives us

$$\begin{aligned}
|\Delta|\mathcal{H}(F) = \prod_{i=1}^{n+1} \|\Delta_i \mathbf{L}_i\| &= (\|\mathbf{M}_1\|^2 + \|\mathbf{M}_2\|^2)\cdots(\|\mathbf{M}_{2r-1}\|^2 + \|\mathbf{M}_{2r}\|^2) \\
&\quad \times \|\mathbf{M}_{2r+1}\|\cdots\|\mathbf{M}_{n+1}\| \\
&\le 2^r (n^{3/2}\mu_n)^2 \prod_{j=n-r+2}^{n} n\mu_j^2 \times \prod_{j=r+1}^{n-r+1} \sqrt{n}\mu_j \\
&= 2^r n^{(n+6)/2} \mu_n^2 \frac{\prod_{j=1}^n \mu_j^2}{\prod_{j=1}^r \mu_j^2 \prod_{j=r+1}^{n-r+1} \mu_j} \\
&\le 2^r n^{(n+6)/2} \mu_n^2 \frac{\prod_{j=1}^n \mu_j^2}{\mu_1^{n+1}} \\
&\le 2^{4n+r} n^{(n+10)/2} \lambda_n^2 \frac{c_2^2(n) 2^{-2r} |\Delta|^2}{(2^{n+1}\lambda_1)^{n+1}} \\
&\le c_3(n)|\Delta|^{1+4/(n+1)}.
\end{aligned}$$

When $r = 0$ we get

$$\begin{aligned}
|\Delta|\mathcal{H}(F) = \prod_{i=1}^{n+1} \|\Delta_i \mathbf{L}_i\| &= \prod_{i=1}^{n+1} \|\mathbf{M}_i\| \\
&\le n^{3/2}\mu_n \prod_{i=1}^{n} \sqrt{n}\mu_i \\
&= n^{(n+5)/2} 2^{2n} \lambda_n c_2(n) \prod_{i=1}^{n} \lambda_i \\
&\le n^{(n+9)/2} 2^{2n} c_2(n) |\Delta|^{1+2/(n+1)}.
\end{aligned}$$

□

Lemma 3. *Let* $\mathbf{x} \neq \mathbf{0}$. *Then there is an index* j *such that*

$$\frac{\prod_{i\neq j} |L_i(\mathbf{x})|}{|\det\left((\mathbf{L}_i)_{i\neq j}\right)|} \le c_4(n) \frac{|F(\mathbf{x})|}{\|\mathbf{x}\| |\Delta|^{1/(n+1)}},$$

where $c_4(n) = n^{3n/2} c_2^2(n)$.

Proof. Consider the n linear forms $M_2(\mathbf{X}), \ldots, M_{n+1}(\mathbf{X})$. By Lemma 4 of [T1] there is an i_0 between 2 and $n+1$ with

$$\frac{|M_{i_0}(\mathbf{x})|}{\|\mathbf{M}_{i_0}\|} \ge n^{-n/2} \|\mathbf{x}\| \frac{|\det(\mathbf{M}_i)_{i>1}|}{\prod_{i>1} \|\mathbf{M}_i\|}.$$

By (5) and Lemma 1 we have

$$\begin{aligned}\prod_{i>1} \|\mathbf{M}_i\| &\le n^{n/2} \prod_{i=1}^{n} \mu_i \\ &= n^{n/2} c_2(n) \prod_{i=1}^{n} \lambda_i \\ &\le n^{n/2} c_2(n) |\det(\mathbf{M}_i)_{i>1}|.\end{aligned}$$

Thus,

$$|M_{i_0}(\mathbf{x})| \ge n^{-n} c_2^{-1}(n) \|\mathbf{x}\| \|\mathbf{M}_{i_0}\|. \tag{8}$$

Now Hadamard's inequality together with (5), (6) and (6′) gives us

$$\prod_{i>1} \|\mathbf{M}_i\| \ge |\det(\mathbf{M}_i)_{i>1}| \ge \prod_{i=1}^{n+1} \lambda_i = c_2^{-1}(n) \prod_{i=1}^{n+1} \mu_i.$$

In particular, if $\|\mathbf{M}_{i_0}\| < n^{-(n-1)/2} c_2^{-1}(n) \mu_1$, then

$$\prod_{i \ne 1, i_0} \|\mathbf{M}_i\| > n^{(n-1)/2} \prod_{i=2}^{n} \mu_i.$$

But this would contradict Lemma 1, which implies that

$$\prod_{i \ne 1, i_0} \|\mathbf{M}_i\| \le n^{(n-1)/2} \prod_{i=2}^{n} \mu_i.$$

We conclude that

$$\|\mathbf{M}_{i_0}\| \ge n^{-(n-1)/2} c_2^{-1}(n) \mu_1 = n^{-(n-1)/2} c_2^{-1}(n) 2^{n+1} \lambda_1. \tag{9}$$

There is a j with $|\Delta_j L_j(\mathbf{x})| \ge |M_{i_0}(\mathbf{x})|$. Using (7), (8) and (9) gives us

$$\begin{aligned}|\Delta F(\mathbf{x})| = \prod_{i=1}^{n+1} |\Delta_i L_i(\mathbf{x})| &\ge |M_{i_0}(\mathbf{x})| \prod_{i \ne j} |\Delta_i L_i(\mathbf{x})| \\ &\ge n^{-n} c_2^{-1}(n) \|\mathbf{x}\| \|\mathbf{M}_{i_0}\| \prod_{i \ne j} |\Delta_i L_i(\mathbf{x})| \\ &\ge c_4^{-1}(n) \|\mathbf{x}\| |\Delta|^{1/(n+1)} \prod_{i \ne j} |\Delta_i L_i(\mathbf{x})| \\ &= c_4^{-1}(n) \|\mathbf{x}\| |\Delta|^{1/(n+1)} \frac{|\Delta| \prod_{i \ne j} |L_i(\mathbf{x})|}{|\Delta_j|} \\ &= c_4^{-1}(n) \|\mathbf{x}\| |\Delta|^{1/(n+1)} \frac{|\Delta| \prod_{i \ne j} |L_i(\mathbf{x})|}{|\det\big((\mathbf{L}_i)_{i \ne j}\big)|}.\end{aligned}$$

□

3. Theorem 1 for Small Discriminants

We prove Theorem 1 in the case where $|D(F)| \le m^{(n+1)!}$, i.e., when $|\Delta| \le m^{n+1}$. We call this a *small discriminant*. According to the discussion in the Introduction, it suffices to prove Theorem 1 in this case and rely on Theorem 2 in the case when the discriminant is not small. Set

$$B_0 = \frac{m^{1/n}}{|\Delta|^{1/n(n+1)}}.$$

Note that $B_0 \ge 1$ since the discriminant is small. For indices $l \ge 0$ let $B_l = e^l B_0$, $C_l = eB_l$ and

$$A_l = c_4(n)\frac{B_0^n}{B_l} = c_4(n)e^{-l}B_0^{n-1}.$$

Let S_0 denote the number of integral solutions to (1) with sup norm no more than B_0, and let V_0 denote the volume of the set of $\mathbf{x} \in \mathbb{R}^n$ that satisfy (1) and have sup norm at most B_0. Then Lemma 14 of [T1] gives

$$|S_0 - V_0| \le n(n+1)(2B_0+1)^{n-1}. \tag{10}$$

It remains to deal with larger solutions. We will use the following.

[T1, Lemma 7]. *Let $K_1(\mathbf{X}), \dots, K_n(\mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ be n linearly independent linear forms in n variables. Denote the corresponding coefficient vectors by $\mathbf{K}_1, \dots, \mathbf{K}_n$. Let $A, B, C > 0$ with $C > B$. Consider the set of $\mathbf{x} \in \mathbb{R}^n$ satisfying*

$$\frac{\prod_{i=1}^n |K_i(\mathbf{x})|}{|\det(\mathbf{K}_1^{tr}, \dots, \mathbf{K}_n^{tr})|} \le A \tag{11}$$

and also $B \le \|\mathbf{x}\| \le C$. If $BC^{n-1} \ge e^{n-1}n!n^{n/2}A$, then this set lies in the union of less than

$$n^3\left(\log\left(BC^{n-1}/n!n^{n/2}A\right)\right)^{n-2}$$

convex sets of the form

$$\begin{gathered}\{\mathbf{y} \in \mathbb{R}^n \colon |K_i'(\mathbf{y})| \le a_i \text{ for } i = 1, \dots, n\},\\ |\det\left((\mathbf{K}_1')^{tr}, \dots, (\mathbf{K}_n')^{tr}\right)| = 1,\\ \|\mathbf{K}_i'\| = 1, \qquad i = 1, \dots, n,\end{gathered} \tag{12}$$

with

$$\prod_{i=1}^n a_i < e^n n! n^{n/2}\frac{CA}{B}.$$

If $BC^{n-1} < e^{n-1}n!n^{n/2}A$, then this set lies in the union of no more than $n!$ convex sets of this form.

Now by Lemma 3, if $\mathbf{x}$ is any solution to (1) with $\|\mathbf{x}\| \ge B_l$, then there is an index j such that

$$\begin{aligned}\frac{\prod_{i\ne j}|L_i(\mathbf{x})|}{|\det\left((\mathbf{L}_i)_{i\ne j}\right)|} &\le c_4(n)\frac{m}{\|\mathbf{x}\||\Delta|^{1/(n+1)}}\\ &\le c_4(n)\frac{B_0^n}{B_l}\\ &= A_l.\end{aligned}$$

Also,

$$n^3\left(\log\left(B_lC_l^{n-1}/n!n^{n/2}A_l\right)\right)^{n-2} \le n^3\left(\log(B_0e^{2l+(n-1)(l+1)}\right)^{n-2}$$
$$= n^3\big((l+1)(n+1)+\log B_0\big)^{n-2}$$

and

$$n^3\big((l+1)(n+1)+\log B_0\big)^{n-2} \ge n!$$

Recalling there are $n+1$ possible indices j to deal with, we see that the total volume of the set of $\mathbf{x} \in \mathbb{R}^n$ with (1) and with $B_l \le \|\mathbf{x}\| \le C_l$ is no more than $n^3(n+1)\big((l+1)(n+1)+\log B_0\big)^{n-2}$ times the volume of a convex set of the form (12) with

$$\prod_{i=1}^n a_i < e^n n! n^{n/2}\frac{C_lA_l}{B_l} = e^{n+1-l}n!n^{n/2}A_0.$$

The volume of such a convex set is less than

$$2^n n!\prod_{i=1}^n a_i < 2^n e^{n+1-l}(n!)^2 n^{n/2}A_0 \ll e^{-l}A_0$$

by Lemma 9 of [T1]. We have

$$\sum_{l=0}^{\infty} n^3(n+1)\big((l+1)(n+1)+\log B_0\big)^{n-2}e^{-l} \ll (1+\log B_0)^{n-2}.$$

All told, then, the total volume of all solutions $\mathbf{x}$ to (1) with $\|\mathbf{x}\| \ge B_0$ is $\ll A_0(1+\log B_0)^{n-2}$. The set of all such solutions overlaps some with the solutions of sup norm no greater than B_0; so the total volume of the set of $\mathbf{x} \in \mathbb{R}^n$ with (1), which is $m^{n/(n+1)}V(F)$, is somewhat less than the sum of V_0 and the volume estimated above. In particular, the volume estimated above is greater than the difference $m^{n/(n+1)}V(F) - V_0$. This together with (10) gives

$$|S_0 - m^{n/(n+1)}V(F)| \ll B_0^{n-1} + A_0(1+\log B_0)^{n-2} \ll B_0^{n-1}(1+\log B_0)^{n-2}.$$

We still must estimate the remaining integral solutions to (1). Our proof of Theorem 1 in the case of small discriminant will be completed once we prove the following.

Lemma 5. *The integral solutions* $\mathbf{x}$ *to* (1) *with* $\|\mathbf{x}\| \ge B_0$ *lie in the union of a set of cardinality* S *satisfying*

$$S \ll B_0^{n-1}(1+\log B_0)^{n-2}$$

and $\ll (1+\log m)^{n-1}$ *proper subspaces defined over* $\mathbb{Q}$. *Further, the number of integral solutions in such a subspace is* $\ll m^{(n-1)/(n+1)}$.

Proof. Consider our argument for estimating the volume of all solutions of length at least B_0. Recall that at one point we used the volume of the convex sets of the form (12). To estimate the number of integral points in such a convex set, we may also use Lemma 9 of [T1], which implies that this number is no greater than $3^n 2^{n(n-1)/2}$ times the volume, *unless* such a set does not contain n linearly independent integral points. Our set of cardinality S is made up of exactly those integral points coming from the convex sets that do contain n linearly independent points.

It remains to consider those convex sets which do not contain n linearly independent integral points. In this case, the integral points in the convex set are contained in a proper subspace defined over $\mathbb{Q}$. We thus count the proper subspaces needed by counting the convex sets. Obviously we cannot continue on ad infinitum as when estimating the volume; we must stop at some point and use the subspace theorem to estimate the number of proper subspaces needed to account for the remaining integral solutions. We determine this point now.

Let $l_0 = [2\log(c_4(n)B_0^n)]$ and $l_1 = [\log(c_3(n)m^5)] + l_0$. Suppose $\mathbf{x} \in \mathbb{Z}^n$ is a solution to (1). Write $\mathbf{x} = g\mathbf{x}'$, where g is a positive integer and $\mathbf{x}'$ is primitive. By the homogeneity of F, we have

$$m \geq |F(\mathbf{x})| = g^{n+1}|F(\mathbf{x}')| \geq g^{n+1},$$

since F cannot vanish at $\mathbf{x}'$. This gives $g \leq m^{1/(n+1)}$; in particular, if $\|\mathbf{x}\| > C_{l_1}$, then

$$\begin{aligned} \|\mathbf{x}'\| = g^{-1}\|\mathbf{x}\| &> m^{-1/(n+1)}C_{l_1} \\ &\geq m^{-1/(n+1)}c_3(n)m^5 C_{l_0} \\ &> C_{l_0}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \|\mathbf{x}'\| = g^{-1}\|\mathbf{x}\| &> m^{-1/(n+1)}C_{l_1} \\ &\geq m^{-1/(n+1)}c_3(n)m^5 C_{l_0} \\ &> c_3(n)m^4 \\ &\geq \mathcal{H}(F) \end{aligned} \tag{15}$$

by Lemma 2 and the hypothesis that the discriminant is small.

By Lemma 3, (14), and using $C_{l_0} \geq e^{l_0+1}$, we have

$$\begin{aligned} \frac{\prod_{i\neq j}|L_i(\mathbf{x}')|}{|\det\big((\mathbf{L}_i)_{i\neq j}\big)|} &\leq c_4(n)\frac{B_0^n}{\|\mathbf{x}'\|} \\ &< c_4(n)\frac{B_0^n}{\|\mathbf{x}'\|^{1/2}C_{l_0}^{1/2}} \\ &\leq \frac{1}{\|\mathbf{x}'\|^{1/2}} \end{aligned}$$

for some index j. Further, by Lemma 2 of [T1], each $\mathbf{L}_i$ has field height $H(\mathbf{L}_i) \leq \mathcal{H}(F)$ and has coordinates in a number field of degree no greater than $n+1$. So (15) says that $\|\mathbf{x}'\| \geq H(\mathbf{L}_i)$ for each $\mathbf{L}_i$. By a quantitative version of the subspace theorem due to Evertse [E], the primitive solutions $\mathbf{x}'$ to

$$\frac{\prod_{i\neq j}|L_i(\mathbf{x}')|}{|\det\big((\mathbf{L}_i)_{i\neq j}\big)|} < \frac{1}{\|\mathbf{x}'\|^{1/2}}$$

with $\|\mathbf{x}'\| \geq H(\mathbf{L}_i)$ lie in the union of no more than

$$2^{60n^2+7n}\log(4n+4)\log\log(4n+4)$$

proper rational subspaces. Thus, the integral solutions $\mathbf{x}$ to (1) with $\|\mathbf{x}\| > C_{l_1}$ lie in the union of $\ll 1$ proper rational subspaces.

As shown above, the solutions $\mathbf{x}$ to (1) with $B_l \le \|\mathbf{x}\| \le C_l$ lie in the union of no more than

$$n^3(n+1)\big((l+1)(n+1)+\log B_0\big)^{n-2}$$

convex sets of the form (12). Each such convex set not containing n linearly independent integral points accounts for a proper subspace of such points. Hence, the integral solutions $\mathbf{x}$ with $B_0 \le \|\mathbf{x}\| \le C_{l_1}$ not already counted in S lie in the union of less than

$$\begin{aligned}\sum_{l=0}^{l_1} n^3(n+1)&\big((l+1)(n+1)+\log B_0\big)^{n-2}\\ &\le (l_1+1)n^3(n+1)\big((l_1+1)(n+1)+\log B_0\big)^{n-2}\\ &\ll (1+\log m)^{n-1}\end{aligned}$$

proper rational subspaces. This completes the proof of the first part of Lemma 5.

Suppose W is a proper subspace of $\mathbb{R}^n$ defined over $\mathbb{Q}$ of dimension n'. Then there is a $T \in \mathrm{GL}_n(\mathbb{Z})$ with

$$T : W \cap \mathbb{Z}^n \to \{(z_1, \dots, z_n) \in \mathbb{Z}^n : z_i = 0 \text{ for } i > n'\}.$$

The form $G = F \circ T^{-1}$ is an equivalent form, and F restricted to W is equivalent to G restricted to $\mathbb{R}^{n'}$. In this manner, we see that considering integral solutions to (1) for F restricted to a proper rational subspace is equivalent to considering integral solutions to (1) for a form in fewer variables. By Theorem 2 of [T1], the number of such integral solutions is $\ll m^{n'/(n+1)} \le m^{(n-1)/(n+1)}$. This completes the proof of the second part of Lemma 5. $\square$

4. Proof of Theorem 2

Suppose $|D(F)|^{1-\varepsilon} \ge m^{n!}$, i.e., $|\Delta|^{1-\varepsilon} \ge m$. This time set $B_0 = m^{1/(n+1)}$. Then, as above,

$$|V_0 - S_0| \le n(n+1)(2B_0+1)^{n-1} \ll m^{(n-1)/(n+1)}.$$

As explained in the introduction, we have $m^{n/(n+1)}V(F) \ll m^{(n-1)/(n+1)}$ when $|\Delta| \ge m$. In particular, we have $V_0 \ll m^{(n-1)/(n+1)}$. Thus, we see that $S_0 \ll m^{(n-1)/(n+1)}$.

By Lemma 3, for any solution $\mathbf{x}$ to (1) with $\|\mathbf{x}\| \ge B_0$, there is an index j such that

$$\begin{aligned}\frac{\prod_{i\ne j}|L_i(\mathbf{x})|}{|\det\big((\mathbf{L}_i)_{i\ne j}\big)|} &\le c_4(n)\frac{m}{\|\mathbf{x}\||\Delta|^{1/(n+1)}}\\ &\le c_4(n)\frac{m}{B_0 m^{1/(n+1)}|\Delta|^{\varepsilon/(n+1)}}\\ &= c_4(n)\frac{m^{(n-1)/(n+1)}}{|\Delta|^{\varepsilon/(n+1)}}.\end{aligned} \tag{16}$$

This time let

$$l_1 = \max\{[2\log(c_4(n)m)], [\log\big(c_3(n)|\Delta|^{4/(n+1)}\big)]\}.$$

Then

$$C_{l_1} \ge B_0 \max\{\big(c_4(n)m\big)^2, \mathcal{H}(F)\} = m^{1/(n+1)} \max\{\big(c_4(n)m\big)^2, \mathcal{H}(F)\}$$

by Lemma 2.

Suppose $\mathbf{x}$ is an integral solution to (1) with $\|\mathbf{x}\| > C_{l_1}$. As above, $\mathbf{x} = g\mathbf{x}'$ for some positive integer $g \le m^{1/(n+1)}$ and some primitive solution $\mathbf{x}'$ to (1). By Lemma 3, there is an index j with

$$\begin{aligned}\frac{\prod_{i\neq j}|L_i(\mathbf{x}')|}{|\det\big((\mathbf{L}_i)_{i\neq j}\big)|} &\le c_4(n)\frac{m}{\|\mathbf{x}'\||\Delta|^{1/(n+1)}}\\ &\le \frac{c_4(n)m}{\|\mathbf{x}'\|}\\ &< \frac{1}{\|\mathbf{x}'\|^{1/2}},\end{aligned}$$

since $\|\mathbf{x}'\| > \big(c_4(n)m\big)^2$. Further, $\|\mathbf{x}'\| \ge \mathcal{H}(F)$. Exactly as we argued in the proof of Lemma 5, we see that the integral solutions $\mathbf{x}$ to (1) with $\|\mathbf{x}\| > C_{l_1}$ lie in the union of $\ll 1$ proper rational subspaces. Hence, the number of such solutions is $\ll m^{(n-1)/(n+1)}$.

All that is left are the integral solutions $\mathbf{x}$ with $B_0 \le \|\mathbf{x}\| \le C_{l_1}$. To deal with them we use the following.

[T1, Lemma 7′]. *Let $K_1(\mathbf{X}),\dots,K_n(\mathbf{X}) \in \mathbb{C}[\mathbf{X}]$ and $\mathbf{K}_1,\dots,\mathbf{K}_n$ be as above. Let $A, C > 0$ and $D > 1$. If $C^n \ge D^n n! A$, then the solutions $\mathbf{x}$ to* (11) *with $\|\mathbf{x}\| \le C$ lie in the union of less than*

$$n\left(\log_D\left(C^n/n!A\right)\right)^{n-1}$$

convex sets of the form (12) *with*

$$\prod_{i=1}^n a_i < D^n n! A.$$

If $C^n < D^n n! A$, then such solutions lie in the union of no more than $n!$ convex sets of this form.

Set $A = c_4(n)m^{(n-1)/(n+1)}/|\Delta|^{\varepsilon/(n+1)},\ B = B_0,\ C = C_{l_1}$ and $D = |\Delta|^{\varepsilon/n(n+1)}$. Then

$$\begin{aligned}n\left(\log_D\left(C^n/n!A\right)\right)^{n-1} &\le n\left(\frac{n(l_1+\log m)}{\log(|\Delta|^{\varepsilon/(n+1)})}\right)^{n-1}\\ &\ll \left(\frac{\log m + \log|\Delta|}{\varepsilon\log|\Delta|}\right)^{n-1}\\ &\ll \varepsilon^{-(n-1)},\end{aligned}$$

since $\log|\Delta| \ge \log m$. By this lemma and (16), the integral solutions $\mathbf{x}$ to (1) with $B_0 \le \|\mathbf{x}\| \le C_{l_1}$ are in the union of $\ll \varepsilon^{-(n-1)}$ convex sets of the form (12) with

$$\prod_{i=1}^n a_i < D^n n! A \ll m^{(n-1)/(n+1)}.$$

As in the proof of Lemma 5, those convex sets that contain n linearly independent integral points contain $\ll m^{(n-1)/(n+1)}$ integral points. For the other convex sets, all the integral points they contain are in a proper subspace, and the total number of solutions in such a subspace is $\ll m^{(n-1)/(n+1)}$. Thus, the total number of integral solutions $\mathbf{x}$ with $B_0 \le \|\mathbf{x}\| \le C_{l_1}$ is $\ll \varepsilon^{-(n-1)}m^{(n-1)/(n+1)}$.

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