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UNIVERSAL DEFORMATION RINGS AND KLEIN FOUR DEFECT GROUPS

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ABSTRACT. In this paper, the universal deformation rings of certain modular representations of a finite group are determined. The representations under consideration are those which are associated to blocks with Klein four defect groups and whose stable endomorphisms are given by scalars. It turns out that these universal deformation rings are always subquotient rings of the group ring of a Klein four group over the ring of Witt vectors.

1. INTRODUCTION

Suppose k is a field of positive characteristic p, $W = W(k)$ is the ring of infinite Witt vectors over k, and G is a profinite group. In [16], [17], Mazur developed a deformation theory of finite-dimensional representations V of G in case k is finite, using results of Schlessinger [21]. He proved that if G satisfies a certain finiteness condition, then V has a universal deformation ring $R(G, V)$ in case V is absolutely irreducible. In [12], de Smit and Lenstra used an argument of Faltings to prove for arbitrary profinite groups G that a finite-dimensional representation V of G over k has a universal deformation ring $R(G, V)$ in case the endomorphism ring of V is just k. The ring $R(G, V)$ is defined as follows. Suppose $\mathcal{C}$ is the category of all topological local commutative W-algebras R with residue field k which are the projective limits of their discrete Artinian quotients. Then a lift of V over an object R in $\mathcal{C}$ is an RG-module M which is free over R so that $k \otimes_R M \cong V$ as kG-modules. An isomorphism class of lifts of V over R is called a deformation of V over R. The deformation functor $\mathcal{F}_V : \mathcal{C} \to \mathrm{Sets}$ sends an object R in $\mathcal{C}$ to the set of all deformations of V over R. Then V has a universal deformation ring $R(G, V)$ in $\mathcal{C}$ if the functor $\mathcal{F}_V$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R(G, V), -)$; in other words, if $R(G, V)$ represents $\mathcal{F}_V$. For more information on deformations and deformation rings, see [12] and [16]. In number theory, deformation rings are at the center of work by many authors concerning Galois representations, modular forms, elliptic curves and diophantine geometry (see, for example, [25], [23], [6], [9], and their references).

Suppose now that k is algebraically closed. To answer questions about the ring structure of universal deformation rings, it is natural to start with the case of a finite group G. In [3], the universal deformation ring of a representation V of a finite group G over k has been determined explicitly in case the stable endomorphism ring

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of V is k and the unique non-projective indecomposable summand of V belongs to a block of kG with cyclic defect groups. The obtained isomorphism types of $R(G, V)$ in this case lead to the following general question.

Question 1.1. Suppose G is a finite group and V is a finitely generated kG-module with stable endomorphism ring k. Is $R(G, V)$ a subquotient ring of the group ring WD over W of a defect group D of the block B of kG associated to V?

In [3], it follows that this question has a positive answer in case D is cyclic. In this paper we prove:

Theorem 1.2. *Question 1.1 has a positive answer in case D is a Klein four group.*

This case provides a more stringent test of Question 1.1 than the cyclic block case in [3], since the representation theory of a block B with Klein four defect group D is considerably more difficult. For example, there are infinitely many non-isomorphic indecomposable B-modules, some of which are parametrized by one-parameter families, whereas a cyclic block has only finitely many indecomposable modules up to isomorphism. It is also more difficult to determine all the indecomposable B-modules which have stable endomorphism ring k; this was relatively easy to determine for cyclic blocks. We prove Theorem 1.2 by determining explicitly all the universal deformation rings.

Theorem 1.2 supports the idea that Question 1.1 is the correct question to consider concerning the dependence of the universal deformation rings $R(G, V)$ on the local structure of G. In case D is non-abelian, it is natural to ask if Question 1.1 can be refined, since WD is no longer a commutative ring. One possibility would be to replace D by its maximal abelian quotient. It turns out that this is not possible in general. A counterexample is given when k has characteristic 2, G is the symmetric group S_4, and V is an irreducible 2-dimensional representation of G over k (see Proposition 4.2 and Remark 4.4). Note that in this case the defect group D is a dihedral group of order 8.

To analyze Question 1.1 for more complicated defect groups, a case-by-case study as done in this paper is inefficient and often not possible. In case D is abelian, there are strong conjectures posed by Broué and others (see, e.g., [7], [19]) which would establish derived equivalences between blocks of kG with defect group D and associated blocks of $kN_G(D)$. Such equivalences should be useful in studying Question 1.1 when D is abelian. Issues of this kind arise in Remark 3.7 of this paper. In particular, Rickard has shown in [18] that there is a splendid derived equivalence between the principal blocks of the alternating groups A_4 and A_5 when $p = 2$. In this paper, we determine the universal deformation rings of all V belonging to these blocks over k and having stable endomorphism ring equal to k. It is a natural problem now to find whether these calculations are consistent with universal deformation rings being preserved by (splendid) derived equivalences. As a first step to study this problem, Mazur's deformation theory has been extended in [5] to objects in derived categories.

The sections of this article are as follows. In Section 2, we recall some basic definitions and results. In particular, Proposition 2.5 shows that Morita equivalences between blocks of group rings over W preserve universal deformation rings. In Section 3, we prove Theorem 1.2 using a case-by-case analysis. In Section 4, we determine the universal deformation rings of the irreducible representations of kS_4 in case the characteristic of k is 2.

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Throughout the paper, k is an algebraically closed field of characteristic $p > 0$ and G is a finite group.

2. Preliminaries

In this section, we recall a few basic definitions and results and prove that Morita equivalences between blocks of group rings over W preserve universal deformation rings. As in Section 1, let $\mathcal{C}$ be the category of all topological local commutative W-algebras R such that $R/\max(R) = k$ and R is the projective limit of its discrete Artinian quotients. The morphisms in $\mathcal{C}$ are continuous W-algebra homomorphisms. Suppose G is a finite group and V is a finitely generated kG-module. Recall from Section 1 that a lift of V over an object R in $\mathcal{C}$ is an RG-module M which is free over R so that $k \otimes_R M$ is isomorphic to V as a kG-module. The isomorphism class $[M]$ of M as an RG-module is called a deformation of V over R, and the set of such deformations is denoted by $\mathrm{Def}_G(V, R)$. The deformation functor $\mathcal{F}_V : \mathcal{C} \to \mathrm{Sets}$ sends an object R in $\mathcal{C}$ to $\mathrm{Def}_G(V, R)$ and a morphism $f : R \to R'$ in $\mathcal{C}$ to the map $\mathrm{Def}_G(V, R) \to \mathrm{Def}_G(V, R')$ defined by $[M] \mapsto [R' \otimes_{R,f} M]$. In case there exists an object $R(G, V)$ in $\mathcal{C}$ and a lift $U(G, V)$ of V over $R(G, V)$ so that for each R in $\mathcal{C}$ and for each lift M of V over R there is a unique morphism $\alpha : R(G, V) \to R$ in $\mathcal{C}$ with $[M] = [R \otimes_{R(G,V),\alpha} U(G, V)]$, then $R(G, V)$ is called the universal deformation ring of V and $[U(G, V)]$ is called the universal deformation of V. In other words, $R(G, V)$ represents the functor $\mathcal{F}_V$ in the sense that $\mathcal{F}_V$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R(G, V), -)$. By a result of Faltings (see [12, Prop. 7.1]), V has a universal deformation ring $R(G, V)$ in case $\mathrm{End}_{kG}(V) = k$.

For the convenience of the reader, we now recall some results from [3]. Suppose Ω denotes the Heller operator for kG (see, for example, [1, §20]). Define $d^j(V) = \dim_k \hat{H}^j(G, \mathrm{Hom}_k(V, V))$ for $j \in \mathbb{Z}$, where $\hat{H}^j$ denotes the j-th Tate cohomology group. In particular, $\hat{H}^0(G, \mathrm{Hom}_k(V, V)) = \underline{\mathrm{End}}_{kG}(V)$, and for $j > 0$, $\hat{H}^j(G, \mathrm{Hom}_k(V, V)) = \mathrm{Ext}^j_{kG}(V, V)$.

Proposition 2.1 ([3, Prop. 2.1])**.** *Suppose V is a finitely generated kG-module with $d^0(V) \le 1$. Then V has a Noetherian universal deformation ring $R(G, V)$.*

Lemma 2.2 ([3, Cor. 2.5])**.** *Let V be a finitely generated kG-module.*

(i) *There is an isomorphism $\underline{\mathrm{End}}_{kG}(V) \cong \underline{\mathrm{End}}_{kG}(\Omega(V))$ of k-vector spaces. In particular, $d^0(V) = d^0(\Omega(V))$.*
(ii) *If $d^0(V) \le 1$, then $R(G, V)$ and $R(G, \Omega(V))$ are isomorphic.*
(iii) *If $d^0(V) = 0$, then $R(G, V) = W$.*

Lemma 2.3 ([3, Cor. 2.8])**.** *Suppose V is a finitely generated kG-module with $d^0(V) = 1$. Then there is a non-projective indecomposable kG-module V_0 such that $d^0(V_0) = 1$, V is isomorphic to $V_0 \oplus P$ for some projective kG-module P and $R(G, V) \cong R(G, V_0)$.*

Remark 2.4. Suppose V is a 1-dimensional representation of G over k. According to Mazur [16], the universal deformation ring of V is $R(G, V) = WG^{ab,p}$, where $G^{ab,p}$ is the maximal abelian p-quotient of G. Similarly to [1, pp. 67–68], it follows that the vertices of V are Sylow p-subgroups of G. This implies that the defect groups of the block to which V belongs are also Sylow p-subgroups of G. Hence Question 1.1 has a positive answer in case V is 1-dimensional over k.

We now show that universal deformation rings are preserved by Morita equivalences between blocks of group rings over W. Since every block of kG can be lifted to a block of WG, every finitely generated indecomposable kG-module belongs to a unique block of WG.

Proposition 2.5. *Let V be a finitely generated indecomposable kG-module with $d^0(V) = 1$ so that V belongs to the block B_W of WG. Suppose B_W is Morita equivalent to a block b_W of WH for some finite group H, and V' is the kH-module corresponding to V under this Morita equivalence. Then $R(G,V)$ and $R(H,V')$ are isomorphic.*

Proof. Suppose the Morita equivalence is given as

$$Q \otimes_{b_W} - \;:\; b_W\text{-mod} \to B_W\text{-mod} \tag{2.1}$$

where Q is a B_W-b_W-bimodule which is a finitely generated projective generator of B_W-mod, and $\mathrm{End}_{B_W}(Q) \cong b_W^{op}$. Let $R \in \mathrm{Ob}(\mathcal{C})$, and let B_R (respectively b_R) be the block of RG (respectively of RH) corresponding to B_W (respectively to b_W). Then $B_R = R \otimes_W B_W$ and $b_R = R \otimes_W b_W$. It follows that $R \otimes_W Q$ is a B_R-b_R-bimodule which is a finitely generated projective generator of B_R-mod. We now show that $\mathrm{End}_{B_R}(R \otimes_W Q) \cong b_R^{op}$ by showing that $\mathrm{End}_{RG}(R \otimes_W Q) \cong R \otimes_W \mathrm{End}_{WG}(Q)$ as R-algebras. Since W is a principal ideal domain, it can easily be seen that

$$\mathrm{End}_{kG}(k \otimes_W Q) \cong k \otimes_W \mathrm{End}_{WG}(Q) \tag{2.2}$$

as k-algebras. We define an R-algebra homomorphism

$$\beta \;:\; R \otimes_W \mathrm{End}_{WG}(Q) \to \mathrm{End}_{RG}(R \otimes_W Q)$$

by

$$r \otimes f \mapsto \mathrm{mult}(r) \otimes f\ .$$

Using (2.2), it follows that β induces an isomorphism

$$\mathrm{id} \otimes \beta \;:\; k \otimes_R R \otimes_W \mathrm{End}_{WG}(Q) \to k \otimes_R \mathrm{End}_{RG}(R \otimes_W Q)\ .$$

Because Q is a summand of a free WG-module of finite rank, $R \otimes_W Q$ is a summand of a free RG-module of finite rank. Since G is finite and W and R are objects in $\mathcal{C}$, it follows that $\mathrm{End}_{WG}(Q)$ (respectively $\mathrm{End}_{RG}(R \otimes_W Q)$) is a finitely generated free W-module (respectively a finitely generated free R-module). Thus β is an R-algebra isomorphism. Hence (2.1) induces a Morita equivalence

$$(R \otimes_W Q) \otimes_{b_R} - \;:\; b_R\text{-mod} \to B_R\text{-mod}$$

and $V \cong (R \otimes_W Q) \otimes_{b_R} V'$. Suppose now that M' is a lift of V' over R. Then M' is a b_R-module. Since M' is finitely generated free over R and $R \otimes_W Q$ is projective over b_R, it follows that $M = (R \otimes_W Q) \otimes_{b_R} M'$ is finitely generated free over R. Moreover, $M \otimes_R k = (R \otimes_W Q) \otimes_{b_R} (M' \otimes_R k) \cong (R \otimes_W Q) \otimes_{b_R} V' \cong V$. Hence M is a lift of V. We thus obtain a bijection between the deformations of V' over R and the deformations of V over R. This implies that the two deformation functors $\mathcal{F}_{V'}$ and $\mathcal{F}_V$ are naturally isomorphic, which completes the proof of Proposition 2.5. □

3. Blocks with Klein four defect groups

In this section, we prove Theorem 1.2. The (tame) blocks having Klein four defect groups are classified, for example, in [2, §6.6]. Throughout the remainder of this section, let k have characteristic 2, let D be a Klein four group, and suppose B is a block of kG with defect group D. Then the number of simple B-modules, up to isomorphism, is either 1 or 3. By [2, §6.6], B has one of the following three forms:

(I) B is isomorphic to $\mathrm{Mat}_n(kD)$ for some n, or
(II) B is Morita equivalent to the group ring kA_4 of the alternating group of degree 4, or
(III) B is Morita equivalent to the principal block $B_0(kA_5)$ of the group ring kA_5 of the alternating group of degree 5.

3.1. **Case (I).**

We first look at $B = kD$. There is a unique simple kD-module, up to isomorphism, namely, the trivial simple kD-module, which we will denote by S.

Lemma 3.1. *Let V be an indecomposable kD-module with $\underline{\mathrm{End}}_{kD}(V) = k$. Then $V \cong \Omega^i(S)$ for some integer i and $R(D, V) = WD$.*

Proof. By Mazur [16] we have $R(G, S) = WD$. By Lemma 2.2, it follows that $R(D, \Omega^i(S)) = WD$ for all i. Since k has characteristic 2 and D is an abelian 2-group, it follows from [11, Thm. 10.1] and from [8] that the only indecomposable kD-modules with stable endomorphism ring equal to k are of the form $\Omega^i(S)$ for some integer i. □

We now suppose that B is a block of kG which is isomorphic to $\mathrm{Mat}_n(kD)$ for some n. Then there is a unique simple B-module, up to isomorphism, which we will denote by T.

Proposition 3.2. *Let V be an indecomposable B-module with $\underline{\mathrm{End}}_{kG}(V) = k$. Then $V \cong \Omega^i(T)$ for some integer i and $R(G, V) = WD$.*

Proof. Since B and kD are Morita equivalent, the B-modules of the form $\Omega^i(T)$ are the only B-modules that have stable endomorphism ring equal to k. Let now B_W be the block of WG corresponding to B. Then B_W and WD are Morita equivalent by [15, Cor 1.4]. Since S and T correspond to each other under this Morita equivalence, Proposition 3.2 follows from Lemma 3.1 and Proposition 2.5. □

3.2. **Case (II).**

We first look at $B = kA_4$. There are 3 non-isomorphic simple kA_4-modules, which we will denote by S_0, S_1, S_2, where S_0 is the trivial simple kA_4-module.

Lemma 3.3. *Let V be an indecomposable kA_4-module with $\underline{\mathrm{End}}_{kA_4}(V) = k$. Then either*

(a) *$V \cong \Omega^i(S_j)$, $j = 0, 1, 2$, for some integer i and $R(A_4, V) = W$, or*
(b) *V is uniserial of length 2 and $R(A_4, V) = k$.*

Proof. The simple kA_4-modules S_0, S_1, S_2 are the inflations of the simple $k(\mathbb{Z}/3\mathbb{Z})$-modules to A_4. Hence they are 1-dimensional over k, and it follows by Mazur [16] that the universal deformation ring of S_j, $j = 0, 1, 2$, is the group ring over W of the maximal abelian 2-quotient of A_4. Thus $R(A_4, S_j) = W$ for $j = 0, 1, 2$. By Lemma 2.2(ii), it follows that $R(A_4, \Omega^i(S_j)) = W$ for $j = 0, 1, 2$ and for all i.

We have 6 uniserial kA_4-modules of length 2, namely, the kA_4-modules U_{ij} that have descending composition series (S_i, S_j), where $i \neq j$ is in $\{0,1,2\}$. It is obvious that the stable endomorphism ring of U_{ij} is k. Since $\mathrm{Ext}^1_{kA_4}(U_{ij}, U_{ij}) = 0$, it follows that $R(A_4, U_{ij})$ is a quotient of W. We now show that $R(A_4, U_{ij}) = k$. The 6 uniserial modules U_{ij} are in the Ω-orbit of either U_{12} or U_{21}. Hence it is enough to show that $R(A_4, U_{12}) = k = R(A_4, U_{21})$. We use that A_4 is a subgroup of A_5 of index 5. There are two simple 2-dimensional kA_5-modules Z_1 and Z_2 which define two isomorphisms

$$\tau_i : A_5 \to \mathrm{SL}_2(\mathbb{F}_4) \subset \mathrm{GL}_2(\mathbb{F}_4)$$

for $i = 1, 2$. The Borel subgroup ST of $\mathrm{SL}_2(\mathbb{F}_4)$ consisting of upper triangular matrices is isomorphic to A_4. When we consider τ_i, $i = 1, 2$, restricted to the full pre-image of ST under τ_i, we obtain two isomorphisms

$$\rho_i : A_4 \to ST \subset \mathrm{GL}_2(\mathbb{F}_4)$$

for $i = 1, 2$, which correspond to two uniserial $\mathbb{F}_4A_4$-modules of length 2. Since $\mathbb{F}_4$ is a splitting field for A_4, it follows that ρ_1, ρ_2 can be identified with two uniserial kA_4-modules. Because the images of ρ_i, $i = 1, 2$, are in $\mathrm{SL}_2(\mathbb{F}_4)$, we conclude that these two uniserial kA_4-modules are U_{12} and U_{21}. We now use the argumentation of [4, Section 6] to determine the universal deformation rings of these modules. Let $\tilde{A}_5$ be the double cover of A_5, i.e., $\tilde{A}_5 = \mathrm{SL}_2(\mathbb{F}_5)$. Then Z_1 and Z_2 have lifts over W when viewed as $k\tilde{A}_5$-modules. Let

$$\lambda_i : \tilde{A}_5 \to \mathrm{GL}_2(W/4W)\ ,$$

$i = 1, 2$, be the corresponding lift of Z_i over $W/4W$ and let H_i be the image of λ_i. For $i = 1, 2$, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \{\pm 1\} & \to & \tilde{A}_5 & \to & A_5 & \to & 1 \\ & & \downarrow & & \downarrow \lambda_i & & \downarrow \tau_i & & \\ 1 & \to & 1 + 2\mathrm{Mat}_2(W/4W) & \to & H_i & \to & \mathrm{SL}_2(\mathbb{F}_4) & \to & 1. \end{array} \tag{3.3}$$

In the bottom row of this diagram, one has an isomorphism

$$\delta : 1 + 2\mathrm{Mat}_2(W/4W) \to \mathrm{Mat}_2(k)^+ \tag{3.4}$$

sending $1+2\alpha$ mod 4 to α mod 2, where $\mathrm{Mat}_2(k)^+$ is the additive group of $\mathrm{Mat}_2(k)$. Let $\beta_i \in H^2(A_5, \mathrm{M}_2(k)^+)$ be the extension class defined by the bottom row of (3.3) and the inverse of the map δ in (3.4). It follows from the proof of [4, Prop. 6.1] that the extension class β_i is nontrivial for $i = 1, 2$. We define γ_i to be the restriction $\mathrm{Res}^{A_4}_{A_5}(\beta_i)$, $i = 1, 2$. Then $\gamma_i \in H^2(A_4, \mathrm{Mat}_2(k)^+)$ is the extension class defined by the inverse of the map δ in (3.4), and by the bottom row of the diagram

$$\begin{array}{ccccccccc} 1 & \to & 1 + 2\mathrm{Mat}_2(W/4W) & \to & H_i & \to & \mathrm{SL}_2(\mathbb{F}_4) & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \to & 1 + 2\mathrm{Mat}_2(W/4W) & \to & H_i' & \to & ST & \to & 1 \end{array} \tag{3.5}$$

where the top row is the bottom row of Diagram (3.3) and H_i' is the full pre-image of ST in H_i. The existence of a lift of U_{12}, respectively U_{21}, over $W/4W$ is equivalent to the splitting of the bottom row of (3.5), i.e., to the triviality of γ_i, for $i = 1, 2$. By [22, Prop. VIII.3.4], we have

$$\mathrm{Cores}^{A_5}_{A_4}(\mathrm{Res}^{A_4}_{A_5}(\beta_i)) = [A_5 : A_4] = 5\,.$$

Hence, since 5 is relatively prime to $2 = \mathrm{char}(k)$, we conclude that $\gamma_i = \mathrm{Res}_{A_5}^{A_4}(\beta_i)$ is nontrivial. This means that $R(A_4, U_{ij}) = k$ for $i \neq j$ in $\{0, 1, 2\}$.

It remains to show that there are no other indecomposable kA_4-modules V that have stable endomorphism ring equal to k. We use that $kA_4 \cong k\mathcal{Q}/I$ as k-algebras, where the quiver $\mathcal{Q}$ and the ideal I of the path algebra $k\mathcal{Q}$ are given as (see [13, Cor. V.2.4.1])

$$\mathcal{Q} \quad = \quad \begin{tikzcd} \overset{0}{\bullet} \arrow[rr, shift right, "\beta"'] \arrow[dr, shift right, "\kappa"'] & & \overset{1}{\bullet} \arrow[ll, shift right, "\gamma"'] \arrow[dl, shift right, "\delta"'] \\ & \underset{2}{\bullet} \arrow[ul, shift right, "\lambda"'] \arrow[ur, shift right, "\eta"'] & \end{tikzcd}$$

$$I \quad = \quad (\delta\beta, \lambda\delta, \beta\lambda, \kappa\gamma, \eta\kappa, \gamma\eta, \gamma\beta - \lambda\kappa, \beta\gamma - \eta\delta, \delta\eta - \kappa\lambda)\ .$$

According to [13, Lemma II.7.4], the remaining indecomposable kA_4-modules lie either in a 3-tube, but not at the mouth of the 3-tube, or in a 1-tube of the stable Auslander-Reiten quiver of kA_4. We first look at the case of a module X lying in a 3-tube, but not at the mouth of it. The modules at the mouth are exactly the uniserial modules U_{ij}, where $i \neq j$ in $\{0, 1, 2\}$. There are k-algebra automorphisms of kA_4 that interchange the two 3-tubes, respectively permute the modules at the mouth of a given 3-tube. Thus it is enough to show that the stable endomorphism ring is not k of a module X of length $2n \geq 4$ which has the following form (where we use the notation of [13, p. 62]):

$$X = \begin{array}{ccccccccccc} S_0 & & S_2 & & S_1 & & S_0 & & \cdots S_i & & \\ & S_1 & & S_0 & & S_2 & & S_1 & & & S_j \end{array}$$

where (S_i, S_j) are any of $\{(S_0, S_1), (S_2, S_0), (S_1, S_2)\}$. Recall that S_0, S_1 and S_2 are 1-dimensional over k. Since X has length at least 4, it is enough to show that an endomorphism ξ of X that sends the first module S_0 in the top isomorphically to the second module S_0 in the socle does not factor through a projective module. By [14], the endomorphisms of X that factor through a projective kA_4-module are linear combinations of three types of endomorphisms: (i) endomorphisms that send two consecutive modules in the top of X simultaneously to two consecutive modules in the socle of X, (ii) endomorphisms that send a module S_1 in the top of X to the first module S_1 in the socle of X, and (iii) endomorphisms that send the last module S_i in the top of X to a module S_i in the socle of X. It follows from this description that the endomorphism ξ of X cannot factor through a projective module.

We now consider the indecomposable kA_4-modules that lie in 1-tubes. By [13, II.7.0], these modules are all band modules. Since the only band is $\beta\kappa^{-1}\delta\gamma^{-1}\lambda\eta^{-1}$, we call these band modules $M(\mu, n)$, where n is a positive integer and $\mu \in k \setminus \{0\}$. Then $M(\mu, n)$ is $6n$-dimensional over k, and we have the following actions:

$$\beta \text{ acts as } \quad \begin{pmatrix} 0 & J_n(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \kappa \text{ acts as } \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\delta \text{ acts as } \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&I_n&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \gamma \text{ acts as } \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&I_n&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$\lambda \text{ acts as } \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&I_n\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \eta \text{ acts as } \begin{pmatrix} 0&0&0&0&0&I_n\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

where $J_n(\mu)$ is the $n \times n$ Jordan block with eigenvalue μ, and I_n is the $n \times n$ identity matrix. According to [14], the endomorphisms of $M(\mu, n)$ are all $6n \times 6n$ matrices of the form

$$\begin{pmatrix} A&0&0&E&0&0\\ 0&A&0&0&0&0\\ 0&0&A&0&0&D\\ 0&0&0&A&0&0\\ 0&C&0&0&A&0\\ 0&0&0&0&0&A \end{pmatrix}$$

so that C, D, E are arbitrary $n \times n$ matrices, and A is an upper triangular $n \times n$ matrix with equal entries in each diagonal. Hence $\dim_k \mathrm{End}_{kA_4}(M(\mu, n)) = 3n^2 + n$. A matrix calculation shows that all endomorphisms Θ_i of $M(\mu, n)$ factoring through the projective indecomposable module $P(S_i)$, $i = 0, 1, 2$, have the form

$$\Theta_0 = \begin{pmatrix} 0&0&0&J_n(\mu)H&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&H\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&K\\ 0&0&0&0&0&0\\ 0&KJ_n(\mu)&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$\Theta_2 = \begin{pmatrix} 0&0&0&L&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&L&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

where H, K, L are $n \times n$ matrices. The matrices of type Θ_2 which are sums of matrices of type Θ_0 and Θ_1 are exactly those for which $H = K$ and $J_n(\mu)H = KJ_n(\mu)$. This yields matrices of type Θ_2 for which L is an upper triangular matrix with equal entries in each diagonal. Thus $\dim_k \underline{\mathrm{End}}_{kA_4}(M(\mu, n)) = (3n^2 + n) - (3n^2 - n) = 2n$. This completes the proof of Lemma 3.3. $\square$

We now consider the case that B is a block of kG that is Morita equivalent to kA_4. Then there are 3 non-isomorphic simple B-modules, which we will denote by T_0, T_1, T_2 so that S_j corresponds to T_j under the Morita equivalence.

Proposition 3.4. *Let V be an indecomposable B-module with $\underline{\mathrm{End}}_{kG}(V) = k$. Then either*

(a) *$V \cong \Omega^i(T_j)$, $j = 0, 1, 2$, for some integer i and $R(G, V) = W$, or*
(b) *V is uniserial of length 2 and $R(G, V) = k$.*

Proof. Since B and kA_4 are Morita equivalent, the only B-modules with stable endomorphism ring equal to k are either of the form $\Omega^i(T_j)$, $j = 0, 1, 2$, or they are uniserial of length 2. Let now B_W be the block of WG corresponding to B. Then B_W and WA_4 are Morita equivalent by [15, Cor. 1.4]. Since S_j and T_j (respectively uniserial modules of length 2) correspond to each other under this Morita equivalence, Proposition 3.4 follows from Lemma 3.3 and Proposition 2.5. □

3.3. **Case (III).** We first assume that B is the principal block $B_0(kA_5)$ of kA_5. There are 3 non-isomorphic simple $B_0(kA_5)$-modules, which we will denote by S_0, S_1, S_2, so that S_0 is the trivial simple kA_5-module. Further, $B_0(kA_5)$ has 4 non-isomorphic uniserial modules of length 2, which we will denote by

$$U_{01} = \begin{matrix} S_0 \\ S_1 \end{matrix}\ ,\ U_{02} = \begin{matrix} S_0 \\ S_2 \end{matrix}\ ,\ U_{10} = \begin{matrix} S_1 \\ S_0 \end{matrix}\ ,\ U_{20} = \begin{matrix} S_2 \\ S_0 \end{matrix}\ .$$

Lemma 3.5. *Let V be an indecomposable $B_0(kA_5)$-module with $\underline{\mathrm{End}}_{kA_5}(V) = k$. Then either*

(a) *$V \cong \Omega^i(S_0)$ for some integer i and $R(A_5, V) = W$, or*
(b) *$V \cong \Omega^i(U_j)$, $j = 01, 02, 10, 20$, for some integer i and $R(A_5, V) = W$, or*
(c) *$V \cong \Omega^i(S_j)$, $j = 1, 2$ and $i = 0, 2, 4$, and $R(A_5, V) = k$.*

Proof. It follows from Mazur [16] that the universal deformation ring of S_0 is $R(A_5, S_0) = W$. By [4, Prop. 6.1], the universal deformation ring of S_j, $j = 1, 2$, is $R(A_5, S_j) = k$. Note that S_j, $j = 1, 2$, lie at the mouths of 3-tubes in the stable Auslander-Reiten quiver of $B_0(kA_5)$.

We now look at U_j, $j = 01, 02, 10, 20$. Since $\mathrm{Ext}^1_{kA_5}(U_j, U_j) = 0$, the universal deformation ring $R(A_5, U_j)$ is a quotient of W. Because $\Omega^{-2}(U_{01}) = U_{20}$ and $\Omega^{-2}(U_{02}) = U_{10}$, we now concentrate on $j = 01, 02$. Let F be the quotient field of W. The decomposition matrix corresponding to the principal block $B_0(kA_5)$ looks as follows (see [13, p. 295]):

	S_0	S_1	S_2
X_0	1	0	0
X_1	1	1	0
X_2	1	0	1
X_3	1	1	1

where X_0, X_1, X_2, X_3 are the irreducible FA_5-modules belonging to $B_0(kA_5)$. Since the exponent of A_5 is not a multiple of 4, F is a splitting field for A_5 and all its subgroups. Let P_0 be a lift over W of the kA_5-projective cover of S_0. By [10, Thm. 18.26], the F-character of P_0 is given by $F \otimes_W P_0 = X_0 \oplus X_1 \oplus X_2 \oplus X_3$. Hence the F-character of X_1 lies inside the F-character of P_0. By [24, Thm. 1 and Cor.], there exists a pure submodule Q_1 of P_0 so that P_0/Q_1 affords the F-character of X_1, and $(P_0/Q_1)/2(P_0/Q_1)$ is indecomposable. Since the top of P_0 is S_0, it follows that $(P_0/Q_1)/2(P_0/Q_1) \cong U_{01}$. Thus P_0/Q_1 is a lift of U_{01} over W. Analogously, we obtain a lift P_0/Q_2 of U_{02} over W. This means that $R(A_5, U_j) = W$.

Restriction from $A_5 = \mathrm{SL}_2(\mathbb{F}_4)$ to its Borel subgroup $ST \cong A_4$ of upper triangular matrices induces a stable equivalence between $B_0(kA_5)$ and kA_4 (see, for example, [20, Example 1]). Note that $\mathrm{Res}^{A_4}_{A_5}(S_0)$ is the trivial simple kA_4-module, and $\mathrm{Res}^{A_4}_{A_5}(S_i)$, $i = 1, 2$, is a uniserial kA_4-module of length 2. More precisely, there exists an ordering of the two nontrivial simple kA_4-modules Y_1 and Y_2 so that $\mathrm{Res}^{A_4}_{A_5}(S_1) = \begin{matrix} Y_2 \\ Y_1 \end{matrix}$ and $\mathrm{Res}^{A_4}_{A_5}(S_2) = \begin{matrix} Y_1 \\ Y_2 \end{matrix}$. It follows that $\mathrm{Res}^{A_4}_{A_5}(U_{01}) = \Omega(Y_1)$, $\mathrm{Res}^{A_4}_{A_5}(U_{02}) = \Omega(Y_2)$, $\mathrm{Res}^{A_4}_{A_5}(U_{10}) = \Omega^{-1}(Y_2)$, and $\mathrm{Res}^{A_4}_{A_5}(U_{20}) = \Omega^{-1}(Y_1)$. Hence it follows from Lemma 3.3 that the modules in parts (a), (b) and (c) of the statement of Lemma 3.5 are the only indecomposable $B_0(kA_5)$-modules that have stable endomorphism ring equal to k. This completes the proof of Lemma 3.5. $\square$

We now consider the case that B is a block of kG that is Morita equivalent to the principal block $B_0(kA_5)$. Then there are 3 non-isomorphic simple B-modules, which we will denote by T_0, T_1, T_2 so that S_j corresponds to T_j under the Morita equivalence. Further, there are 4 non-isomorphic uniserial B-modules of length 2, which we will denote by

$$U_{01}{}' = \begin{matrix} T_0 \\ T_1 \end{matrix}\ ,\ U_{02}{}' = \begin{matrix} T_0 \\ T_2 \end{matrix}\ ,\ U_{10}{}' = \begin{matrix} T_1 \\ T_0 \end{matrix}\ ,\ U_{20}{}' = \begin{matrix} T_2 \\ T_0 \end{matrix}\ .$$

Proposition 3.6. *Let V be an indecomposable B-module with $\underline{\mathrm{End}}_{kG}(V) = k$. Then either*

(a) *$V \cong \Omega^i(T_0)$ for some integer i and $R(G, V) = W$, or*
(b) *$V \cong \Omega^i(U_j{}')$, $j = 01, 02, 10, 20$, for some integer i and $R(G, V) = W$, or*
(c) *$V \cong \Omega^i(T_j)$, $j = 1, 2$ and $i = 0, 2, 4$, and $R(G, V) = k$.*

Proof. Since B and $B_0(kA_5)$ are Morita equivalent, the only B-modules with stable endomorphism ring equal to k are the modules occurring in parts (a), (b) and (c) of the statement of Proposition 3.6. Let now B_W be the block of WG corresponding to B. Then B_W and $B_0(WA_5)$ are Morita equivalent by [15, Cor. 1.4]. Since S_j and T_j (respectively U_j and $U_j{}'$) correspond to each other under this Morita equivalence, Proposition 3.6 follows from Lemma 3.5 and Proposition 2.5. $\square$

Theorem 1.2 follows from Lemma 2.3 and Propositions 3.2, 3.4 and 3.6.

Remark 3.7. It was shown in [18] that WA_4 and $B_0(WA_5)$ are splendidly derived equivalent. This means that the derived equivalence is given by a split endomorphism two-sided tilting complex of summands of permutation $(A_4 \times A_5)$-modules induced from diagonal subgroups of $P \times P$, where P is a common Sylow 2-subgroup of A_4 and A_5.

Note that the same universal deformation rings occur for modules V of these blocks over k having stable endomorphisms equal to k. Moreover, the stable Auslander-Reiten quivers of kA_4 and of $B_0(kA_5)$ are isomorphic as graphs. The modules which lie at the same position in the respective stable Auslander-Reiten quivers have isomorphic universal deformation rings.

4. The symmetric group S_4 of degree 4

Let k have characteristic 2. We want to consider the irreducible kS_4-modules and determine their universal deformation rings. The principal block of kS_4 is kS_4 itself, which has as defect groups dihedral groups of order 8. In particular, the defect groups are not abelian. We need the following result.

Lemma 4.1. *Let R be a W-algebra that surjects onto W so that $R/2R \cong k[t]/(t^2)$. Then either R is isomorphic to a W-subalgebra of $W(\mathbb{Z}/2\mathbb{Z})$, or R is isomorphic to $W[t]/(t^2)$.*

Proof. It follows from the assumptions that R is a quotient ring of a power series ring $W[[s]]$ in one variable. Hence we obtain a short exact sequence

$$0 \to J \to W[[s]] \xrightarrow{\tau} R \to 0 \,.$$

We first show that we can choose s so that $J \subseteq (s)$. Since R surjects onto W and $\tau : W[[s]] \to R$ is surjective, we obtain a short exact sequence

$$0 \to K \to W[[s]] \xrightarrow{\psi} W \to 0$$

with $J \subseteq K$. On the other hand, $K = \mathrm{Ker}(\psi)$ is generated by a linear polynomial of the form $as + b$ where a is a unit in W. So $K = (s + c)$ with $c = a^{-1}b$. If we replace s by $s + c$, then we can assume that $K = (s)$, and thus $J \subseteq (s)$. Next we show that $J + 2W[[s]] = (s^2, 2)$ as ideals. It follows from the assumptions that $J + 2W[[s]] \supseteq (s^2, 2)$. Now let $x \in J$; so $x = a_1 s + s^2 f(s)$ where f is some element of $W[[s]]$. We have to show that a_1 is a multiple of 2. If this were not the case, a_1 would be a unit in W, and thus $u = a_1 + sf(s)$ would be a unit in $W[[s]]$. Hence $x = s \cdot u$, which is impossible since $R/2R \cong k[t]/(t^2)$.

Using the assumptions on $R = W[[s]]/J$ and $J \subset (s)$, it follows now that J is generated by a square polynomial of the form $s^2 + rs$ where $r \in W$ is a multiple of 2. If $r = 0$, this implies that $R = W[[s]]/(s^2) \cong W[t]/(t^2)$. Now suppose $r \neq 0$. Then we define a W-algebra homomorphism ρ by

$$\begin{array}{rcl} \rho \,:\, R \,=\, W[[s]]\,/\,(s(s+r)) & \longrightarrow & W \times W \\ \overline{1} & \mapsto & (1,1) \\ \overline{s} & \mapsto & (-r, 0) \end{array}$$

which is injective. Let (x, y) be in the image of ρ, i.e.,

$$(x, y) = \rho(b_0 \overline{s} + b_1) = (b_1 - b_0 r, b_1) \,.$$

Then $y - x = b_0 r$ is a multiple of 2. Hence the image of ρ is a W-subalgebra of

$$\{(x, y) \in W \times W \mid x \equiv y \bmod 2W\} \cong W(\mathbb{Z}/2\mathbb{Z}) \,.$$

□

There are two non-isomorphic simple kS_4-modules, given by the trivial kS_4-module E_0 and by a module E_1 of k-dimension 2. Both E_0 and E_1 are inflated from irreducible $kS_3 \cong k(S_4/K)$-modules, where K denotes the normal subgroup of S_4 isomorphic to a Klein four group. In fact, E_1 is irreducible and projective as a kS_3-module.

Proposition 4.2. *The universal deformation rings of E_0 and E_1 as kS_4-modules are*

$$R(S_4, E_0) = W(\mathbb{Z}/2\mathbb{Z}) \quad \textit{and} \quad R(S_4, E_1) = W[t]/(t^2) \,.$$

Proof. It follows from [16] that $R(S_4, E_1) = WS_4^{ab,2} = W(\mathbb{Z}/2\mathbb{Z})$. Viewing E_1 as a kS_3-module leads to $R(S_3, E_1) = W$ by Lemma 2.2(iii). Hence, since S_3 is a quotient group of S_4, $R(S_4, E_1)$ surjects onto $R(S_3, E_1) = W$.

Denote $R(S_4, E_1)/2R(S_4, E_1)$ by R'. We now show that $R' \cong k[t]/(t^2)$. By using, for example, the k-algebra in [13, p. 63], which is Morita equivalent to kS_4 by [13, Cor. V.2.5.1], we get

$$\mathrm{Ext}^1_{kS_4}(E_1, E_1) = k \ ,$$

which implies $R' = k[t]/(t^r)$ for some r. There is a uniserial kS_4-module M that is given as $M = \begin{matrix} E_1 \\ E_1 \end{matrix}$. If we let t act as the shift down by one, it follows that M is a free $k[t]/(t^2)$-module of rank 2 which is a lift of E_1 over $k[t]/(t^2)$. Hence there exists a k-algebra homomorphism

$$\phi : R' \to k[t]/(t^2)$$

corresponding to M. Since M is indecomposable as a kS_4-module, it follows that ϕ is surjective. Suppose ϕ is not a k-algebra isomorphism. Then there exists a surjective k-algebra homomorphism $\phi_1 : R' \to k[t]/(t^3)$ so that $\pi\phi_1 = \phi$ where $\pi : k[t]/(t^3) \to k[t]/(t^2)$ is the natural projection. Let M_1 be a lift of E_1 over $k[t]/(t^3)$. Then M_1 is a lift of M over $k[t]/(t^3)$ and $t^2 M_1 \cong E_1$. Hence we have a short exact sequence of $k[t]/(t^3)S_4$-modules

$$0 \to t^2 M_1 \to M_1 \to M \to 0 \ . \tag{4.6}$$

Suppose this sequence splits as a sequence of kS_4-modules. Then $M \cong E_1 \oplus M$ as kS_4-modules, and t acts on $\begin{pmatrix} \lambda \\ v \end{pmatrix} \in E_1 \oplus M \cong M_1$ as multiplication by

$$U_t = \begin{pmatrix} 0 & \alpha \\ 0 & \mu_t \end{pmatrix}$$

where $\alpha : M \to E_1$ is a surjective kS_4-module homomorphism, and μ_t is the multiplication by t on M. Since $t^2 M_1 \cong E_1$, there exists a nonzero $\begin{pmatrix} \lambda \\ v \end{pmatrix} \in E_1 \oplus M \cong M_1$ so that $U_t^2 \begin{pmatrix} \lambda \\ v \end{pmatrix}$ is not zero. But

$$U_t^2 \begin{pmatrix} \lambda \\ v \end{pmatrix} = \begin{pmatrix} \alpha(\mu_t(v)) \\ \mu_t^2(v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a contradiction. Hence the short exact sequence (4.6) does not split as a sequence of kS_4-modules. Therefore, M_1 is an indecomposable kS_4-module. The description of the indecomposable kS_4-modules as string and band modules (see, for example, [13, §II.3] using [13, Cor. V.2.5.1]), shows that an indecomposable kS_4-module of the form M_1 does not exist. Hence $\phi : R' \to k[t]/(t^2)$ must be a k-algebra isomorphism.

By Lemma 4.1, it follows that either $R(S_4, E_1) \cong W[t]/(t^2)$, or $R(S_4, E_1)$ is isomorphic to a subquotient ring of $W(\mathbb{Z}/2\mathbb{Z})$. If the latter is the case, the proof of Lemma 4.1 shows that

$$R(S_4, E_1) \cong W[[s]]/(s(s + 2r_1)) = W[[s]]/((s + r_1)^2 - r_1^2) \cong W[[t]]/(t^2 - r_1^2)$$

for some nonzero $r_1 \in W$. Because there are two different surjective W-algebra homomorphisms $R(S_4, E_1) \to W$, there are two non-isomorphic lifts of E_1 as a

kS_4-module over W. We show now that this is not possible. Let F be the quotient field of W. Then

$$\begin{aligned} FS_4 &\cong F \times F \times \mathrm{Mat}_2(F) \times \mathrm{Mat}_3(F) \times \mathrm{Mat}_3(F) \ , \\ F(S_4/K) \cong FS_3 &\cong F \times F \times \mathrm{Mat}_2(F) \ . \end{aligned}$$

Since E_1 is irreducible over kS_4, it follows for lifts X of E_1 over W that $F \otimes_W X$ is an irreducible FS_4-module of rank 2. Since there is only one matrix ring of rank 2 in the product giving FS_4 which corresponds to the matrix ring of rank 2 in the product of $F(S_4/K) \cong FS_3$, it follows that X is actually a WS_3-module, i.e., it has trivial action by K. Hence all lifts of E_1 as a kS_4-module over W are actually lifts of E_1 as a kS_3-module over W. Since $R(S_3, E_1) = W$, this implies that all lifts of E_1 as a kS_4-module over W are isomorphic. Therefore, there is only one surjective W-algebra homomorphism $R(S_4, E_1) \to W$, and $R(S_4, E_1) = W[t]/(t^2)$. □

Corollary 4.3. *Question 1.1 has a positive answer in case V is a simple kS_4-module.*

Proof. We have to show that $W[t]/(t^2)$ is isomorphic to a subquotient ring of WD_8, where D_8 denotes a dihedral group of order 8. Let F be the quotient field of W. Then

$$FD_8 \cong F \times F \times F \times F \times \mathrm{Mat}_2(F) \ .$$

Let $a \in \mathrm{Mat}_2(F)$ with $a^2 = 0$, and let $x \in FD_8$ be such that x corresponds to $(0,0,0,0,a)$ in the above product. Then we can multiply x with a large enough power of 2 to obtain an element $y \in WD_8$ with $y^2 = 0$. Hence $W[t]/(t^2)$ is isomorphic to a subquotient ring of WD_8. □

Remark 4.4. Proposition 4.2 shows that Question 1.1 cannot be refined for non-abelian defect groups D by replacing D by its maximal abelian quotient.

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