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A BERGER-GREEN TYPE INEQUALITY FOR COMPACT LORENTZIAN MANIFOLDS

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Abstract. We give a Lorentzian metric on the null congruence associated with a timelike conformal vector field. A Liouville type theorem is proved and a boundedness for the volume of the null congruence, analogous to a well-known Berger-Green theorem in the Riemannian case, will be derived by studying conjugate points along null geodesics. As a consequence, several classification results on certain compact Lorentzian manifolds without conjugate points on its null geodesics are obtained. Finally, several properties of null geodesics of a natural Lorentzian metric on each odd-dimensional sphere have been found.

1. Introduction

In [13], L. W. Green solved the famous Blaschke conjecture in dimension two. A key tool to prove it was the Berger inequality, which asserts

$$\mathrm{area}(M,g) \geq \frac{2a^2}{\pi}\chi(M),$$

$\chi(M)$ being the Euler-Poincaré characteristic of M, for a 2-dimensional compact Riemannian manifold (M,g) without conjugate points before a fixed distance a in the parameter of any (unit) geodesic, and the equality holds only in case (M,g) has constant sectional curvature $\frac{\pi^2}{a^2}$. This inequality was later generalized to higher dimensions by Berger and independently by Green [6, Proposition 5.64] as follows:

$$\mathrm{Vol}(M,g) \geq \frac{a^2}{\pi^2 n(n-1)}\int_M S\, d\mu_g, \tag{1.1}$$

where (M,g) is a compact Riemannian manifold of dimension n, scalar curvature S, and without conjugate points before a fixed distance a in the parameter of any geodesic. Moreover, equality holds if and only if (M,g) has constant sectional curvature $\frac{\pi^2}{a^2}$. Note that the Berger inequality is a direct consequence from (1.1), via the Gauss-Bonnet theorem. The Berger-Green inequality can be equivalently written as follows:

$$\mathrm{Vol}(UM,\hat{g}) \geq \frac{a^2}{\pi^2(n-1)}\int_{UM} \widetilde{\mathrm{Ric}}\, d\mu_{\hat{g}}, \tag{1.2}$$

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where UM denotes the unit tangent bundle of (M, g), $\hat{g}$ is the restriction to UM of the Sasaki metric on the tangent bundle TM, $\widetilde{\mathrm{Ric}}$ denotes the quadratic form associated with the Ricci tensor of (M, g) and $d\mu_{\hat{g}}$ is the canonical measure associated with $\hat{g}$ (the kinematic measure).

On the other hand, the Blaschke conjecture for spheres of any dimension remained unsolved until 1978, when a nice proof was found using results of M. Berger, J. Kazdan, A. Weinstein and C. T. Yang (see [19] and references therein or [6, Appendix D]).

Inspired by this approach, in this paper we study conjugate points along null geodesics on Lorentzian manifolds. Remember that no null geodesic of any 2-dimensional Lorentzian manifold has conjugate points [4, Lemma 10.45]. On the other hand, it is not difficult to see that there are no conjugate points along null geodesics in any Lorentzian manifold of constant sectional curvature. For a general Lorentzian manifold, conjugate points along null (or more generally nonspacelike) geodesics have been systematically studied [4, Chapters 10 and 11], [23, Chapter 10], even from different approaches [10]. In the compact Lorentzian case, as far as we know, not much has been studied about the behaviour of conjugate points along null geodesics. We will introduce in this paper a *new integral inequality* on a compact Lorentzian manifold that admits a timelike conformal vector field which extends to the Lorentzian case the Berger-Green inequality. We will apply this inequality to get new information about conjugate points along null geodesics.

Given an $n(\geq 3)$-dimensional Lorentzian manifold (M, g), the null congruence associated with a timelike vector field K, $C_K M$, is a natural codimension-two orientable submanifold of the tangent bundle TM (see section 2 for the definition and main properties). It was used to characterize Robertson-Walker space-times [15], and to study infinitesimal null isotropy [21]. The null congruence has the property that, for each null tangent vector v, there exists a unique $\lambda \in \mathbb{R}$ such that $\lambda v \in C_K M$; thus, it can be seen as the manifold of all null tangent directions.

The null congruence can be endowed with a Lorentzian metric $\hat{g}$, namely, the restriction to $C_K M$ of the Sasaki metric on TM induced from the Lorentzian metric g. Several geometric properties of $(C_K M, \hat{g})$ are similar to the well-known ones of the unit tangent bundle of a Riemannian manifold; for instance, $C_K M$ is a fibre bundle with fibre a sphere $\mathbb{S}^{n-2}$ (note that n is the dimension of M) and the natural projection from $C_K M$ onto M is a semi-Riemannian submersion in the sense of Gray [12] and O'Neill [23, Definition 7.44].

The unit tangent bundle is invariant by the geodesic flow. Nevertheless, $C_K M$ is not invariant by the geodesic flow in general, but it is if and only if the vector field K is assumed to be conformal (Proposition 2.5). This fact is fundamental for our approach, because it permits us to choose a distinguished affine parameter for each null geodesic, which we will call *affine K-parameter* (see discussion after Proposition 2.5). Moreover, the existence of a timelike conformal vector field permits us to prove (without assuming compactness) a Liouville type theorem, i.e., the canonical measure associated with the Lorentzian metric $\hat{g}$ on $C_K M$ is preserved by the geodesic flow (Theorem 2.6). This will be a technical key fact in reaching our main goal in this paper.

On the other hand, the study of conformal vector fields in Lorentzian Geometry is a topic of special importance. Historically, it has been developed mainly under assumptions of interest in physics. From a mathematical viewpoint, the particular case of Killing vector fields has appeared as a useful tool to get classification theorems in some areas of Lorentzian Geometry, [17] and [29]; moreover, standard Lorentzian space forms (those with a time-orientable double semi-Riemannian covering admitting a timelike Killing vector field) have been studied in [22]. In the compact Lorentzian case, the assumption of the existence of a timelike conformal vector field yields to geodesic completeness [25] (remember that a compact Lorentzian manifold may be incomplete). Finally, the existence of a timelike conformal vector field has also been used to classify compact Ricci-flat Lorentzian manifolds [26], [27], and to study compact spacelike hypersurfaces of constant mean curvature [2].

The main result of this paper is the integral inequality (Theorem 3.2),

$$\mathrm{Vol}(C_K M, \hat{g}) \geq \frac{a^2}{\pi^2(n-2)} \int_{C_K M} \widetilde{\mathrm{Ric}}\, d\mu_{\hat{g}} \tag{1.3}$$

where null geodesics of (M, g) are assumed to have no conjugate point in $[0, a)$. Moreover, the equality can be characterized in terms of the null sectional curvature. In Lorentzian geometry, sectional curvature plays quite a different role than in Riemannian geometry due to the fact that it cannot be defined on null (i.e., degenerate) tangent planes. Harris introduced the notion of null sectional curvature and pointed out its relationship with the usual sectional curvature [14]. If a null congruence has been fixed, then the null sectional curvature can be seen as a map on null tangent planes (see comment after Remark 2.8).

The remainder of this paper is devoted to showing several applications of the previous integral inequality (1.3). In Theorem 3.5 we give another version of (1.3) which implies, as a particular case, the Berger-Green inequality in the Riemannian case (Corollary 3.9). If the conformal vector field K is assumed to be Killing and null geodesics are assumed to have no conjugate point in $[0, a)$, then (1.3) may be notably improved. Moreover, in this case, equality holds if and only if $g(K, K)$ is constant and the universal Lorentzian covering of (M, g) is globally isometric to the semi-Riemannian product $(\mathbb{R} \times \mathbb{S}^{n-1}(\frac{ah}{\pi}), -dt^2 + g_{can})$, with $h = 1/\sqrt{-g(K, K)}$, (Corollary 3.8).

Moreover, we prove in Corollary 3.10 that if a compact Lorentzian manifold (M, g) of constant scalar curvature S_0 admits a timelike Killing vector field K and has no conjugate point on its null geodesics, then $S_0 \leq 0$, and $S_0 = 0$ implies that the first Betti number of M is not zero and the Levi-Civita connection of g is Riemannian. So, we obtain a proper extension of Kamishima's theorem [17, Theorem A(1)] (see Remark 3.11 for more details).

As a practical application, we show in the last section of this paper several nice properties of the null geodesics of a natural Lorentzian metric on the odd-dimensional sphere $\mathbb{S}^{2n+1}$. We prove the existence of a conjugate point along any null geodesic. Furthermore, the first conjugate point along a null geodesic occurs at the same value $\frac{\pi}{2\sqrt{2}}$ of its affine K-parameter, which is independent of the point and the null direction. Moreover, the topology of the lightlike conjugate locus of any point of $\mathbb{S}^{2n+1}$ is analyzed (Proposition 4.4). For the special case $\mathbb{S}^3$, we notice that its null sectional curvature is 8, and it is the unique odd-dimensional Lorentzian sphere with constant null sectional curvature (Remark 4.5).

2. Null congruence and geodesic vector field

Let (M,g) be an $n(\geq 2)$-dimensional Lorentzian manifold, that is, a (connected) smooth manifold M endowed with a nondegenerate metric g with signature $(-,+,.....,+)$. We shall write ∇ for its Levi-Civita connection, R for its Riemannian curvature tensor (our convention on the curvature tensor is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$), Ric for its Ricci tensor and $\widetilde{\mathrm{Ric}}$ for its associated quadratic form. As usual, the causal character of a tangent vector $v \in T_pM$ is timelike (resp. null, spacelike) if $g(v,v) < 0$ (resp. $g(v,v) = 0$ and $v \neq 0$, $g(v,v) > 0$ or $v = 0$). If $v \in T_pM$, then γ_v will denote the unique geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. It is well known that the causal character of $\gamma'(t)$, for any geodesic γ of (M,g), does not depend on the parameter t. In particular, a null geodesic γ of (M,g) is a geodesic such that $\gamma'(t)$ is a null vector. A vector field $K \in \mathfrak{X}(M)$ is said to be timelike if K_p is timelike for all $p \in M$. A timelike or null tangent vector $v \in T_pM$ is said to be future (resp. past) with respect to K if $g(v,K_p) < 0$ (resp. $g(v,K_p) > 0$). We will write $U = hK$ where $h = [-g(K,K)]^{-\frac{1}{2}}$ and so $g(U,U) = -1$.

Unless we indicate otherwise, (M,g) will denote a Lorentzian manifold that admits a timelike vector field K. Recall [15], [21] that the *null congruence associated with* K is the set

$$C_KM = \{v \in TM : g(v,v) = 0 \text{ and } g(v,K_{\pi v}) = 1\},$$

where $\pi : TM \longrightarrow M$ is the natural projection. We put $(C_KM)_p = C_KM \cap T_pM$.

In this section we develop several properties of the null congruence. The first important result is Theorem 2.6, which is a Liouville type theorem that allows us to handle properly some integrals on the null congruence.

Proposition 2.1. *C_KM is an orientable imbedded submanifold of TM with dimension $2(n-1)$.*

Proof. Let $F : TM \longrightarrow \mathbb{R}^2$ be the map defined by $F(v) = (g(v,v), g(v,K_{\pi v}))$. It is clear that F is smooth and C_KM is the inverse image of its regular value $(0,1)$. TM being an orientable manifold, it suffices to construct two pointwise independent vector fields along the imbedding $C_KM \hookrightarrow TM$, $\mathbf{A}$ and $\mathbf{K}$, such that

$$T_vTM = T_vC_KM \oplus W_v \text{ for all } v \in C_KM, \tag{2.1}$$

where $W_v = Span\,\{\mathbf{A}_v, \mathbf{K}_v\}$, [1, Proposition 6.5.8]. In fact, $\mathbf{A}$ and $\mathbf{K}$ can be chosen such that

$$\mathbf{A}_v = (v)_v \text{ and } \mathbf{K}_v = (K_{\pi v})_v,$$

where $(\ )_v : T_{\pi v}M \longrightarrow T_vT_{\pi v}M$ denotes the natural identification. □

Proposition 2.2. *(C_KM, π, M) is a fibre bundle with fibre type $\mathbb{S}^{n-2}$ and structure group $O(n-1)$.*

Proof. Let $E_1, E_2, ..., E_{n-1}, E_n = K$ be a local basis on an open subset $O \subset M$ such that $g(E_i,E_j) = \delta_{ij}$ and $g(E_i,K) = 0$ for every $1 \leq i,j \leq n-1$. We can define a local trivialization as follows:

$$\Psi : \pi^{-1}(O) \longrightarrow O \times \mathbb{S}^{n-2}, \quad v \mapsto \big(\pi v, 1/\sqrt{-v_n}(v_1,...,v_{n-1})\big),$$

where $v = \sum_{i=1}^n v_iE_i(\pi v)$. The result follows after choosing a suitable family of such local trivializations [16, Chapter 5]. □

Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇. The *connector* c associated with ∇ is defined by

$$c : TTM \longrightarrow TM, \qquad X \mapsto \frac{\nabla \alpha}{dt} \mid_0,$$

where α is a curve in TM with $\alpha'(0) = X$ and $\frac{\nabla \alpha}{dt}$ is the covariant derivative of the vector field α along the curve $\pi \circ \alpha$ on M. The map c is well defined and the pair (c, π) is a vector bundle morphism from $\pi_{TM} : TTM \longrightarrow TM$ to $\pi : TM \longrightarrow M$, π_{TM} being the natural projection. Moreover, if $(u)_v \in T_vTM$ where $u \in T_{\pi v}M$, then $c((u)_v) = u$.

We consider the Sasaki metric $\hat{g}$ on TM defined from g, [32, Section 4.5] and [6, Section 1.K]

$$\hat{g}(X, Y) = g(d\pi(X), d\pi(Y)) + g(c(X), c(Y)) \quad \text{ for all } X, Y \in T_vTM.$$

Recall that $\pi : (TM, \hat{g}) \longrightarrow (M, g)$ is a semi-Riemannian submersion in the terminology of Gray [12] or O'Neill [23, Definition 7.44]. Let c_v be the restriction $c \mid_{T_v(TM)}$ for every $v \in TM$. We get the following $\hat{g}$-orthogonal decomposition:

$$T_vTM = \text{Ker } c_v \oplus T_vT_{\pi v}M.$$

If (M, g) is a Riemannian manifold, then $(TM, \hat{g})$ is also a Riemannian manifold; but if (M, g) is Lorentzian, then $(TM, \hat{g})$ is a semi-Riemannian with index 2. In the sequel, $\hat{g}$ also will represent the induced metric on C_KM from the Sasaki one of TM.

Proposition 2.3. *$(C_KM, \hat{g})$ is a Lorentzian manifold and the restriction of π to C_KM is a semi-Riemannian submersion with spacelike fibres.*

Proof. In the same notation as in the proof of Proposition 2.1, we get $\hat{g}(\mathbf{A}, \mathbf{A}) = 0$, $\hat{g}(\mathbf{A}, \mathbf{K}) = 1$ and $\hat{g}(\mathbf{K}, \mathbf{K}) < 0$. Therefore, at any $v \in C_KM$, W_v is a Lorentzian vector space with the induced metric from $\hat{g}$, and it is not difficult to show that the direct sum decomposition (2.1) is an orthogonal decomposition. This gives that $(C_KM, \hat{g})$ is a Lorentzian manifold.

For all $p \in M$, $T_v(C_KM)_p = \{w \in T_pM : g(w, v) = g(w, K_p) = 0\}$, which implies that the induced metric from $\hat{g}$ on $(C_KM)_p$ is Riemannian. On the other hand, if we denote by $[T_v(C_KM)_p]^{\perp}$ the orthogonal complement of $T_v(C_KM)_p$ in T_vC_KM, then $[T_v(C_KM)_p]^{\perp} = \text{Ker } c_v$, and therefore, $\hat{g}(X, Y) = g(d\pi(X), d\pi(Y))$ holds for every $X, Y \in [T_v(C_KM)_p]^{\perp}$. □

Next consider that (M, g) is a compact Lorentzian manifold. Then, it is easily seen that C_KM must be also compact. As a nice application of the previous result, if $\dim M \geq 3$, a slight variation of [5, Proposition A.III.5] permits us to obtain the Fubini type result,

$$\int_{C_KM} f \, d\mu_{\hat{g}} = \int_M \left\{ \int_{(C_KM)_p} f \, d\mu_p \right\} d\mu_g, \tag{2.2}$$

where $f \in C^0(C_KM)$, $d\mu_{\hat{g}}$ is the canonical measure induced from the Lorentzian metric $\hat{g}$ of C_KM (equivalently, the canonical measure induced from the $\hat{g}$-volume in the following Proposition 2.4) and $d\mu_p$ (resp. $d\mu_g$) is the canonical measure associated with $\hat{g}\,|_{(C_K M)_p}$(resp. g). In particular, for every $f \in C^0(M)$ we obtain

$$\int_{C_K M} (f \circ \pi)\, d\mu_{\hat{g}} = \int_M (f \ \mathrm{vol}_K)\, d\mu_g,$$

where $\mathrm{vol}_K : M \longrightarrow \mathbb{R}$ is defined by $\mathrm{vol}_K(p) = \mathrm{Vol}\left((C_K M)_p, \hat{g}\right)$. It is not difficult to show that $((C_K M)_p, \hat{g})$ and $\left(\mathbb{S}^{n-2}, h^2(p) g_{can}\right)$ are isometric Riemannian manifolds where $(\mathbb{S}^{n-2}, g_{can})$ is the canonical Riemannian unit sphere; so

$$\mathrm{vol}_K(p) = \omega_{n-2} h^{n-2}(p), \tag{2.3}$$

where ω_{n-2} denotes $\mathrm{Vol}(\mathbb{S}^{n-2}, g_{can})$. In particular, we have

$$\mathrm{Vol}(C_K M, \hat{g}) = \omega_{n-2} \int_M h^{n-2}\, d\mu_g. \tag{2.4}$$

Recall that the cotangent bundle T^*M carries a natural symplectic structure $d\alpha$ where α is the 1-form $\alpha(Y) = -\pi_{T^*M}(Y)\,[dq(Y)]$ with $Y \in TT^*M$, and $\pi_{T^*M} : TT^*M \longrightarrow T^*M$, and $q : T^*M \longrightarrow M$ are the natural projections.

If (M, g) is a semi-Riemannian manifold, then g determines a vector bundle isomorphism $\flat$ from TM onto T^*M by putting $v \mapsto g(v, \ )$. We call α_g the pull-back by $\flat$ of α; so

$$\alpha_g(X) = -g(v, d\pi(X)), \tag{2.5}$$

where $X \in T_v(TM)$. Thus, $d\alpha_g$ is a symplectic form on TM. The geodesic vector field, induced from the metric g on M, is the vector field $Z_g \in \mathfrak{X}(TM)$ given by

$$i_{Z_g}(d\alpha_g) = dE, \tag{2.6}$$

where $E(v) = \frac{1}{2} g(v, v)$. An important property of this vector field is

$$d\pi(Z_g) = \pi_{TM}(Z_g), \tag{2.7}$$

where $\pi_{TM} : TTM \longrightarrow TM$ is the natural projection.

Finally, the flow $\{\Phi_t\}$ of Z_g, given by $\Phi_t(v) = \gamma_v'(t)$, is called the *geodesic flow* of (M, g). If (M, g) is assumed to be complete, then Z_g is a complete vector field.

Proposition 2.4. *Let (M, g) be a Lorentzian manifold that admits a timelike vector field K. Then the $\hat{g}$-volume of $(C_K M, \hat{g})$ is given, up to a constant multiple, by the $2(n-1)$ form*

$$\beta_g \wedge \alpha_g \wedge (d\alpha_g)^{n-2},$$

where $\beta_g(X) = -g(K_{\pi v}, d\pi(X))$ for every $X \in T_v C_K M$ and α_g also denotes its restriction to $C_K M$.

Proof. It is well known that $\Omega = (d\alpha_g)^n$ is, up to a constant multiple, the $\hat{g}$-volume of $(TM, \hat{g})$. So $i_{\mathbf{K}} i_{\mathbf{A}} \Omega$ is a volume form on $C_K M$; moreover, it is the $\hat{g}$-volume, up to a constant multiple. In fact, we take the following vector fields along the inclusion $C_K M \hookrightarrow TM$,

$$\xi_{\mathbf{1}} = \frac{\mathbf{K}}{\sqrt{-\hat{g}(\mathbf{K}, \mathbf{K})}} \quad \text{and} \quad \xi_{\mathbf{2}} = \sqrt{-\hat{g}(\mathbf{K}, \mathbf{K})}\,\mathbf{A} + \xi_{\mathbf{1}}, \tag{2.8}$$

and a straightforward computation gives that $\xi_{\mathbf{1}}$ and $\xi_{\mathbf{2}}$ are $\hat{g}$-orthonormal with $\xi_{\mathbf{1}}$ timelike and $i_{\mathbf{K}} i_{\mathbf{A}} \Omega = i_{\xi_1} i_{\xi_2} \Omega$.

Now, we get $\mathfrak{L}_{\mathbf{A}} \alpha_g = i_{\mathbf{A}} d\alpha_g + d i_{\mathbf{A}} \alpha_g = i_{\mathbf{A}} d\alpha_g$. On the other hand, a direct calculation gives us $\mathfrak{L}_{\mathbf{A}} \alpha_g = \alpha_g$. Therefore, $i_{\mathbf{A}} \Omega = n \alpha_g \wedge (d\alpha_g)^{n-1}$. Thus $i_{\mathbf{K}} i_{\mathbf{A}} \Omega =$

$n\left[i_{\mathbf{K}}\alpha_g \wedge (d\alpha_g)^{n-1} - (n-1)\alpha_g \wedge i_{\mathbf{K}} d\alpha_g \wedge (d\alpha_g)^{n-2}\right]$. We end the proof by taking into account that $i_{\mathbf{K}}\alpha_g = 0$ and $i_{\mathbf{K}} d\alpha_g = \mathfrak{L}_{\mathbf{K}}\alpha_g = \beta_g$. □

Since $\hat{g}(Z_g, \mathbf{A}) = \hat{g}(Z_g, \mathbf{K}) = 0$, we obtain from (2.1) that $Z_g(v) \in T_v C_K M$ for every $v \in C_K M$ and every timelike vector field K. Nevertheless, $C_K M$ may be not invariant by any local geodesic flow. We will characterize this fact in the following result.

Recall that a conformal vector field on a semi-Riemannian manifold (M, g) is a vector field K such that $\mathfrak{L}_K g = \rho g$, where ρ is a (necessarily smooth) function on M. Equivalently, this condition can be written

$$g(\nabla_X K, Y) + g(X, \nabla_Y K) = \rho g(X, Y), \tag{2.9}$$

for every $X, Y \in \mathfrak{X}(M)$. When $\rho = 0$, the conformal vector field is Killing. It is easy to check that for an arbitrary timelike vector field K, $C_K M$ is invariant by the differential of the flow of K if and only if K is Killing.

Proposition 2.5. *Let (M, g) be a Lorentzian manifold that admits a timelike vector field K. Then, $C_K M$ is invariant by any local geodesic flow if and only if K is a conformal vector field.*

Proof. If K is a conformal vector field, then for every $v \in C_K M$, we get from (2.9) that $\frac{d}{dt} g(\gamma_v^{'}, K_{\gamma_v}) = g(\gamma_v^{'}, \frac{\nabla K_{\gamma_v}}{dt}) = \frac{1}{2}(\rho \circ \gamma_v) g(\gamma_v^{'}, \gamma_v^{'}) = 0$; thus $g(\Phi_t(v), K_{\gamma_v(t)}) = g(v, K_{\pi v}) = 1$. Conversely, if $C_K M$ is invariant by any local geodesic flow, then an easy computation gives $(\mathfrak{L}_K g)(v, v) = 0$ for every $v \in C_K M$. Thus, from [9, Theorem 1], there exists $\rho \in C^\infty(M)$ such that $\mathfrak{L}_K g = \rho g$. □

Note that if K is a timelike conformal vector field, then every null geodesic γ_v of (M, g) with $\gamma_v'(0) = v \in C_K M$, gives rise to a null geodesic γ_v' of $(C_K M, \hat{g})$. Furthermore, each null geodesic β of (M, g) may be reparametrized to obtain a null geodesic α that satisfies $\alpha'(t) \in C_K M$ for all t. In fact, from Proposition 2.5, it can be shown that $g(\beta', K_\beta) = a \in \mathbb{R}$, $a \neq 0$. Thus, if we put $\alpha(t) = \beta(\frac{t}{a})$ we achieve $g(\alpha', K_\alpha) = 1$. Thus, in the sequel, we always assume that a null geodesic γ of (M, g) satisfies $g(\gamma', K_\gamma) = 1$ and then we will call the parameter t of γ the *affine K-parameter*.

Let $x : [0, a] \times (-\delta, \delta) \to M$, $x = x(t, s)$, be a smooth variation of a curve γ. We write $\partial_t x$, $\partial_s x$ for $dx(\frac{\partial}{\partial t})$, $dx(\frac{\partial}{\partial s})$ respectively, and denote covariant derivation of vector fields along x, with respect to the directions $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial s}$ by ∇_t, ∇_s, respectively. It is well known that $\nabla_t \partial_s x = \nabla_s \partial_t x$ (see, for instance, [23, Proposition 4.44]).

In the sequel of this paper, the following Liouville type result will be a key fact.

Theorem 2.6. *Let (M, g) be a Lorentzian manifold that admits a timelike conformal vector field K. Then*

$$\operatorname{div}(Z_g \mid_{C_K M}) = 0,$$

where div *denotes the divergence operator on $(C_K M, \hat{g})$.*

Proof. In this proof Z_g will be written for $Z_g \mid_{C_K M}$. From Proposition 2.4 it suffices to show that $\mathfrak{L}_{Z_g}(\beta_g \wedge \alpha_g \wedge (d\alpha_g)^{n-2}) = 0$.

From (2.5), (2.6) and (2.7), taking into account that $\alpha_g(Z_g) = -2E$ vanishes on $C_K M$, the above condition is equivalent to

$$\left(\mathfrak{L}_{Z_g}\beta_g\right) \wedge \alpha_g \wedge (d\alpha_g)^{n-2} = 0. \tag{2.10}$$

Let $v \in C_K M$ and $X \in T_v C_K M$. We take a curve λ on $C_K M$ such that $\lambda(0) = v$ and $\lambda'(0) = X$. The following formula gives a variation of $\gamma_v' : [0, \varepsilon] \longrightarrow C_K M$ for some $\varepsilon > 0$ sufficiently small:

$$x : [0, \varepsilon] \times (-\delta, \delta) \longrightarrow C_K M, \quad (t, s) \mapsto \Phi_t(\lambda(s)) = \gamma'_{\lambda(s)}(t).$$

Observe that all the longitudinal curves of x are null geodesics of $(C_K M, \hat{g})$.

One easily checks that $d\Phi_\varepsilon(X) = d\Phi_\varepsilon(\lambda'(0)) = \widetilde{J}(\varepsilon)$ where $\widetilde{J}$ is the variation vector field of x. Now, in a natural way, x provides us with a variation $\pi \circ x$ of $\gamma_v : [0, \varepsilon] \longrightarrow M$ with null geodesics as longitudinal curves. Thus, we can get

$$\left(\Phi_\varepsilon^* \beta_g\right)_v (X) = (\beta_g)_{\Phi_\varepsilon(v)}(d\Phi_\varepsilon(X)) = -g(J(\varepsilon), K_{\gamma_v(\varepsilon)}),$$

where J is the variation vector field of $\pi \circ x$; so

$$\mathfrak{L}_{Z_g} \beta_g(X) = -\frac{d}{dt} g(J, K_{\gamma_v}) \mid_0 . \tag{2.11}$$

Since $\partial_t(\pi \circ x)(0, s) = \lambda(s) \in C_K M$, we obtain $g(\partial_t(\pi \circ x), K_{\pi \circ x}) = 1$, and thus

$$g\big(\nabla_s \partial_t(\pi \circ x)(t, s), K_{\pi \circ x}(t, s)\big) + g\big(\partial_t(\pi \circ x)(t, s), \nabla_s K_{\pi \circ x}(t, s)\big) = 0.$$

If we take $t, s = 0$ in the previous equation, then

$$g\Big(K_{\pi v}, \frac{\nabla J}{dt} \mid_0 \Big) - g\Big(J(0), \frac{\nabla K_{\gamma_v}}{dt} \mid_0 \Big) + (\rho \circ \pi)(v) g\left(v, J(0)\right) = 0, \tag{2.12}$$

where $\mathfrak{L}_K g = \rho\, g$. But $J(0) = d\pi(X)$ and

$$c(X) = \frac{\nabla \lambda}{ds} \mid_0 = \nabla_s \partial_t(\pi \circ x)(0, 0) = \nabla_t \partial_s(\pi \circ x)(0, 0) = \frac{\nabla J}{dt} \mid_0 .$$

On the other hand, $\hat{g}(X, \mathbf{K}_v) = g(c(X), K_{\pi v}) = 0$. Thus, (2.12) and (2.11) yield $\mathfrak{L}_{Z_g} \beta_g = (\rho \circ \pi) \alpha_g$. □

Remark 2.7. Observe that our proof in Theorem 2.6 cannot be directly deduced from the classical Liouville theorem in the Riemannian case [6, Proposition 1.56]. In fact, the $\hat{g}$-volume of $(C_K M, \hat{g})$ introduced in Proposition 2.4, clearly depends on K, as is shown by the presence of the 1-form β_g in its definition. Compare with [3, Appendix].

Assume that (M, g) is a compact Lorentzian manifold and K is a timelike conformal vector field. In this case, $Z_g \mid_{C_K M}$ is a complete vector field. Therefore, as a consequence of Theorem 2.6, we have

$$\int_{C_K M} (f \circ \Phi_t)\, d\mu_{\hat{g}} = \int_{C_K M} f\, d\mu_{\hat{g}} \tag{2.13}$$

for every $f \in C^0(C_K M)$ and $t \in \mathbb{R}$.

Remark 2.8. In [21] the null congruence has been defined as only associated with a unit timelike vector field U; so, from (2.3), $\mathrm{vol}_U(p)$ would not depend on p. Nevertheless, if we take U instead of K in our computations, then U may be nonconformal. As we have shown, to be conformal is crucial in getting Proposition 2.5 and Theorem 2.6 (see also [15]).

Now, we close this section by analyzing the well-known notion of null sectional curvature [4, Definition A.6], [15]. Let (M, g) be a Lorentzian manifold, $\dim M \geq 3$. Given a null vector v and a null plane σ containing v, the null sectional curvature with respect to v of the plane σ is defined to be $\mathcal{K}_v(\sigma) = g(R(u, v)v, u)/g(u, u)$, where $u \neq 0$ is any nonnull (and therefore spacelike) vector in σ. This is independent of the choice of the nonzero spacelike vector u, but it does depend quadratically on the choice of v in σ. However, if a null congruence associated with a timelike vector field K has been fixed and we make the choice of $v \in C_K M \cap \sigma$, then the null sectional curvature can be thought of as a function on null planes. In this paper we always use null sectional curvature under that assumption and we will call it the *K-normalized null sectional curvature.*

It is well known that null sectional curvature is identically zero if and only if the sectional curvature is constant [14, Proposition 2.3]. On the other hand, a semi-Riemannian product $I \times B$ of a negative definite open interval $(I, -dt^2)$ and a Riemannian manifold (B, g_B) has constant $\frac{\partial}{\partial t}$-normalized null sectional curvature c if and only if (B, g_B) has constant sectional curvature c.

3. The integral inequality

In this section we prove a boundedness property for the volume of the null congruence associated with a compact Lorentzian manifold that admits a timelike conformal vector field and with no conjugate point along any null geodesic before a fixed length of its affine K-parameter. As mentioned in the introduction, this will resemble the inequality of Berger and Green in the Riemannian case. In fact, in the particular case that our Lorentzian manifold is the product of $(\mathbb{S}^1, -g_{can})$ and an arbitrary compact Riemannian manifold (B, g_B), we obtain, as a consequence of our result, the classical theorem of Berger and Green for (B, g_B).

We recall that, if $\gamma_v : [0, a] \longrightarrow M$ is a null geodesic of (M, g) such that there are no conjugate points of $\gamma_v(0)$ in $[0, a)$, then the *Hessian form* $H^{\perp}_{\gamma_v}$ is positive semidefinite, i.e.,

$$H^{\perp}_{\gamma_v}(V, V) = \int_0^a \Big[g\Big(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}\Big) - g(R(V, \gamma_v')\gamma_v', V) \Big] dt \geq 0,$$

for every piecewise smooth vector field V along γ_v such that $V(0) = 0$, $V(a) = 0$ and $g(\gamma_v', V) = 0$, [23, pp. 290, 291].

A standard argument permits us to show:

Lemma 3.1. *Let (M, g) be a Lorentzian manifold, and let $\gamma_v : [0, a] \longrightarrow M$ be a null geodesic such that $\gamma_v(0)$ has no conjugate point in $[0, a)$. Let V be a vector field along γ_v such that $V(0) = 0$, $V(a) = 0$, $g(\gamma_v', V) = 0$, and $H^{\perp}_{\gamma_v}(V, V) = 0$. Then $\frac{\nabla^2 V}{dt^2} + R(V, \gamma_v')\gamma_v' = f\,\gamma_v'$ where f is a smooth function.* □

Now, we are ready to prove the main result of this paper:

Theorem 3.2. *Let (M, g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold that admits a timelike conformal vector field K. If there exists $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0, a] \longrightarrow M$, with $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0, a)$, then*

$$\mathrm{Vol}(C_K M, \hat{g}) \geq \frac{a^2}{\pi^2 (n-2)} \int_{C_K M} \widetilde{\mathrm{Ric}}\, d\mu_{\hat{g}}. \tag{3.1}$$

Moreover, equality holds if and only if (M, g) has K-normalized null sectional curvature $\frac{\pi^2}{a^2}$.

Proof. Let $\gamma_v : [0, a] \longrightarrow M$ be a null geodesic of (M, g) such that $v \in C_K M$.

We take a set of parallel vector fields $E_3, ..., E_n$ along γ_v such that $g(E_i, E_j) = \delta_{ij}$ and $g(\gamma_v', E_i) = 0$ for every i, j. Thus, we have [23, Proof of Lemma 8.9]

$$\widetilde{\mathrm{Ric}}(\gamma_v') = \sum_{i=3}^{n} g(R(E_i, \gamma_v')\gamma_v', E_i).$$

If we put $X_i(t) = \sin \frac{\pi t}{a} E_i(t)$, then

$$\sum_{i=3}^{n} H_{\gamma_v}^{\perp}(X_i, X_i) = (n-2)\frac{\pi^2}{a^2}\int_0^a \cos^2 \frac{\pi t}{a} dt - \int_0^a \sin^2 \frac{\pi t}{a} \widetilde{\mathrm{Ric}}(\gamma_v') dt \geq 0. \tag{3.2}$$

Now, we integrate both sides in the previous inequality on the space $C_K M$. Taking into account that $\gamma_v'(t) = \Phi_t(v)$, Fubini's theorem and (2.13), we get

$$(n-2)\frac{\pi^2}{a^2}\mathrm{Vol}(C_K M, \hat{g}) \int_0^a \cos^2 \frac{\pi t}{a} dt \geq \int_{C_K M} \widetilde{\mathrm{Ric}}\, d\mu_{\hat{g}} \int_0^a \sin^2 \frac{\pi t}{a} dt.$$

Finally, (3.1) follows from $\int_0^a \cos^2 \frac{\pi t}{a} dt = \int_0^a \sin^2 \frac{\pi t}{a} dt > 0$.

We are left with the proof of the if and only if part in the equality case. Note that equality holds in (3.1) if and only if

$$H_{\gamma_v}^{\perp}(X_i, X_i) = 0 \quad \text{for every } i; \tag{3.3}$$

so, from Lemma 3.1 we have $\frac{\nabla^2 X_i}{dt^2} + R(X_i, \gamma_v')\gamma_v' = f_i\, \gamma_v'$, where the functions f_i are smooth. Consequently,

$$g(R(X_i, \gamma_v')\gamma_v', X_i)(t) = \frac{\pi^2}{a^2} \sin^2 \frac{\pi t}{a}, \quad t \in [0, a].$$

On the other hand, a direct computation gives

$$g(R(X_i, \gamma_v')\gamma_v', X_i)(t) = \sin^2 \frac{\pi t}{a} g(R(E_i, \gamma_v')\gamma_v', E_i)(t), t \in [0, a],$$

and hence $g(R(E_i, \gamma_v')\gamma_v', E_i) = \pi^2/a^2$, for every $i = 3, ..., n$, which implies that $\mathcal{K}_v(\sigma) = \pi^2/a^2$ for every null plane with $v \in \sigma$. The converse is trivial. □

Remark 3.3. We point out that if the equality holds in (3.1), then (M, g) has U-normalized null sectional curvature $-\frac{\pi^2}{a^2} g(K, K)$. That is, the U-normalized null sectional curvature of (M, g) is a point function. So, (M, g) is *infinitesimally null-isotropic relative to* U in the terminology of L. Koch-Sen in [21]. In that paper and in [15], it is shown that if U is a unit timelike vector field on a Lorentzian manifold (M, g), with $\dim M \geq 3$, then (M, g) is infinitesimally null-isotropic relative to U if and only if it is *infinitesimally isotropic relative to* U, which means [18] that

$$R(X, Y)V = k\,\{g(Y, V)X - g(X, V)Y\} \quad \text{and} \quad R(X, U)U = \theta X,$$

for all $X, Y, V \in U^{\perp}$ where $k, \theta \in C^{\infty}(M)$. Moreover, (M, g) is infinitesimally isotropic relative to U if and only if all planes containing U have sectional curvature $-\theta$, and all planes perpendicular to U have sectional curvature k. In this case we get $\nu = \theta + k$, where $\nu : M \longrightarrow \mathbb{R}$ is the function defined by $\mathcal{K}_v(\sigma)$ with $v \in C_U M$ and σ is a null plane such that $v \in \sigma$, [21]. In [18] H. Karcher proved that, when $\dim M \geq 4$, a Lorentzian manifold (M, g) is infinitesimally isotropic relative to U, with everywhere nonzero null sectional curvature, if and only if:

1. The distribution $U^{\perp}$ is integrable.
2. The integral manifolds of $U^{\perp}$ are totally umbilic and have constant curvature.
3. (M, g) is locally conformal to a flat Lorentzian space.

We would like to point out that conclusion 1 does not remain true if it is assumed $\dim M = 3$ (see Remark 4.5 below).

Next we are going to show several technical results which permit us to give another interpretation for the right-hand side of integral formula (3.1). Let T be a symmetric bilinear form on T_pM. We write $\widetilde{T}(v) = T(v, v)$ for its associated quadratic form and A_T for the linear operator given by $T(u, v) = g(A_T(u), v)$.

Lemma 3.4. *Let (M, g) be an $n(\geq 3)$-dimensional Lorentzian manifold admitting an arbitrary timelike vector field K. Take a point $p \in M$.*

(a) If φ is a linear 1-form on T_pM, then

$$\int_{(C_KM)_p} \varphi \, d\mu_p = -\omega_{n-2}\varphi(K_p)h(p)^n.$$

(b) If T is a symmetric bilinear form on T_pM, then

$$\int_{(C_KM)_p} \widetilde{T} \, d\mu_p = \omega_{n-2} h(p)^n \Big[\frac{nh(p)^2\widetilde{T}(K_p) + \text{trace } A_T}{(n-1)}\Big].$$

Proof. (a) To check this formula, we are going to study the semi-Riemannian submanifold $((C_KM)_p, g)$ of (T_pM, g) (note that $\hat{g} \mid_{T_pM}= g$). We take $\xi_{\mathbf{1}}$ and $\xi_{\mathbf{2}}$ as they were defined in (2.8) but restricted to $(C_KM)_p$. Observe that

$$(T_v(C_KM)_p)^{\perp} = Span\, \{\xi_{\mathbf{1}}(v), \xi_{\mathbf{2}}(v)\} \subset T_vT_pM.$$

The *Gauss formula* for the submanifold $(C_KM)_p$ in T_pM is written as follows:

$$\nabla^0_X Y = \nabla_X Y - h(p)^{-1} g(X, Y)\xi_{\mathbf{2}}, \tag{3.4}$$

where $X, Y \in \mathfrak{X}((C_KM)_p)$ and ∇^0 (respectively ∇) is the induced connection on $(C_KM)_p \subset T_pM$ (resp. the Levi-Civita connection of $g \mid_{(C_KM)_p}$).

Denote by Hess and Hess^0 the Hessian operators of $(C_KM)_p$ and T_pM, respectively. It is easily seen from (3.4), taking into account $\text{Hess}^0(\varphi) = 0$, that

$$\text{Hess } \varphi(X, Y) = -h(p)^{-1} g(X, Y)\xi_{\mathbf{2}}(\varphi). \tag{3.5}$$

A straightforward computation from (3.5) gives for the *Laplacian* of φ in $(C_KM)_p$,

$$(\triangle\varphi)(v) = -(n-2)h(p)^{-1}\xi_{\mathbf{2}\,v}(\varphi). \tag{3.6}$$

Since φ is lineal, $\mathbf{A}(\varphi) = \varphi$ and $\mathbf{K}(\varphi) = \varphi(K_p)$. Thus, we get

$$\triangle\varphi = (n-2)[g(K_p, K_p)\varphi - \varphi(K_p)]. \tag{3.7}$$

Our integral formula in (a) can be directly achieved from (3.7) by taking into account the classical Green divergence theorem and (2.3).

(b) If $X, Y \in \mathfrak{X}((C_KM)_p)$, then it is easily seen that

$$Y(\widetilde{T}) = 2T(Y, \mathbf{A}) \quad \text{and} \quad X(Y\widetilde{T}) = 2T(\nabla^0_X Y, \mathbf{A}) + 2T(X, Y). \tag{3.8}$$

From the Gauss formula (3.4) we obtain

$$X(Y\widetilde{T}) = 2T(\nabla_X Y, \mathbf{A}) - 2h(p)^{-1} g(X, Y)T(\mathbf{A}, \xi_{\mathbf{2}}) + 2T(X, Y). \tag{3.9}$$

Therefore,

$$\text{Hess } \widetilde{T}(X, Y) = -2h(p)^{-1} g(X, Y)T(\mathbf{A}, \xi_{\mathbf{2}}) + 2T(X, Y). \tag{3.10}$$

Let $\{a_1, ..., a_{n-2}\}$ be an orthonormal basis of $T_v(C_K M)_p$. Then $\{a_1, ..., a_{n-2}, \xi_{\mathbf{1}}(v), \xi_{\mathbf{2}}(v)\}$ is also an orthonormal basis of $T_v T_p M \equiv T_p M$ with $\xi_{\mathbf{1}}(v)$ timelike. Now, by using (3.10),

$$(\triangle \widetilde{T})(v) = -2(n-2)h(p)^{-2}\widetilde{T}(v) - 2(n-2)T(K_p, v) + 2\sum_{i=1}^{n-2} \widetilde{T}(a_i). \tag{3.11}$$

On the other hand,

$$\text{trace } A_T = \sum_{i=1}^{n-2} \widetilde{T}(a_i) + h(p)^{-2}\widetilde{T}(v) + 2T(v, K_p),$$

which permits us to rewrite (3.11) as follows:

$$(\triangle \widetilde{T})(v) = -2(n-1)h(p)^{-2}\widetilde{T}(v) - 2nT(K_p, v) + 2\text{trace}A_T;$$

thus,

$$\int_{(C_K M)_p} \Big\{(1-n)h(p)^{-2}\widetilde{T} - nT(K_p, \ )\Big\} d\mu_p + \text{trace}A_T \ \text{vol}_K(p) = 0. \tag{3.12}$$

We end the proof by taking into account (a) and (2.3). □

Observe that if $\mathbb{S}^k$ is a usual k-dimensional unit sphere of $\mathbb{R}^{n+1}$, $1 \le k \le n$, i.e., $\mathbb{S}^k = \big\{(x_1, ..., x_{n+1}) : \sum_{i=1}^{k+1} x_i^2 = 1 \text{ and } x_{k+2} = \cdots = x_{n+1} = 0\big\}$, and φ is any 1-form on $\mathbb{R}^{n+1}$, then clearly $\int_{\mathbb{S}^k} \varphi \, d\mu = 0$, where $d\mu$ is the canonical measure of $\mathbb{S}^k$. In particular, on the contrary to Lemma 3.4 (a), it follows that $\int_{(UM)_p} \varphi \, d\mu_p = 0$, for any Riemannian manifold (M, g), where $p \in M$, $(UM)_p$ is the unit sphere in $T_p M$ and φ is any 1-form on $T_p M$. Note that $(C_K M)_p$ is not centered at the origin of $T_p M$; in fact, $u \in (C_K M)_p$ implies $-u \notin (C_K M)_p$.

Recalling now the integral formula (2.2) and using Lemma 3.4 for the Ricci tensor at any point $p \in M$, we get

$$\int_{C_K M} \widetilde{\text{Ric}} \, d\mu_{\hat{g}} = \frac{\omega_{n-2}}{n-1} \int_M \Big[n\widetilde{\text{Ric}}(K) - g(K, K)S\Big] h^{n+2} \, d\mu_g. \tag{3.13}$$

Now, taking into account this formula, we are in a position to rewrite Theorem 3.2 as follows.

Theorem 3.5. *Let (M, g) be an $n(\ge 3)$-dimensional compact Lorentzian manifold that admits a timelike conformal vector field K. If there exists $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0, a] \longrightarrow M$, with $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0, a)$, then*

$$\int_M h^{n-2} \, d\mu_g \ge \frac{a^2}{\pi^2 (n-1)(n-2)} \int_M \Big[n\widetilde{\text{Ric}}(U) + S\Big] h^n \, d\mu_g. \tag{3.14}$$

Moreover, equality holds if and only if (M, g) has U-normalized null sectional curvature $\frac{\pi^2}{a^2 h^2}$. □

Taking $T(u, v) = -g(R(u, U_{\pi v})U_{\pi v}, v)$, where U is an arbitrary unit timelike vector field, it is clear that for every $v \in C_U M$, $\widetilde{T}(v)$ is the sectional curvature of the nondegenerate plane $Span\,\{v, U_{\pi v}\}$. Thus, we obtain from Lemma 3.4 (b):

Corollary 3.6. *Let (M,g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold that admits a unitary timelike vector field U. Then,*

$$\int_{C_U M} \widetilde{T}\, d\mu_{\hat{g}} = -\frac{\omega_{n-2}}{n-1}\int_M \widetilde{\mathrm{Ric}}(U)\, d\mu_g.$$

□

In order to apply the previous result to unit timelike vector fields of geometric interest, note that if K is a timelike conformal vector field with $\mathfrak{L}_K g = \rho g$, then its normalized U satisfies $(\mathfrak{L}_U g)(X,Y) = \rho g(X,Y)$ for all $X, Y \in U^{\perp}$. Recall that a unit timelike vector field is said to be a *spatially conformal reference frame* if it satisfies that condition [11]. Note that not all spatially conformal reference frames can be obtained as previously [27].

By using Corollary 3.6 we can rewrite [27, Theorem 3.6] as follows:

If U is a spatially conformal reference frame on an $n(\geq 3)$-dimensional compact Lorentzian manifold (M,g), then

$$\int_{C_U M} \widetilde{T}\, d\mu_{\hat{g}} \leq 0,$$

and equality holds if and only if $\nabla_v U = 0$ for every $v \in U^{\perp}$.

On the other hand, from (3.13) and Corollary 3.6 we deduce that the total scalar curvature of (M,g) satifies

$$\int_M S\, d\mu_g = \frac{n-1}{\omega_{n-2}}\int_{C_U M} \left[\widetilde{\mathrm{Ric}} + n\widetilde{T}\right]\, d\mu_{\hat{g}}$$

for every unit timelike vector field U.

We have considered, in all previous developments, timelike conformal vector fields. Next, we focus our attention on the special case of timelike Killing vector fields.

Lemma 3.7. *Let (M,g) be an n-dimensional compact Lorentzian manifold that admits a timelike Killing vector field K. Then*

$$\int_M h^n \widetilde{\mathrm{Ric}}(U)\, d\mu_g \geq 0, \tag{3.15}$$

and equality holds if and only if U is parallel.

Proof. Since $\operatorname{div} U = 0$, the classical Raychaudhuri equation [28, p. 120] for U reduces to

$$\operatorname{div}(\nabla_U U) = \widetilde{\mathrm{Ric}}(U) + \mathrm{trace}\ (A_U'^2), \tag{3.16}$$

where A'_U is the linear operator of $U_p^{\perp}$ given by $A'_U(v) = -\nabla_v U$.

From (3.16) we can compute $\operatorname{div}(h^n \nabla_U U)$ and using the divergence theorem, we get

$$\int_M h^n \widetilde{\mathrm{Ric}}(U)\, d\mu_g = -\int_M \left[h^n \mathrm{trace}\ (A_U'^2) + (\nabla_U U)(h^n)\right]\, d\mu_g. \tag{3.17}$$

On the other hand, the adjoint operator of A'_U with respect to $g\mid_{U^{\perp}}$ is $-A'_U$ and $\operatorname{grad} h = -h^3 \nabla_K K$. Therefore, if $e_1,\ldots,e_{n-1}$ is an orthonormal basis of $U_p^{\perp}$, then

$$\mathrm{trace}\left(A_U'^2\right) = -\sum_{i=1}^{n-1} g\left(A'_U(e_i), A'_U(e_i)\right) \leq 0,$$

at any $p \in M$, and $(\nabla_U U)(h^n) = -nh^n g(\nabla_U U, \nabla_U U) \leq 0$.

These inequalities imply the announced sign in (3.15). Equality holds if and only if $A'_U = 0$ and $\nabla_U U = 0$. ◻

Note that if the equality holds in (3.15), then h must be constant.

Corollary 3.8. *Let (M, g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold that admits a timelike Killing vector field K. If there is $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0, a] \longrightarrow M$, $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0, a)$, then*

$$\int_M h^{n-2}\, d\mu_g \geq \frac{a^2}{\pi^2(n-1)(n-2)} \int_M S h^n\, d\mu_g. \tag{3.18}$$

Moreover, equality holds if and only if h is constant and the universal covering of (M, g) is isometric to the semi-Riemannian product $\left(\mathbb{R} \times \mathbb{S}^{n-1}(\frac{ah}{\pi}), -dt^2 + g_{can}\right)$.

Proof. The inequality (3.18) follows from (3.14) and Lemma 3.7. Moreover, equality holds if and only if U is parallel and (M, g) has U-normalized null sectional curvature $\frac{\pi^2}{a^2 h^2}$. In this case, the result can be deduced from the de Rham-Wu decomposition theorem [31]. ◻

Corollary 3.8 contains, as a particular case, the classical Berger-Green inequality in Riemannian geometry, ([13], [6, Proposition 5.64]).

Corollary 3.9. *Let (B, g) be an $n(\geq 2)$-dimensional compact Riemannian manifold with scalar curvature S. Suppose no unit geodesic $\gamma : [0, a] \longrightarrow B$ has a conjugate point in $[0, a)$. Then*

$$\mathrm{Vol}(B, g) \geq \frac{a^2}{\pi^2 n(n-1)} \int_B S\, d\mu_g. \tag{3.19}$$

Moreover, equality holds if and only if (B, g) has constant sectional curvature $\frac{\pi^2}{a^2}$.

Proof. This result easily follows if Corollary 3.8 specializes to the Lorentzian manifold $(M, g_L) = (\mathbb{S}^1 \times B, -g_{can} + g)$, taking as K the lift to $\mathbb{S}^1 \times B$ of the vector field $z \mapsto iz$ on $\mathbb{S}^1 \subset \mathbb{C}$. ◻

Corollary 3.10. *Let (M, g) be an $n(\geq 3)$-dimensional compact Lorentzian manifold of constant scalar curvature $S = S_0$. If (M, g) admits a timelike Killing vector field K and has no conjugate point on its null geodesics, then $S_0 \leq 0$, and $S_0 = 0$ implies that U is parallel, the first Betti number of M is not zero and the Levi-Civita connection of g is Riemannian.*

Proof. Since there are no conjugate points on null geodesics, (3.14) is valid for all positive a, which implies

$$\int_M \left[n\widetilde{\mathrm{Ric}}(U) + S_0 \right] h^n\, d\mu_g \leq 0. \tag{3.20}$$

Thus, from (3.15) we get $S_0 \leq 0$. If $S_0 = 0$, then it follows from (3.20) and (3.15) that U is parallel. Therefore, the Riemannian metric given by $g_R(X, Y) = g(X, Y) + 2g(X, U)g(Y, U)$ for every $X, Y \in \mathfrak{X}(M)$ and the Lorentzian metric g have the same Levi-Civita connection. Let $\omega(X) = g(X, U)$ be the 1-form metrically equivalent to U, ω is closed and if there is $f \in C^\infty(M)$ such that $\omega = df$, then $K_{p_0} = 0$ at every critical point p_0 of f. Therefore, ω cannot be exact. ◻

Remark 3.11. Kamishima proved [17, Theorem A] that if a compact Lorentzian manifold M with constant sectional curvature $c \in \mathbb{R}$ admits a timelike Killing vector field, then it is complete and $c \leq 0$. Moreover, if $c = 0$, then M is affinely diffeomorphic to a Riemannian manifold with nonzero first Betti number. This result was widely extended in [27, Corollaries 3.8, 3.9]. Since a Lorentzian manifold of constant sectional curvature has no conjugate point along its null geodesics, Corollary 3.10 is a proper extension, with a completely new technique, of Kamishima's theorem.

4. Application to Lorentzian odd-dimensional spheres

We consider $\mathbb{R}^{2n+2}$ identified with $\mathbb{C}^{n+1}$ as usual: $(x_1, ..., x_{2n+2}) = (z_1, ..., z_{n+1})$, $z_j = x_j + ix_{n+1+j}$; so, $\mathbb{S}^{2n+1} = \left\{ z = (z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} |z_j|^2 = 1 \right\}$.

Let $U \in \mathfrak{X}(\mathbb{S}^{2n+1})$ be given by $U_z = iz$ at any $z \in \mathbb{S}^{2n+1}$. For the canonical Riemannian metric g_{can} of $\mathbb{S}^{2n+1}$, U is Killing and $g_{can}(U, U) = 1$. Therefore, $\nabla_U U = 0$, where ∇ is the Levi-Civita connection of g_{can}, that is, the integral curves of U are geodesics for g_{can}.

We define on $\mathbb{S}^{2n+1}$ a Lorentzian metric g by

$$g(X, Y) = g_{can}(X, Y) - 2g_{can}(X, U)g_{can}(Y, U), \tag{4.1}$$

for all $X, Y \in \mathfrak{X}(\mathbb{S}^{2n+1})$, [33], [24]. The vector field U satisfies $g(U, U) = -1$, and it is Killing for the Lorentzian metric g. Thus $\widetilde{\nabla}_U U = 0$, where $\widetilde{\nabla}$ is the Levi-Civita connection of g, and so, its integral curves are timelike geodesics for g. Moreover, it is not difficult to show that

$$\widetilde{\nabla}_X Y = \nabla_X Y - 2g_{can}(X, U)\nabla_Y U - 2g_{can}(Y, U)\nabla_X U. \tag{4.2}$$

Since the inclusion map

$$(\mathbb{S}^{2n+1}, g_{can}) \hookrightarrow (\mathbb{S}^{2m+1}, g_{can}), \ (z_1, ..., z_{n+1}) \mapsto (z_1, ..., z_{n+1}, 0, ..., 0),$$

for $m > n$, is clearly totally geodesic, a direct computation using (4.2) shows that, if g_{can} is changed for g, the same inclusion is a totally geodesic Lorentzian submanifold.

We consider the Hopf fibration $\pi : (\mathbb{S}^{2n+1}, g_{can}) \longrightarrow (\mathbb{C}P^n, g_{FS}), \ z \mapsto [z]$, where $\mathbb{C}P^n$ is the complex projective space endowed with its classical Fubini-Study metric g_{FS} of constant holomorphic sectional curvature 4 [20, p. 273]. The Hopf fibration is a principal bundle with structural group $\mathbb{S}^1$, and also a Riemannian submersion with totally geodesic fibres. If g_{can} is replaced by the Lorentzian metric g, then π is a semi-Riemannian submersion from $(\mathbb{S}^{2n+1}, g)$ to $(\mathbb{C}P^n, g_{FS})$ with timelike totally geodesic fibres.

It is useful to have an alternative definition for the metric g in (4.1). In fact, we can also introduce g from the Riemannian metric g_{can} by putting

$$g_{|\mathcal{V}} = -g_{can|\mathcal{V}}, \ g_{|\mathcal{H}} = g_{can|\mathcal{H}}, \ g(\mathcal{V}, \mathcal{H}) = 0, \tag{4.3}$$

where $\mathcal{V}$ and $\mathcal{H}$ are respectively the vertical and the horizontal distributions for the canonical connection of the Hopf fibration.

Given a Lorentzian manifold (M, g), a unit timelike vector field U on M and $p \in M$, recall that (M, g) is said to be *spatially isotropic* with respect to U at p if for every two unit vectors $u_1, u_2 \in U_p^{\perp}$, there is an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u_1) = u_2$. It is said that (M, g) is *spatially* *isotropic* with respect to U if it is *spatially isotropic* with respect to U at every point, [28, p. 47]. An easy algebraic argument gives

Lemma 4.1. *Let (M, g) be a Lorentzian manifold that admits a unit timelike vector field U. The following assertions are equivalent:*

1. *(M, g) is spatially isotropic with respect to U.*
2. *For all $p \in M$ and for all $u, v \in (C_U M)_p$ there is an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u) = v$.* □

Proposition 4.2. *$(\mathbb{S}^{2n+1}, g)$ is spatially isotropic with respect to U and the unitary group $U(n+1)$ acts transitively by g-isometries on $\mathbb{S}^{2n+1}$.*

Proof. It is well known that $U(n+1)$ acts transitively by g_{can}-isometries on $\mathbb{S}^{2n+1}$. Moreover, the vector field U is $U(n+1)$-invariant. Therefore, $U(n+1)$ also acts by g-isometries on $\mathbb{S}^{2n+1}$. Thus, it is sufficient to check that $(\mathbb{S}^{2n+1}, g)$ is spatially isotropic with respect to U at the point $e_{n+1} = (0, ..., 1) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$.

Consider $U(n) = \{A \in U(n+1) : Ae_{n+1} = e_{n+1}\}$. If $u, v \in U^{\perp}_{e_{n+1}} \subset \mathbb{C}^{n+1}$, then $u = (u_1, ..., u_n, 0)$ and $v = (v_1, ..., v_n, 0)$. Hence, $u, v \in \mathbb{S}^{2n-1}$; so, there exists $A \in U(n)$ such that $Au = v$, giving the required isometry. □

We now give explicit parametrizations for the null geodesics. Because of Proposition 4.2 it is enough to find the null geodesics starting from the specific point $p_0 = (1, ..., 0) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Note that $v \in (C_U \mathbb{S}^{2n+1})_{p_0}$ if and only if $v = (-i, v_2, ..., v_{n+1})$ with $\sum_{j=2}^{n+1} v_j \overline{v_j} = 1$. On the other hand, (4.2) yields

$$\nabla_{\gamma_v'} \gamma_v' = -4 \nabla_{\gamma_v'} U. \tag{4.4}$$

The Gauss equation for the canonical Riemannian unit sphere permits us to write (4.4) as follows:

$$\frac{d^2 \gamma_v}{dt^2} + 4i \frac{d\gamma_v}{dt} - 2\gamma_v = 0.$$

We obtain for $\gamma_v = (\Theta_1^v, ..., \Theta_{n+1}^v)$ with $\Theta_j^v : \mathbb{R} \to \mathbb{C}$ $(1 \leq j \leq n+1)$,

$$\Theta_1^v(t) = \frac{2 - \sqrt{2}}{4} e^{(-2-\sqrt{2})it} + \frac{2 + \sqrt{2}}{4} e^{(-2+\sqrt{2})it},$$

and

$$\Theta_j^v(t) = \frac{\sqrt{2} i v_j}{4} \left[e^{(-2-\sqrt{2})it} - e^{(-2+\sqrt{2})it} \right] \quad (j \geq 2).$$

The following result is similar to the well-known Morse-Schönberg theorem [8, Theorem 2.14], and its proof follows from a slight variation to the one in the Riemannian case. Compare with [14, Proposition 2.6].

Proposition 4.3. *Let (M, g) be a Lorentzian manifold, $\delta > 0$, and $\gamma : [0, \lambda] \longrightarrow M$ a null geodesic such that $\mathcal{K}_{\gamma'(t)}(\sigma) \leq \delta$ for all null planes σ with $\gamma'(t) \in \sigma$. If $\gamma(0)$ and $\gamma(\lambda)$ are conjugate points along γ, then $\lambda \geq \frac{\pi}{\sqrt{\delta}}$.* □

As an application of our integral inequalities, we give the following result about the behaviour of conjugate points along null geodesics in odd-dimensional Lorentzian spheres.

Proposition 4.4. *For every null geodesic γ_v of $(\mathbb{S}^{2n+1}, g)$ with $v \in C_U\mathbb{S}^{2n+1}$, $\gamma_v(0)$ and $\gamma_v(\frac{\pi}{2\sqrt{2}})$ are conjugate points and γ_v has no conjugate point on $[0, \frac{\pi}{2\sqrt{2}})$. Moreover, the past lightlike conjugate locus of each point $p \in \mathbb{S}^{2n+1}$ is an imbedded sphere $\mathbb{S}^{2n-1}$.*

Proof. A slight variation of [7, Proposition 9.70] and (4.3) gives us for the Ricci tensor and the scalar curvature of $(\mathbb{S}^{2n+1}, g)$,

$$\widetilde{\mathrm{Ric}}(U) = 2n \text{ and } S = S_{FS} \circ \pi + 2n, \tag{4.5}$$

where S_{FS} is the scalar curvature of $\mathbb{C}P^n$. But, we know that $S_{FS} = 4n(n+1)$ and hence, as a direct consequence of Corollary 3.10 and Proposition 4.2, there exists $\lambda \in (0, +\infty)$ such that for every null geodesic γ_v with $v \in C_U\mathbb{S}^{2n+1}$, $\gamma_v(\lambda)$ is the first conjugate point of $\gamma_v(0)$ along γ_v.

From Proposition 4.2, $\theta(v) = -\mathcal{K}(v, U_{\pi v})$ is a constant function on $C_U\mathbb{S}^{2n+1}$. Therefore, Corollary 3.6 and (4.5) give $\theta = 1$. On the other hand, if $X, Y \in U^\perp$ are orthogonal and unitary vector fields on $\mathbb{S}^{2n+1}$, then the O'Neill formulas for a semi-Riemannian submersion, [23, Theorem 7.47], can be claimed to get

$$\mathcal{K}(X, Y) = \mathcal{K}_{FS}(d\pi(X), d\pi(Y)) - \tfrac{3}{4} g(\mathcal{V}[X, Y], \mathcal{V}[X, Y]), \tag{4.6}$$

where we write $\mathcal{K}_{FS}$ for the sectional curvature of $(\mathbb{C}P^n, g_{FS})$, and $\mathcal{V}$ denotes the vertical operator. By a straightforward computation, we get

$$\mathcal{K}(X, Y) = 2\mathcal{K}_{FS}(d\pi(X), d\pi(Y)) - 1. \tag{4.7}$$

Let σ be a degenerate plane $\sigma = \mathrm{Span}\{v, X_p\}$ with $v \in (C_U\mathbb{S}^{2n+1})_p$. We can write $v = Y_p - U_p$. One easily checks that

$$\mathcal{K}_v(\sigma) = 1 + \mathcal{K}(X_p, Y_p) - 2g_{can}(\widetilde{R}(X_p, Y_p)X_p, U_p), \tag{4.8}$$

where $\widetilde{R}$ denotes the Riemannian curvature tensor of $(\mathbb{S}^{2n+1}, g)$. From (4.2) we obtain $g_{can}(\widetilde{R}(X_p, Y_p)X_p, U_p) = 0$; thus, taking into account (4.7), the above formula (4.8) reduces to

$$\mathcal{K}_v(\sigma) = 2\mathcal{K}_{FS}(d\pi(X_p), d\pi(Y_p)); \tag{4.9}$$

so, using the well-known range of variation of the sectional curvature of $\mathbb{C}P^n$, we have $2 \leq \mathcal{K}_v(\sigma) \leq 8$.

Now,

$$\frac{\pi^2}{8} \leq \lambda^2 \leq \frac{(2n-1)\pi^2}{4(n+1)} \tag{4.10}$$

follows directly from (4.5), Theorem 3.5 and Proposition 4.3. Therefore, we find $\lambda = \frac{\pi}{2\sqrt{2}}$ for $(\mathbb{S}^3, g)$. Taking into account that $(\mathbb{S}^3, g)$ is a totally geodesic Lorentzian submanifold of any $(\mathbb{S}^{2n+1}, g)$, we get the first assertion.

Finally, for any $u, v \in (C_U\mathbb{S}^{2n+1})_p$ with $u \neq v$, $\gamma_u(T) = \gamma_v(T)$ if and only if $T = \frac{q\pi}{\sqrt{2}}$ for some $q \in \mathbb{Z}$; so the last assertion follows from $\lambda < \frac{\pi}{\sqrt{2}}$. □

Note that Proposition 4.4 has a dual version for future null geodesics.

Remark 4.5. We end this section by observing several special geometric facts in the 3-dimensional case. From Lemma 4.1, if a 3-dimensional Lorentzian manifold (M, g) is spatially isotropic with respect to U, then it is infinitesimally null-isotropic relative to U (Remark 3.3). This statement shows, taking into account Proposition

4.2, that the hypothesis $\dim M \geq 4$ in Karcher's theorem [18] cannot be weakened to $\dim M \geq 3$, because of nonintegrability of the distribution $U^{\perp}$.

We point out that it is possible to get the previous assertion in a different and direct way. Indeed, using (4.9) one derives that the U-normalized null sectional curvature of $(\mathbb{S}^{2n+1}, g)$ is a point function if and only if $n = 1$, and $\mathcal{K}_v(v^{\perp}) = 8$ for any $v \in C_U\mathbb{S}^3$.

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