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KÄHLER-EINSTEIN METRICS FOR SOME QUASI-SMOOTH LOG DEL PEZZO SURFACES

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Abstract. Recently Johnson and Kollár determined the complete list of anticanonically embedded quasi-smooth log del Pezzo surfaces in weighted projective 3-spaces. They also proved that many of those surfaces admit a Kähler-Einstein metric, and that some of them do not have tigers.

The aim of this paper is to settle the question of the existence of Kähler-Einstein metrics and tigers for those surfaces for which the question was left open. In order to do so, we will use techniques developed earlier by Nadel, Demailly and Kollár.

1. Introduction

Recently Johnson and Kollár determined the complete list of anticanonically embedded quasi-smooth log del Pezzo surfaces in weighted projective 3-spaces [JK]. Then, using techniques developed earlier by Demailly and Kollár [DK], they proved that many of those surfaces admit a Kähler-Einstein metric, and that some of them do not have tigers. Later Boyer, Galicki and Nakamaye showed the existence of Kähler-Einstein metrics for a general surface of degree 16 in weighted projective space with weights $1, 3, 5, 8$ [BGN1]. Moreover, it follows from their ([BGN2], Corollary 5.3) that a general surface of degree 18 in weighted projective space with weights $2, 3, 5, 9$ also admits a Kähler-Einstein metric. These were two of the log del Pezzo surfaces for which the question was left open by Johnson and Kollár.

The aim of this paper is to settle the question of the existence of Kähler-Einstein metrics and tigers for those surfaces in [JK] for which the question was still open. In order to show the existence of Kähler-Einstein metrics for these surfaces, we will use a criterion described in the work of Nadel [Na] and Demailly and Kollár [DK].

2. Anticanonically embedded quasi-smooth log del Pezzo surfaces in weighted projective 3-spaces

For positive integers q_i let $\mathbb{P}(q_0, q_1, q_2, q_3)$ denote the *weighted projective 3-space* with weights q_0, q_1, q_2, q_3. (See [Dol] or [Fle] for basic definitions and results.) When there is no ambiguity, we abbreviate $\mathbb{P} = \mathbb{P}(q_0, q_1, q_2, q_3)$. We always assume that any 3 of the q_i are relatively prime. Let x_0, x_1, x_2, x_3 denote the corresponding weighted projective coordinates. $\mathbb{P}(q_0, q_1, q_2, q_3)$ can be viewed as the quotient

$$\mathbb{C}^4 \setminus \{0\} \;/\; \mathbb{C}^*(q_0, q_1, q_2, q_3),$$

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where $\mathbb{C}^*(q_0, q_1, q_2, q_3)$ denotes the $\mathbb{C}^*$-action on $\mathbb{C}^4 \setminus \{0\}$ given by

$$(z_0, z_1, z_2, z_3) \mapsto (t^{q_0} z_0, t^{q_1} z_1, t^{q_2} z_2, t^{q_3} z_3).$$

The action of $\mathbb{C}^*$ on $\mathbb{C}^4 \setminus \{0\}$ induces the action $\mathbb{Z}_{q_i}(q_j, q_k, q_l)$ on $V_i = \{z_i = 1\} \subset \mathbb{C}^4$ which, after identifying V_i with $\mathbb{C}^3$, can be written as

$$(z_j, z_k, z_l) \mapsto (\xi^{q_j} z_j, \xi^{q_k} z_k, \xi^{q_l} z_l),$$

where ξ is a primitive q_ith root of unity. Hence the affine chart $U_i \subset \mathbb{P}$ where $x_i \neq 0$ can be identified with

$$\mathbb{C}^3 / \mathbb{Z}_{q_i}(q_j, q_k, q_l).$$

These are called the *orbifold charts*. Write $p_i : \mathbb{C}^3 \to U_i$ for the natural projection.

$\mathbb{P}(q_0, q_1, q_2, q_3)$ has an index q_i singularity at $P_i = (x_j = 0)_{j \neq i}$, and an index (q_i, q_j) singularity along the line $(x_k = 0)_{k \neq i,j}$.

For every $d \in \mathbb{Z}$ there is a rank 1 sheaf $\mathcal{O}_{\mathbb{P}}(d)$, and the sections of $\mathcal{O}_{\mathbb{P}}(d)$ consist of the homogeneous polynomials $f(x_0, x_1, x_2, x_3)$ of weighted degree d.

Let $X = X_d$ be a member of $|\mathcal{O}_{\mathbb{P}}(d)|$. If X does not contain any of the singular lines of $\mathbb{P}$, then the adjunction formula

$$K_X \cong \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + X)|_X \cong \mathcal{O}_{\mathbb{P}}(d - (q_0 + q_1 + q_2 + q_3))|_X$$

holds.

A *log del Pezzo* surface is a projective surface with quotient singularities such that its anticanonical class is ample.

If X does not contain any of the singular lines, then, by the adjunction formula, its anticanonical class is ample if and only if $d < q_0 + q_1 + q_2 + q_3$. Here we are interested in the case when d is as large as possible, i.e., $d = q_0 + q_1 + q_2 + q_3 - 1$. In this case we say that X is *anticanonically embedded* in $\mathbb{P}(q_0, q_1, q_2, q_3)$.

We say that X is *quasi-smooth* if the pre-image of X under the quotient map

$$\mathbb{C}^4 \setminus \{0\} \to \mathbb{C}^4 \setminus \{0\} \ / \ \mathbb{C}^*(q_0, q_1, q_2, q_3) = \mathbb{P}(q_0, q_1, q_2, q_3)$$

is smooth.

We write conditions on the weights q_0, q_1, q_2, q_3 for the general surface $X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ of degree $d = q_0 + q_1 + q_2 + q_3 - 1$ to be quasi-smooth, and for the adjunction formula to hold. These conditions then imply that the general X_d is an *anticanonically embedded quasi-smooth log del Pezzo* surface:

Conditions 2.1. *For X_d to be quasi-smooth (see* [Fle]*, 8.5):*

1. *for every i there is a j and a monomial $x_i^{m_i} x_j$ of degree d;*
2. *for all distinct i and j, either there is a monomial $x_i^{b_i} x_j^{b_j}$ of degree d, or there are monomials $x_i^{c_i} x_j^{c_j} x_k$ and $x_i^{d_i} x_j^{d_j} x_l$ of degree d, with k and l distinct.*

For X_d not to contain any of the singular lines of $\mathbb{P}$ (which implies that the adjunction formula holds):

3. *for all distinct i and j such that $(q_i, q_j) > 1$, there is a monomial $x_i^{b_i} x_j^{b_j}$ of degree d.* □

Notation 2.2. Given a monomial M, and a polynomial f, we write $M \in f$ if the coefficient of M in f is nonzero.

3. Tigers and the existence of Kähler-Einstein metrics

We start this section by giving some definitions from the log category. We refer to [KM] for a detailed introduction.

Definition 3.1. Let X be a normal surface, and D a $\mathbb{Q}$-divisor on X such that $m(K_X + D)$ is Cartier for some $m > 0$. Let $f : Y \to X$ be a proper birational morphism from a smooth surface Y. Then there is a unique $\mathbb{Q}$-divisor $\sum e_i E_i$ on Y such that

$$K_Y \equiv f^*(K_X + D) + \sum e_i E_i \quad \textit{and} \quad f_* \sum e_i E_i = -D.$$

We say that the pair (X, D) is *canonical* (resp. *klt*, resp. *log canonical*) if $e_i \geq 0$ (resp. $e_i > -1$, resp. $e_i \geq -1$) for every f and every i.

Remark 3.2. If (X, D) and (X, D') are both canonical (resp. klt, resp. log canonical), then so is $(X, \alpha D + (1 - \alpha)D')$ for any $0 \leq \alpha \leq 1$. This is a very easy result, but it will be very useful in our applications.

Definition 3.3 ([KMcK]). Let X be a normal surface. A *tiger* on X is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that (X, D) is not klt.

We use the following sufficient condition for the existence of a Kähler-Einstein metric on log del Pezzo surfaces:

Theorem 3.4 ([Na], [DK]). *Let X be a log del Pezzo surface. If there is an $\epsilon > 0$ such that $(X, \frac{2+\epsilon}{3} D)$ is klt for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$, then X admits a Kähler-Einstein metric.* □

In order to prove the existence of a Kähler-Einstein metric (resp. the nonexistence of tigers) for a surface $X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$, we will show that, for all effective $\mathbb{Q}$-divisors $D \equiv -\frac{2+\epsilon}{3} K_X$ (resp $D \equiv K_X$), (X, D) is klt. The definition of a klt pair (X, D), however, involves understanding all resolutions of singularities of (X, D). Instead, we will use the following multiplicity conditions:

Conditions 3.5 ([JK]). *Let $X = X_d$ be a hypersurface of weighted degree d in $\mathbb{P}(q_0, q_1, q_2, q_3)$, and assume that X is quasi-smooth and has only isolated (quotient) singularities. Write*

$$p_i \ : \ \mathbb{C}^3 \ \to \ U_i \ = \ (x_i \neq 0) \ \cong \ \mathbb{C}^3/\mathbb{Z}_{q_i}(q_j, q_k, q_l)$$

for the natural projections to the orbifold charts. For an effective $\mathbb{Q}$-divisor D on X, (X, D) is klt provided that the following 3 conditions hold:

0. *D does not contain any irreducible component with coefficient ≥ 1;*
1. *for every smooth point $P \in X$, $mult_P D \leq 1$;*
2. *for every singular point $P \in X \cap U_i$, $mult_Q p_i^* D \leq 1$, where $p_i(Q) = P$.* □

In our applications, condition (3.5.0) can be verified right away. To check the other 2 conditions, we will bound the multiplicities above by suitable intersection numbers.

3.1. **The Smooth Points.** Let $P \in X^0$, the nonsingular locus of X. Suppose we can find a positive integer l such that, for any effective $\mathbb{Q}$-divisor $D \equiv -K_X$, there is a divisor $F \in |\mathcal{O}_X(l)|$ such that:

1. $F = F' + B$, where F' is an effective Weil divisor, and B is nef;
2. F' passes through P;

3. F' does not contain any irreducible component of D.

Then we can intersect D with F', and obtain:

$$mult_P D \leq (D \cdot F') \leq (D \cdot F) = \frac{ld}{q_0 q_1 q_2 q_3}.$$

We would like this bound to be ≤ 1 in order to verify Condition 3.5.1. Unfortunately in most of the cases we cannot choose an l that works for all $P \in X^0$. Instead, we will find an l that works for a dense open set of X^0, say $X^0 \setminus C$, and then show by other methods that $mult_P D \leq 1$ for $P \in C \cap X^0$.

Lemma 3.6. *Let $\mathbb{P} = \mathbb{P}(q_0, \dots, q_n)$ denote the weighted projectice n-space with weights $q_0, \dots, q_n$. Let l be a positive integer such that for every $i \neq 0$ there are at least 2 distinct monomials in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ of the form $x_0^\alpha x_i^\beta$. Then, for every $P \in \mathbb{P} \setminus (x_0 = 0)$, and every divisor D on $\mathbb{P}$, there is a divisor $F \in |\mathcal{O}_{\mathbb{P}}(l)|$ that can be written as $F = F' + a(x_0 = 0)$, where a is a nonnegative integer, and F' is an effective Weil divisor passing through P and not containing any irreducible component of D.*

Proof. We choose from each irreducible component D_j of D, $D_j \not\subset (x_0 = 0)$, a point $Q_j \in D_j \setminus (x_0 = 0)$ (later we will impose a further condition on the Q_j). We shall prove that there is a polynomial $F_j \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ that vanishes at P but not at Q_j. We then choose F to be a suitable linear combination of the F_j.

Let $Q = Q_j$. Fix representations $P = (1 : a_1 : \dots : a_n)$ and $Q = (1 : b_1 : \dots : b_n)$ in weighted projective coordinates (notice that these representations are not unique). If $a_i = 0 \neq b_i$ or $a_i \neq 0 = b_i$ for some i, then it is easy to find a polynomial in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ that vanishes at P but not at Q. So we assume that for every i, either $a_i = 0 = b_i$ or $a_i \neq 0 \neq b_i$.

For each $i \neq 0$ fix $x_0^{\alpha_i} x_i^{\beta_i}$ and $x_0^{\gamma_i} x_i^{\delta_i}$ distinct monomials in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$, and assume $\delta_i > \beta_i$. Define:

$$H_i \;=\; a_i^{\delta_i} x_0^{\alpha_i} x_i^{\beta_i} \;-\; a_i^{\beta_i} x_0^{\gamma_i} x_i^{\delta_i} \;\in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)).$$

Then $H_i(P) = 0$, and $H_i(Q) = 0$ if and only if $b_i = \zeta_i a_i$, where $\zeta_i^{\delta_i - \beta_i} = 1$.

Therefore, if Q is outside the finite set $\{(1 : \zeta_1 a_1 : \dots : \zeta_n a_n) \mid \zeta_i^{\delta_i - \beta_i} = 1\}$ (and we can certainly impose that condition when we choose the Q_j above), we get a polynomial in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ that vanishes at P but not at Q. ☐

Now we will use Lemma 3.6 to find a linear system $|\mathcal{O}_{X_d}(l)|$ on X_d that will give us a bound for $mult_P D$, at least for $P \in X_d \cap (x_0 \neq 0)$.

Corollary 3.7. *Let $X = X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ be an anticanonically embedded quasi-smooth log del Pezzo surface. Let $\pi_3 : X \to \mathbb{P} = \mathbb{P}(q_0, q_1, q_2)$ denote the projection from $P_3 = (0, 0, 0, 1)$. Assume π_3 has only finite fibers. Let l be a positive integer such that, for $i = 1, 2$, there are at least 2 monomials in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l))$ of the form $x_0^\alpha x_i^\beta$.*

Then, for every $P \in X^0 \setminus (x_0 = 0)$ and every effective $\mathbb{Q}$-divisor $D \equiv -K_X$, there is a divisor $F \in |\mathcal{O}_X(l)|$ that can be written as $F = F' + a(x_0 = 0)$, where a is a nonnegative integer, and F' is an effective Weil divisor passing through P and not containing any irreducible component of D. Hence $mult_P D \leq \frac{ld}{q_0 q_1 q_2 q_3}$.

Proof. Set $P' = \pi_3(P) \in \mathbb{P}$, and $D' = \pi_{3*} D$. By the lemma, there is a divisor $E \in |\mathcal{O}_{\mathbb{P}}(l)|$ that can be written as $E = E' + a(x_0 = 0)$, where a is a nonnegative integer, and E' is an effective Weil divisor passing through P' and not containing any irreducible component of D'. Since π_3 has only finite fibers, we can take F to be $\pi_3^*E \in \pi_3^*|\mathcal{O}_{\mathbb{P}}(l)| \subset |\mathcal{O}_X(l)|$. □

Now we have to deal with the smooth points of X in $(x_0 = 0)$.

Lemma 3.8. *Let $X = X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ be an anticanonically embedded quasi-smooth log del Pezzo surface. Assume that $C = X \cap (x_0 = 0)$ is irreducible and smooth outside the singular locus of X. If $d \le q_1q_2q_3$, then for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$ and every $P \in X^0 \cap C$ we have $mult_P D \le 1$.*

Proof. Fix $P \in X^0 \cap C$. Since C is smooth outside the singular locus of X, $mult_P C = 1$. Given $D \equiv -K_X$, write $D = \alpha C + (1 - \alpha q_0)D'$, where $0 \le \alpha \le 1/q_0$, $D' \equiv -K_X$, and C is not contained in the support of D'. Then

$$mult_P D' \;\le\; (C \cdot D') \;\le\; \frac{d}{q_1q_2q_3} \;\le\; 1,$$

and hence $mult_P D \le \alpha + (1 - q_0\alpha) \le 1$. □

Now we put these results together to obtain conditions on the weights q_i for every effective $\mathbb{Q}$-divisor $D \equiv -K_{X_d}$ to have multiplicity ≤ 1 at the smooth points of X_d.

Lemma 3.9. *Let $X = X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ be an anticanonically embedded quasi-smooth log del Pezzo surface. Let f be the defining polynomial of X_d in $\mathbb{P}(q_0, q_1, q_2, q_3)$. Assume that:*

1. *the curve $C = X \cap (x_0 = 0)$ is irreducible and smooth outside the singular locus of X;*
2. *$x_3^a \in f$ for some $a \in \mathbb{Z}$, $a \ge 1$;*
3. *$d \le min\{q_1, q_2\} \cdot q_3$.*

Then $mult_P D \le 1$ for every smooth point $P \in X$ and every $\mathbb{Q}$-divisor $D \equiv -K_X$.

Proof. Let D be a $\mathbb{Q}$-divisor such that $D \equiv -K_X$.

Conditions 3.9.1 and 3.9.3 together with Lemma 3.8 imply that $mult_P D \le 1$ for every smooth point $P \in C$.

Condition 3.9.2 implies that the projection π_3 has only finite fibers. Thus the hypotheses of Corollary 3.7 are fulfilled for $l = q_0 \cdot max\{q_1, q_2\}$. (Indeed, assuming $q_1 \le q_2$, we have that $x_0^{q_2},\ x_0^{q_2-q_1}x_1^{q_0},\ x_2^{q_0} \in |\mathcal{O}_X(q_0q_2)|$.) Thus we get that $mult_P D \le \frac{ld}{q_0q_1q_2q_3} \le 1$ for every $P \in X^0 \setminus C$. □

3.2. **The Singular Points.**

Estimate 3.10. *Let $X = X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ be a surface of degree d defined by the polynomial f. Let $P \in X \cap U_i$ be a singular point, and let $p_i : (S, Q) \to (X, P)$ be a local orbifold chart. Assume that some monomial of the form $x_i^a x_l^b \in f$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Then, for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$,*

$$mult_Q p_i^* D \quad \le \quad \frac{d}{min(q_j, q_k) \cdot q_l}.$$

Proof. Consider the linear system $|p_i^*(x_j^{q_k}), p_i^*(x_k^{q_j})| \subset p_i^*|\mathcal{O}_X(q_jq_k)|$.

The assumption that $x_i^a x_l^b \in f$ implies that $(x_j = x_k = 0) \not\subset X$. Thus the above linear system does not have any fixed components. So we can intersect a general member of it with $p_i^* D$, obtaining:

$$mult_Q p_i^* D \ \le \frac{q_i}{min(q_j, q_k)}(D \cdot \mathcal{O}_X(q_j q_k)) \ = \frac{d}{min(q_j, q_k) \cdot q_l}.$$

□

When this bound is > 1 but ≤ 2, we can look closer at what happens near Q, that is, we can blow-up this point and then use Shokurov's Inversion of Adjunction:

Theorem 3.11 ([KM], 5.50)**.** *Let S be a smooth surface, $Q \in S$, and B be an effectice $\mathbb{Q}$-divisor such that $mult_Q B \le 2$. Let $\pi : S' \to S$ be the blow-up of Q with exceptional divisor E. If $(\pi_*^{-1}(B))|_E$ is a sum of points, all with coefficients ≤ 1, then (S, B) is log canonical at Q.* □

4. The Main Result

In this section we answer the question of the existence of Kähler-Einstein metrics and tigers for the six surfaces in [JK] for which the question was still unsolved. In Theorem 4.1 below, the existence of a Kähler-Einstein metric was not known for the first two families of surfaces, $X_{10} \subset \mathbb{P}(1, 2, 3, 5)$ and $X_{15} \subset \mathbb{P}(1, 3, 5, 7)$. As for the remaining ones, it was not known if they could have tigers.

Theorem 4.1. *Let $X_d \subset \mathbb{P}(q_0, q_1, q_2, q_3)$ denote any anticanonically embedded quasi-smooth log del Pezzo surface in the weighted projective 3-space with weights q_0, q_1, q_2, q_3. Then:*

1. *$X_{10} \subset \mathbb{P}(1, 2, 3, 5)$ admits a Kähler-Einstein metric.*
2. *$X_{15} \subset \mathbb{P}(1, 3, 5, 7)$ admits a Kähler-Einstein metric provided that the coefficient of the monomial $x_1x_2x_3$ in the equation of X_{15} is nonzero.*
3. *$X_{18} \subset \mathbb{P}(2, 3, 5, 9)$ does not have a tiger.*
4. *$X_{25} \subset \mathbb{P}(3, 5, 7, 11)$ does not have a tiger.*
5. *$X_{28} \subset \mathbb{P}(3, 5, 7, 14)$ does not have a tiger.*
6. *$X_{36} \subset \mathbb{P}(3, 5, 11, 18)$ does not have a tiger.*

Remarks 4.2. 1. The nonexistence of tigers on the last four families of surfaces implies that these surfaces admit a Kähler-Einstein metric.

2. $(x_0 = 0)$ is a tiger on the surfaces in (4.1.1) and (4.1.2).

3. In (4.1.2), if the coefficient of $x_1x_2x_3$ in the equation of X_{15} is zero, then the criterion of Theorem 3.4 does not apply. Indeed, $c \cdot (x_0 = 0)$ is not klt for any $c > 8/15$. However, it is possible that even in this case, X_{15} admits a Kähler-Einstein metric.

Proof. Let f denote the defining polynomial of $X = X_d$ in $\mathbb{P}(q_0, q_1, q_2, q_3)$.

1. $X_{10} \subset \mathbb{P}(1, 2, 3, 5)$.

The polynomial f is a linear combination of the monomials in $|\mathcal{O}_{\mathbb{P}(1,2,3,5)}(10)|$. Since X is quasi-smooth, we can apply Condition 2.1.1 with $i = 1, 2, 3$ and conclude that x_1^5, $x_0x_2^3$, $x_3^2 \in f$.

Let $C = X \cap (x_0 = 0)$. It is easy to see that the only monomials of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ in $|\mathcal{O}_{\mathbb{P}(1,2,3,5)}(10)|$ are x_1^5, $x_1^2x_2^2$, $x_1x_2x_3$, and x_3^2. Hence, after rescaling, the equations for C in $\mathbb{P}(1,2,3,5)$ become:

$$x_0 = 0, \tag{1}$$
$$x_1^5 + ax_1^2x_2^2 + bx_1x_2x_3 + x_3^2 = x_1^5 + (\alpha x_1x_2 + x_3)(\beta x_1x_2 + x_3) = 0.$$

We see that C is irreducible and smooth outside the singular locus of X.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor. We shall show that $(X, \frac{7}{10}D)$ is log canonical. This is more than enough to apply Theorem 3.4 and then conclude that X admits a Kähler-Einstein metric.

First let us consider the smooth points of X. As we have seen above, C is irreducible and smooth on X^0, and $x_3^2 \in f$. So we can apply Lemma 3.9 to conclude that D has multiplicity ≤ 1 at the smooth points of X. Therefore, by Condition 3.5.1, $(X, \frac{7}{10}D)$ is log canonical (in fact klt) at the smooth points of X.

Now let us consider the singular points of X. These are the singular points of $\mathbb{P}(1,2,3,5)$ that lie on X (since X is quasi-smooth). The singular points of $\mathbb{P}(1,2,3,5)$ are $P_1 = (0:1:0:0)$, $P_2 = (0:0:1:0)$, and $P_3 = (0:0:0:1)$. Since $x_1^5, x_3^2 \in f$, the points P_1 and P_3 do not lie on X. Since there is no pure power of x_2 in $|\mathcal{O}_{\mathbb{P}(1,2,3,5)}(10)|$, $P_2 \in X$, and it is the only singular point of X. Let

$$p_2 : \ (\mathbb{C}^3, Q_2 = 0) \ \rightarrow \ (U_2 = (x_2 \neq 0), P_2)$$

denote the natural projection to the orbifold chart.

Write $D = \lambda C + (1-\lambda)D'$, with $0 \leq \lambda \leq 1$, $D' \equiv -K_X$, and C not contained in the support of D'. By Remark 3.2, it is enough to show that both $(X, \frac{7}{10}C)$ and $(X, \frac{7}{10}D')$ are log canonical.

From the equations (1) above, we see that $mult_{Q_2} p_2^* C = 2$, and thus we can use Shokurov's inversion of adjunction. Let the notation be as in Theorem 3.11. We have 2 cases: If $\alpha \neq \beta$, then p_2^*C has a simple node at Q_2, and $(\pi_*^{-1}(p_2^*C))|E$ is a sum of 2 points, each with coefficient 1. In this case, (X, C) is log canonical at P_2. If $\alpha = \beta$, then p_2^*C has only one tangent direction at Q_2, and $(\pi_*^{-1}(p_2^*C))|E$ is a single point, with coefficient 2. In this case, (X, C) is not log canonical at P_2. However, using ([Kol],8.15) and ([Kol],5.20), we see that $(X, \frac{7}{10}C)$ is log canonical at P_2.

Now for D' we have:

$$mult_{Q_2} p_2^* D' \ \leq \ \tfrac{1}{2}(p_2^*D' \cdot p_2^*C) \ \leq \ \tfrac{3}{2}(D' \cdot C) \ = \tfrac{1}{2},$$

and hence $(X, \frac{7}{10}D')$ is log canonical at P_2.

2. $X_{15} \subset \mathbb{P}(1,3,5,7)$.

Let $C = X \cap (x_0 = 0) \equiv -K_X$. The only monomials of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ in $|\mathcal{O}_{\mathbb{P}(1,3,5,7)}(15)|$ are x_1^5, $x_1x_2x_3$, and x_2^3. Condition 2.1.1 for $i = 1,2$ implies that $x_1^5, x_2^3 \in f$. If $x_1x_2x_3 \notin f$, then, after rescaling, the equations of C in $\mathbb{P}(1,3,5,7)$ become:

$$x_0 = 0, \quad x_1^5 + x_2^3 = 0,$$

and $(X, \frac{2}{3}C)$ is not klt. (To see this, notice that $(S = p_3^{-1}(X), p_3^*(\frac{2}{3}C))$ is not klt by ([Kol], 8.15), and then apply ([Kol], 5.20).) In this case, the criterion of Theorem 3.4 cannot be applied. From now on we assume that $x_1x_2x_3 \in f$. Then, after rescaling, the equations of C in $\mathbb{P}(1,3,5,7)$ become:

$$x_0 = 0, \quad x_1^5 + x_1x_2x_3 + x_2^3 = 0. \tag{2}$$

One sees easily that C is irreducible and smooth on X^0.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor. We shall show that (X, D) is log canonical. This is more than enough to apply Theorem 3.4.

As we have seen above, $x_2^3 \in f$, and C is irreducible and smooth outside the singular locus of X. Thus we can apply Lemma 3.9 for $X_{15} \subset \mathbb{P}(1,3,7,5)$, and conclude that (X, D) is log canonical at the smooth points of X.

$P_3 = (0:0:0:1)$ is the only singular point of X. Write $D = \alpha C + (1-\alpha)D'$, with $0 \le \alpha \le 1$, $D \equiv -K_X$, and C not contained in the support of D'. We just need to show that both (X, C) and (X, D') are log canonical at P_3.

From the equations (2) above, we see that p_3^*C has a simple node at Q_3. So $mult_{Q_3} p_3^* C = 2$, and we can use Shokurov's inversion of adjunction. Let the notation be as in Theorem 3.11. Then $(\pi_*^{-1}(p_3^*C))|_E$ is a sum of 2 points, each with coefficient 1. We conclude that (X, C) is log canonical at P_3.

Now for D' we have:

$$mult_{Q_3} p_3^* D' \ \le\ \tfrac{1}{2}(p_3^*D' \cdot p_3^*C) \ \le\ \tfrac{7}{2}(D' \cdot C) \ = \tfrac{1}{2},$$

and hence (X, D') is log canonical at P_3.

3. $X_{18} \subset \mathbb{P}(2,3,5,9)$.

Condition 2.1.1 for $i = 2, 3$ implies that $x_1 x_2^3, x_3^2 \in f$. After rescaling, the equations for $C = X \cap (x_0 = 0)$ can be written as:

$$x_0 = 0, \quad x_3^2 + x_1 x_2^3 + a x_1^6 + b x_3 = 0.$$

C is irreducible and smooth outside the singular locus of X.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor. We will show that (X, D) is klt.

We apply Lemma 3.9 to conclude that D is klt at the smooth points of X.

The singular points of X are $P_2 = (0:0:1:0)$ and $P_\lambda = (0:1:0:\lambda)$, where λ satisfies $\lambda^2 + b\lambda + a = 0$. Estimate 3.10 shows that $mult_{Q_2} p_2^* D \le 1$ (here take $(i,j,k,l) = (2,0,1,3)$), and $mult_{Q_\lambda} p_1^* D \le 1$ (here take $(i,j,k,l) = (1,0,2,3)$). Thus D is klt at the singular points of X.

4. $X_{25} \subset \mathbb{P}(3,5,7,11)$.

Condition 2.1.1 for $i = 1,2,3$ implies that $x_1^5, x_2^2 x_3, x_0 x_3^2 \in f$. After rescaling we can write the equations for $C = X \cap (x_0 = 0) \in |\mathcal{O}_X(3)|$ as:

$$x_0 = 0, \quad x_1^5 + x_2^2 x_3 = 0. \tag{3}$$

We see that C is irreducible and smooth outside the singular locus of X.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor.

We apply Lemma 3.9 to $X_{25} \subset \mathbb{P}(3,7,11,5)$, and conclude that D is klt at the smooth points of X.

The singular points of X are $P_0 = (1:0:0:0)$, $P_2 = (0:0:1:0)$ and $P_3 = (0:0:0:1)$. Estimate 3.10 takes care of P_0 and P_2 (take $(i,j,k,l) = (0,1,2,3)$ and $(2,0,1,3)$ respectively), but it only gives that $mult_{Q_3} p_3^* D \le 25/21$ (here take $(i,j,k,l) = (3,0,1,2)$). We can improve this bound:

From the equations (3) above, we see that $p_3^*C \cong (y_1^5 + y_2^2 = 0) \subset \mathbb{C}^2_{(y_1,y_2)}$, and so $mult_{Q_3} p_3^* C = 2$. Write $D = \alpha C + (1-3\alpha)D'$, where $0 \le \alpha \le 1/3$, $D' \equiv -K_X$, and C is not contained in the support of D'. Then:

$$mult_{Q_3} p_3^* D' \le \tfrac{1}{2}(p_3^*C \cdot p_3^*D') \le \tfrac{11}{2}(\mathcal{O}_X(3) \cdot \mathcal{O}_X(1)) = \tfrac{5}{14}.$$

So $mult_{Q_3} p_3^* D \le 2/3$, and we have taken care of all singular points.

5. $X_{28} \subset \mathbb{P}(3,5,7,14)$.

Condition 2.1.1 for $i = 1, 3$ implies that $x_0x_1^5, x_3^2 \in f$. For $i = 2$, it implies that either $x_2^4 \in f$ or $x_2^2x_3 \in f$. After rescaling, the equations for $C = X \cap (x_0 = 0)$ become:

$$x_0 = 0, \quad x_3^2 + ax_2^2x_3 + bx_2^4 = (x_3 + \alpha x_2^2)(x_3 + \beta x_2^2) = 0.$$

We see that C is the union of 2 irreducible components.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor.

We can apply Corollary 3.7 with $l = 21$ (indeed, $x_3^2 \in f$ implies that π_3 has finite fibers, and $x_0^7,\ x_0^2x_1^3,\ x_2^3 \in |\mathcal{O}_{\mathbb{P}(3,5,7,14)}(21)|$), and conclude that $mult_P D \leq 2/5$ for every smooth point $P \in X \setminus (x_0 = 0)$. Since C is reducible, we cannot apply Lemma 3.9. Instead we apply Corollary 3.7 again with x_0 and x_2 interchanged, and $l = 35$ (π_3 has finite fibers, and $x_2^5,\ x_1^7,\ x_2^2x_0^7 \in |\mathcal{O}_{\mathbb{P}(3,5,7,14)}(35)|$). This shows that for every smooth point $P \in X \setminus (x_2 = 0)$, $mult_P D \leq 2/3$. Since $X \cap (x_0 = 0) \cap (x_2 = 0) = \{P_1\}$, this takes care of all smooth points of X.

The singular points of X are $P_0 = (1:0:0:0)$, $P_1 = (0:1:0:0)$ and $P_\lambda = (0:0:1:\lambda)$, where λ satisfies $\lambda^2 + a\lambda + b = 0$. Estimate 3.10 shows that Condition 3.5.2 for the singular points is satisfied.

6. $X_{36} \subset \mathbb{P}(3, 5, 11, 18)$.

Condition 2.1.1 for $i = 1, 3$ implies that $x_1^5x_2, x_3^2 \in f$. After rescaling, we can write the equations for C as:

$$x_0 = 0, \quad x_1^5x_2 + x_3^2 = 0.$$

We see that C is irreducible and smooth outside the singular locus of X.

Let $D \equiv -K_X$ be an effective $\mathbb{Q}$-divisor.

Lemma 3.9 shows that D is klt at the smooth points of X.

The singular points of X are $P_1 = (0:1:0:0)$, $P_2 = (0:0:1:0)$ and $P_\lambda = (1:0:0:\lambda)$, where λ satisfies $\lambda^2 + a\lambda + b = 0$. Here a and b are the coefficients of $x_0^6x_3$ and x_0^{12} in f respectively, and we still assume that the coefficient of x_3^2 in f is 1. Estimate 3.10 shows that Condition 3.5.2 for the singular points is satisfied. □

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