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MATRIX-WEIGHTED BESOV SPACES

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ABSTRACT. Nazarov, Treil and Volberg defined matrix A_p weights and extended the theory of weighted norm inequalities on L^p to the case of vector-valued functions. We develop some aspects of Littlewood-Paley function space theory in the *matrix weight setting*. In particular, we introduce matrix-weighted homogeneous Besov spaces $\dot{B}_p^{\alpha q}(W)$ and matrix-weighted sequence Besov spaces $\dot{b}_p^{\alpha q}(W)$, as well as $\dot{b}_p^{\alpha q}(\{A_Q\})$, where the A_Q are reducing operators for W. Under any of three different conditions on the weight W, we prove the norm equivalences $\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}$, where $\{\vec{s}_Q\}_Q$ is the vector-valued sequence of φ-transform coefficients of $\vec{f}$. In the process, we note and use an alternate, more explicit characterization of the matrix A_p class. Furthermore, we introduce a weighted version of almost diagonality and prove that an almost diagonal matrix is bounded on $\dot{b}_p^{\alpha q}(W)$ if W is doubling. We also obtain the boundedness of almost diagonal operators on $\dot{B}_p^{\alpha q}(W)$ under any of the three conditions on W. This leads to the boundedness of convolution and non-convolution type Calderón-Zygmund operators (CZOs) on $\dot{B}_p^{\alpha q}(W)$, in particular, the Hilbert transform. We apply these results to wavelets to show that the above norm equivalence holds if the φ-transform coefficients are replaced by the wavelet coefficients. Finally, we construct inhomogeneous matrix-weighted Besov spaces $B_p^{\alpha q}(W)$ and show that results corresponding to those above are true also for the inhomogeneous case.

1. Introduction. Overview of the results

Littlewood-Paley theory gives a unified perspective to the theory of function spaces. Well-known spaces such as Lebesgue, Hardy, Sobolev, Lipschitz spaces, etc. are special cases of either Besov spaces $\dot{B}_p^{\alpha q}$ (homogeneous), $B_p^{\alpha q}$ (nonhomogeneous) or Triebel-Lizorkin spaces $\dot{F}_p^{\alpha q}$ (homogeneous), $F_p^{\alpha q}$ (nonhomogeneous) (e.g., see [15]). The properties of these spaces are characterized by their discrete analogues: the sequence Besov spaces $\dot{b}_p^{\alpha q}$, $b_p^{\alpha q}$ and sequence Triebel-Lizorkin spaces $\dot{f}_p^{\alpha q}$, $f_p^{\alpha q}$ ([4], [3]). Littlewood-Paley theory provides alternate methods for studying singular integrals. The Hilbert transform, the classical example of a singular integral operator, led to the extensive modern theory of Calderón-Zygmund operators, mostly studied on the Lebesgue L^p spaces.

Motivated by the fundamental result of M. Riesz in the 1920s that the Hilbert transform preserves L^p for $1 < p < \infty$, Hunt, Muckenhoupt and Wheeden showed

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that the famous A_p condition on a weight w is the necessary and sufficient condition for the Hilbert transform to be bounded on $L^p(w)$ (1973, [8]). More recent developments deal with matrix-weighted spaces where scalar methods simply could not be applied. In 1996 Treil and Volberg obtained the analogue of the Hunt-Muckenhoupt-Wheeden condition for the matrix case when $p = 2$ ([16]). Soon afterwards, Nazarov and Treil introduced in [12] a new "Bellman function" method to extend the theory to $1 < p < \infty$. In 1997 Volberg presented a different solution to the matrix weighted L^p boundedness of the Hilbert transform via techniques related to classical Littlewood-Paley theory ([17]).

The purpose of this paper is to extend some aspects of Littlewood-Paley function space theory, in particular, the study of Besov spaces and Calderón-Zygmund operators on them, previously obtained with no weights and partially for scalar weights, to the *matrix weight setting*.

We define a new generalized function space: the vector-valued homogeneous Besov space $\dot{B}^{\alpha q}_p(W)$ with matrix weight W. Let $\mathcal{M}$ be the cone of nonnegative definite operators on a Hilbert space $\mathcal{H}$ of dimension m (consider $\mathcal{H} = \mathbb{C}^m$ or $\mathbb{R}^m$), i.e., for $M \in \mathcal{M}$ we have $(Mx, x)_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$. By definition, a *matrix weight* W is an a.e. invertible map $W : \mathbb{R}^n \to \mathcal{M}$. For a measurable $\vec{g} = (g_1, ..., g_m)^{\mathrm{T}} : \mathbb{R}^n \to \mathcal{H}$, let $\|\vec{g}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} \|W^{1/p}(t)\vec{g}(t)\|^p_{\mathcal{H}}\, dt \right)^{1/p}$. If the previous norm is finite, then $\vec{g} \in L^p(W)$. We say that a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ belongs to the class $\mathcal{A}$ of admissible kernels if supp $\hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. Set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ for $\nu \in \mathbb{Z}$.

Definition 1.1 (*Matrix-weighted Besov space* $\dot{B}^{\alpha q}_p(W)$)**.** For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, $\varphi \in \mathcal{A}$ and W a matrix weight, the Besov space $\dot{B}^{\alpha q}_p(W)$ is the collection of all vector-valued distributions $\vec{f} = (f_1, ..., f_m)^{\mathrm{T}}$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n), 1 \leq i \leq m$ (the space of tempered distributions modulo polynomials) such that

$$\|\vec{f}\|_{\dot{B}^{\alpha q}_p(W)} = \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(W)} \right\}_\nu \right\|_{l^q} = \left\| \left\{ \|W^{1/p} \cdot (\varphi_\nu * \vec{f}\,)\|_{L^p} \right\}_\nu \right\|_{l^\alpha_q} < \infty,$$

where $\varphi_\nu * \vec{f} = (\varphi_\nu * f_1, ..., \varphi_\nu * f_m)^{\mathrm{T}}$ and the l^q-norm is replaced by the supremum on ν if $q = \infty$.

The case $p = \infty$ is not of interest to us, since $\dot{B}^{\alpha q}_\infty(W) = \dot{B}^{\alpha q}_\infty$ because $L^\infty(W) = L^\infty$. Since φ is directly involved in the definition of $\dot{B}^{\alpha q}_p(W)$, there seems to be a dependence on the choice of φ: $\dot{B}^{\alpha q}_p(W) = \dot{B}^{\alpha q}_p(W, \varphi)$. Under appropriate conditions on W, Theorem 1.8 below shows that this is not the case. The space $\dot{B}^{\alpha q}_p(W)$ is complete, as is discussed at the end of Section 7.

We also introduce the corresponding weighted sequence (discrete) Besov space $\dot{b}^{\alpha q}_p(W)$:

Definition 1.2 (*Matrix-weighted sequence Besov space* $\dot{b}^{\alpha q}_p(W)$)**.** For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ and W a matrix weight, the space $\dot{b}^{\alpha q}_p(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_Q$, where $\vec{s}_Q = \left(s^{(1)}_Q, ..., s^{(m)}_Q\right)^{\mathrm{T}}$, enumerated by the dyadic cubes Q contained in $\mathbb{R}^n$, such that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_\nu \right\|_{l^q}$$

$$= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \left(\|W^{1/p}(t)\vec{s}_Q\|_{\mathcal{H}} \right) \chi_Q(t) \right\|_{L^p(dt)} \right\}_\nu \right\|_{l_q^\alpha} < \infty,$$

where $|Q|$ is the Lebesgue measure of Q, $l(Q)$ is the side length of Q, and the l^q-norm is again replaced by the supremum on ν if $q = \infty$.

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu k}$ be the dyadic cube $\{(x_1, ..., x_n) \in \mathbb{R}^n : \ k_i \leq 2^\nu x_i < k_i + 1, \ i = 1, ..., n\}$, and $x_Q = 2^{-\nu}k$ the lower left corner of $Q_{\nu k}$. Set $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k) = |Q|^{1/2}\varphi_\nu(x - x_Q)$ for $Q = Q_{\nu k}$. For each $\vec{f}$ with $f_i \in \mathcal{S}'(\mathbb{R}^n)$, we define the *$\varphi$-transform* S_φ as the map taking $\vec{f}$ to the vector-valued sequence $S_\varphi(\vec{f}\,) = \left\{ \left\langle \vec{f}, \varphi_Q \right\rangle \right\}_Q = \left\{ \left(\langle f_1, \varphi_Q \rangle, ..., \langle f_m, \varphi_Q \rangle \right)^{\mathrm{T}} \right\}_Q$ for Q dyadic. We call $\vec{s}_Q(\vec{f}\,) := \left\langle \vec{f}, \varphi_Q \right\rangle$ the *φ-transform coefficients* of $\vec{f}$.

The next question is motivated by the following results:

(i) Frazier and Jawerth ([4], 1985) showed that, in the unweighted scalar case,

$$\|f\|_{\dot{B}_p^{\alpha q}} \approx \|\{s_Q(f)\}_Q\|_{\dot{b}_p^{\alpha q}},$$

where $\{s_Q(f)\}_Q$ are the φ-transform coefficients. A similar equivalence holds if $\{s_Q(f)\}_Q$ are the wavelet coefficients $\{\langle f, \psi_Q \rangle\}_Q$ of f with ψ_Q being smooth, say, Meyer's wavelets (ref. [11]).

(ii) Nazarov, Treil and Volberg ([12], 1996, [17], 1997) obtained

$$\left\| \vec{f} \right\|_{L^p(W)} \approx \left\| \left\{ \left\langle \vec{f}, h_Q \right\rangle \right\} \right\|_{\dot{f}_p^{02}(W)} \quad \text{if } W \in A_p, \tag{1.1}$$

where $\{h_Q\}_Q$ is the Haar system and $\dot{f}_p^{02}(W)$ is the coefficient (sequence Triebel-Lizorkin) space for $L^p(W)$. A particular case of (1.1), when $m = 1$ and w is a scalar weight, is

$$\|f\|_{\dot{B}_2^{02}(w)} = \|f\,\|_{L^2(w)} \approx \|\{\langle f, h_Q \rangle\}\|_{\dot{f}_2^{02}(w)} = \|\{\langle f, h_Q \rangle\}\|_{\dot{b}_2^{02}(w)},$$

where the first equality and the equivalence hold if $w \in A_2$.

For our purposes we will use a condition on W that is equivalent to the matrix A_p condition of [12] (for the proof, refer to Section 3):

Lemma 1.3. *Let W be a matrix weight, $1 < p < \infty$, and let p' be the conjugate of p ($\frac{1}{p} + \frac{1}{p'} = 1$). Then*

$$\int_B \left(\int_B \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \leq c_{p,n} \text{ for every ball } B \subseteq \mathbb{R}^n \tag{1.2}$$

if and only if $W \in A_p$.

In (1.2), $\|W^{1/p}(x)W^{-1/p}(t)\|$ refers to the matrix (operator) norm.

The advantage of condition (1.2) is that it allows us to understand the A_p condition in terms of matrices, avoiding metrics ρ, ρ^* and their averagings as well as reducing operators (for definitions and details refer to Section 3).

Our first result is the norm equivalence between the continuous matrix-weighted Besov space $\dot{B}_p^{\alpha q}(W)$ and the discrete matrix-weighted Besov space $\dot{b}_p^{\alpha q}(W)$ under the A_p condition:

Theorem 1.4. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 < p < \infty$ and $W \in A_p$. Then*

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\|\left\{\vec{s}_Q\left(\vec{f}\right)\right\}_Q\right\|_{\dot{b}_p^{\alpha q}(W)}. \tag{1.3}$$

In some cases, the A_p requirement on W can be relaxed. Recall that a scalar measure μ is called *doubling* if there exists $c > 0$ such that for any $\delta > 0$ and any $z \in \mathbb{R}^n$,

$$\mu(B_{2\delta}(z)) \leq c\,\mu(B_\delta(z)), \tag{1.4}$$

where $B_\delta(z) = \{x \in \mathbb{R}^n : \ |z - x| < \delta\}$.

Definition 1.5 (*Doubling matrix*)**.** A matrix weight W is called a *doubling matrix (of order p, $1 \leq p < \infty$)*, if there exists a constant $c = c_{p,n}$ such that for any $y \in \mathcal{H}$, any $\delta > 0$ and any $z \in \mathbb{R}^n$,

$$\int_{B_{2\delta}(z)} \|W^{1/p}(t)\, y\|_{\mathcal{H}}^p\, dt \leq c \int_{B_\delta(z)} \|W^{1/p}(t)\, y\|_{\mathcal{H}}^p\, dt, \tag{1.5}$$

i.e., the scalar measure $w_y(t) = \|W^{1/p}(t)\, y\|_{\mathcal{H}}^p$ is uniformly doubling and not identically zero (a.e.). If $c = 2^\beta$ is the smallest constant for which (1.5) holds, then β is called the *doubling exponent* of W.

It is known that if $W \in A_p$, then w_y is a scalar A_p weight for any $y \in \mathcal{H}$ and the A_p constant is independent of y (for example, see [17]). This, in turn, implies that w_y is a scalar doubling measure (e.g., see [14]) and the doubling constant is also independent of y. Using decomposition techniques, we prove the equivalence (1.3) under the doubling assumption on W with the restriction that p is large, and with no restriction on p in the case when W is a diagonal matrix:

Theorem 1.6. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W be a doubling matrix of order p with doubling exponent β. Suppose $p > \beta$. Then the norm equivalence* (1.3) *holds. If W is diagonal, then* (1.3) *holds for all $1 \leq p < \infty$.*

The case of a scalar weight is a particular case of the diagonal matrix weight case, and thus, the equivalence (1.3) holds just under the doubling condition. This fact is essentially known (see [5] for the case of $\dot{F}_p^{\alpha q}$); it is proved here for purposes of comparison and generalization to the diagonal matrix case.

Remark 1.7. One of the directions of the norm equivalence uses only the doubling property of W with no restrictions (see Corollary 5.6), but the other direction requires the stated assumptions on W (see Theorem 6.6). Furthermore, the first direction is obtained from a more general norm estimate involving families of "smooth molecules" (see Theorem 5.2).

Summarizing Theorems 1.4 and 1.6, the norm equivalence (1.3) holds under any of the following conditions:

(A1) $W \in A_p$ with $1 \leq p < \infty$,
(A2) W is a doubling matrix of order p with $p > \beta$, where β is the doubling exponent of W,
(A3) W is a diagonal doubling matrix of order p with $1 \leq p < \infty$.

Now we will state the independence of the space $\dot{B}_p^{\alpha q}(W,\varphi)$ from φ:

Theorem 1.8. *Let $\vec{f} \in \dot{B}_p^{\alpha q}(W,\varphi^{(1)})$, $\varphi^{(1)} \in \mathcal{A}$, $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and suppose any of (A1)-(A3) hold. Then for any $\varphi^{(2)} \in \mathcal{A}$,*

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W,\varphi^{(1)})} \approx \left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W,\varphi^{(2)})}.$$

If we use the language of reducing operators (see [17] or Section 3 below), we extend the norm equivalence (1.3) to a different sequence space, namely $\dot{b}_p^{\alpha q}(\{A_Q\})$. For each dyadic cube Q, consider a reducing operator A_Q corresponding to the L^p average over Q of the norm $\|W^{1/p}\cdot\|_{\mathcal{H}}$, i.e.,

$$\|A_Q\vec{u}\|_{\mathcal{H}} \approx \left(\frac{1}{|Q|}\int_Q \|W^{1/p}(t)\vec{u}\|_{\mathcal{H}}^p\,dt\right)^{1/p}$$

for all vector-valued sequences $\vec{u}$. Note that the assumption that W is a.e. invertible guarantees that each A_Q is invertible. Define the sequence space $\dot{b}_p^{\alpha q}(\{A_Q\})$ for $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ as the space containing all vector-valued sequences $\{\vec{s}_Q\}_Q$ with

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \left\|\left\{2^{\nu\alpha}\left\|\sum_{l(Q)=2^{-\nu}}|Q|^{-\frac{1}{2}}\left(\|A_Q\vec{s}_Q\|_{\mathcal{H}}\right)\chi_Q\right\|_{L^p(dt)}\right\}_\nu\right\|_{l^q} < \infty.$$

Theorem 1.9. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$. Suppose W satisfies any of (A1)-(A3). Then*

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\|\left\{\vec{s}_Q\left(\vec{f}\right)\right\}_Q\right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}. \tag{1.6}$$

Next we study operators on $\dot{B}_p^{\alpha q}(W)$ by considering corresponding operators on $\dot{b}_p^{\alpha q}(W)$. In [3] it was shown that almost diagonal operators are bounded on $\dot{b}_p^{\alpha q}$ and, thus, on $\dot{B}_p^{\alpha q}$. In Section 8 we define a class of almost diagonal matrices $\mathbf{ad}_p^{\alpha q}(\beta)$ for the weighted case and show the boundedness of these matrices on $\dot{b}_p^{\alpha q}(W)$ if W is a doubling matrix weight:

Theorem 1.10. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W be a doubling matrix of order p with doubling exponent β. Consider $A \in \mathbf{ad}_p^{\alpha q}(\beta)$. Then $A : \dot{b}_p^{\alpha q}(W) \longrightarrow \dot{b}_p^{\alpha q}(W)$ is bounded.*

We say that a linear continuous operator $T : \mathcal{S} \to \mathcal{S}'$ is *almost diagonal*, $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, if for some pair of mutually admissible kernels (φ,ψ) (see (2.1), Section 2) the matrix $(\langle T\psi_P,\varphi_Q\rangle_{QP})_{Q,P\,\text{dyadic}} \in \mathbf{ad}_p^{\alpha q}(\beta)$ (see Section 8). Combining the boundedness of an almost diagonal matrix with the norm equivalence, we obtain the boundedness of an almost diagonal operator on $\dot{B}_p^{\alpha q}(W)$ under any of (A1)-(A3):

Corollary 1.11. *Let $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. Then T is a bounded operator on $\dot{B}_p^{\alpha q}(W)$ if W satisfies any of (A1)-(A3).*

In Section 9 we consider classical convolution and generalized non-convolution Calderón-Zygmund operators (CZOs). The following criterion is used: if an operator T maps "smooth atoms" into "smooth molecules" (see Sections 5 and 9 for definitions), then T is almost diagonal (Lemma 9.2) and, therefore, bounded on $\dot{B}_p^{\alpha q}(W)$. To show this property for a CZO, the definition of a "smooth molecule" is modified in order to compensate for the growth of the weight W (note the dependence of the decay rate of the molecule on the doubling exponent β), and, thus, more smoothness of a CZO kernel is required (see Theorems 9.14, 9.8). In particular, for example, we obtain the boundedness of the Hilbert transform (when the underlying dimension is $n = 1$) and the Riesz transforms ($n \geq 2$) on $\dot{B}_p^{\alpha q}(W)$ under any of the conditions (A1)-(A3).

In Section 10 we apply the previous results to Meyer's wavelets and Daubechies' DN wavelets with N sufficiently large, to show that, instead of the φ-transform coefficients, one can use the wavelet coefficients for the norm equivalence:

Theorem 1.12. *If W satisfies any of (A1)-(A3), $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, then*

$$\left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)},$$

where $\left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_Q$ are the wavelet coefficients of $\vec{f}$.

So far we have dealt only with homogeneous spaces. However, for a number of applications it is necessary to consider the inhomogeneous distribution spaces (e.g., localized Hardy spaces $H^p_{\text{loc}} = F_p^{02}, 0 < p < \infty$, in particular, $H^2_{\text{loc}} = B_2^{02}$, [7]). In the last section, we "transfer" the theory developed up until now to the inhomogeneous Besov spaces. The main difference is that instead of considering all dyadic cubes, we consider only the ones with side length $l(Q) \leq 1$, and the properties of functions corresponding to $l(Q) = 1$ are slightly changed. Modifying the definitions of the φ-transform and smooth molecules, we show that all the statements from the homogeneous case are essentially the same for the inhomogeneous spaces.

2. Notation and definitions

Let $z \in \mathbb{R}^n$. Recall that $B(z, \delta) = \{x \in \mathbb{R}^n : |z - x| < \delta\} \equiv B_\delta(z)$. If the center z of the ball is not essential, we will write B_δ for simplicity.

For each admissible $\varphi \in \mathcal{A}$, there exists $\psi \in \mathcal{A}$ (see e.g. [3]) such that

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat{\varphi}(2^\nu \xi)} \cdot \hat{\psi}(2^\nu \xi) = 1, \text{ if } \xi \neq 0. \tag{2.1}$$

A pair (φ, ψ) with $\varphi, \psi \in \mathcal{A}$ and the property (2.1) will be referred to as a pair of mutually admissible kernels.

Similarly to φ_Q, define $\psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k)$ for $Q = Q_{\nu k}$. The *inverse φ-transform* T_ψ is the map taking a sequence $s = \{s_Q\}_Q$ to $T_\psi s = \sum_Q s_Q \psi_Q$. In the vector case, $T_\psi \vec{s} = \sum_Q \vec{s}_Q \psi_Q$, where $\vec{s}_Q \psi_Q = (s_Q^{(1)} \psi_Q, ..., s_Q^{(m)} \psi_Q)^{\mathrm{T}}$. The φ-transform decomposition (see [5] for more details) states that for all $f \in \mathcal{S}'/\mathcal{P}$,

$$f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q =: \sum_Q s_Q \psi_Q. \tag{2.2}$$

In other words, $T_\psi \circ S_\varphi$ is the identity on $\mathcal{S}'/\mathcal{P}$. Observe that if $\tilde{\varphi}(x) = \overline{\varphi(-x)}$ (note that $\tilde{\varphi} \in \mathcal{A}$), then $s_Q = \langle f, \varphi_Q \rangle = |Q|^{1/2}(\tilde{\varphi}_\nu * f)(2^{-\nu}k)$.

3. Matrix A_p condition

Although the A_p condition can be formulated for any family of norms ρ_t on a Hilbert space, we will consider only the particular case of norms:

$$\rho_t(x) = \|W^{1/p}(t)x\|, \quad \text{where } x \in \mathcal{H},\ t \in \mathbb{R}^n.$$

Then the dual metric ρ^* is given by

$$\rho_t^*(x) = \sup_{y \neq 0} \frac{|(x,y)|}{\rho_t(y)} = \|W^{-1/p}(t)x\|.$$

Following [17], we introduce the norms $\rho_{p,B}$ through the averagings of the metrics ρ_t over a ball B:

$$\rho_{p,B}(x) = \left(\frac{1}{|B|} \int_B [\rho_t(x)]^p \, dt \right)^{1/p}.$$

Similarly, for the dual metric,

$$\rho_{p',B}^*(x) = \left(\frac{1}{|B|} \int_B [\rho_t^*(x)]^{p'} \, dt \right)^{1/p'}.$$

Definition 3.1 (*Matrix A_p weight*)**.** For $1 < p < \infty$, we say that W is an A_p matrix weight if $W : \mathbb{R}^n \to \mathcal{M}$ is such that W and $W^{-p'/p}$ are locally integrable and there exists $C < \infty$ such that

$$\rho_{p',B}^* \leq C(\rho_{p,B})^* \quad \text{for every ball } B \subseteq \mathbb{R}^n. \tag{3.1}$$

In general, if ρ satisfies (3.1), then ρ is called an A_p-metric. Note that the condition (3.1) is equivalent to

$$\rho_{p,B} \leq C(\rho_{p',B}^*)^*,$$

which means that ρ^* is an $A_{p'}$-metric.

If ρ is a norm on $\mathcal{H}$, then there exists a positive operator A, which is called a reducing operator of ρ, such that

$$\rho(x) \approx \|Ax\| \quad \text{for all } x \in \mathcal{H}.$$

For details we refer the reader to [17]. Let A_B be a reducing operator for $\rho_{p,B}$, and $A_B^{\#}$ for $\rho_{p',B}^*$. Then, $(\rho_{p,B})^*(x) \approx \|A_B^{-1}x\|$. Hence, in the language of reducing operators, the condition (3.1) for the A_p class is

$$\|A_B^{\#} A_B\| \leq C < \infty \quad \text{for every ball } B \subseteq \mathbb{R}^n. \tag{3.2}$$

Observe the following two useful facts. First, if P and Q are two selfadjoint operators in a normed space, then

$$\|PQ\| = \|(PQ)^*\| = \|Q^*P^*\| = \|QP\|. \tag{3.3}$$

Thus, the operators can be commuted as long as we deal with norms.

Second, we need the following lemma:

Lemma 3.2 (Norm lemma). *If $\{e_1, \dots, e_m\}$ is any orthonormal basis in a Hilbert space $\mathcal{H}$, then for any $V \in \mathcal{B}(\mathcal{H})$ and $r > 0$,*

$$\|V\|^r \underset{(r,m)}{\approx} \sum_{i=1}^m \|Ve_i\|^r_{\mathcal{H}}.$$

Proof. With $x_i = (x, e_i)_{\mathcal{H}}$, we get

$$\begin{aligned}\|V\|^r &= \sup_{\|x\|\le 1} \|V\sum_{i=1}^m x_i e_i\|^r_{\mathcal{H}} \\ &\le c_r \sup_{\|x\|\le 1} \sum_{i=1}^m |x_i|^r \|Ve_i\|^r_{\mathcal{H}} \le c_r \sum_{i=1}^m \|Ve_i\|^r_{\mathcal{H}} \le c_r m\, \|V\|^r.\end{aligned}$$

□

Now we are ready to prove the equivalence of (1.2) and the A_p condition.

Proof of Lemma 1.3. By property (3.3) and the Norm Lemma,

$$\begin{aligned}&\int_B \left(\int_B \left\|W^{1/p}(x)W^{-1/p}(t)\right\|^{p'} \frac{dt}{|B|}\right)^{p/p'} \frac{dx}{|B|} \\ &= \int_B \left(\int_B \left\|W^{-1/p}(t)W^{1/p}(x)\right\|^{p'} \frac{dt}{|B|}\right)^{p/p'} \frac{dx}{|B|} \\ &\approx \int_B \left(\int_B \sum_{i=1}^m \left\|W^{-1/p}(t)W^{1/p}(x)e_i\right\|^{p'} \frac{dt}{|B|}\right)^{p/p'} \frac{dx}{|B|} \\ &\approx \sum_{i=1}^m \int_B \left(\int_B \left[\rho_t^*(W^{1/p}(x)\, e_i)\right]^{p'} \frac{dt}{|B|}\right)^{p/p'} \frac{dx}{|B|} \\ &= \sum_{i=1}^m \int_B \left[\rho^*_{p',B}(W^{1/p}(x)\, e_i)\right]^p \frac{dx}{|B|}.\end{aligned}$$

Now, in terms of the reducing operators, the last expression is equivalent to

$$\begin{aligned}\sum_{i=1}^m \int_B \left\|A_B^{\#}(W^{1/p}(x)\, e_i)\right\|^p \frac{dx}{|B|} &\approx \int_B \left\|A_B^{\#}\, W^{1/p}(x)\right\|^p \frac{dx}{|B|} \\ &\approx \sum_{i=1}^m \int_B \left\|W^{1/p}(x)(A_B^{\#}\, e_i)\right\|^p \frac{dx}{|B|} \\ &\approx \sum_{i=1}^m \left[\rho_{p,B}(A_B^{\#}\, e_i)\right]^p \\ &\approx \sum_{i=1}^m \left\|A_B(A_B^{\#}\, e_i)\right\|^p \\ &\approx \left\|A_B\, A_B^{\#}\right\|^p.\end{aligned}$$

Therefore, (1.2) is equivalent to $\left\|A_B^{\#}\, A_B\right\|^p \le c$, i.e., the A_p condition. □

Corollary 3.3 (Symmetry of matrix A_p condition). *The following statements are equivalent:*

(i) $W \in A_p$;

(ii) $W^{-p'/p} \in A_{p'}$;

(iii) $\displaystyle\int_B \left(\int_B \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \le c$ *for every ball* $B \subseteq \mathbb{R}^n$;

(iv) $\displaystyle\int_B \left(\int_B \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^{p} \frac{dx}{|B|} \right)^{p'/p} \frac{dt}{|B|} \le c$ *for every ball* $B \subseteq \mathbb{R}^n$.

Proof. Recall that $\rho \in A_p$ if and only if $\rho^* \in A_{p'}$. In terms of matrix weights, $W \in A_p$ if and only if $W^{-p'/p} \in A_{p'}$ (note that $\rho_t^*(x) = \|(W^{-p'/p})^{1/p'}(t)\, x\|$). By Lemma 1.3, the third statement is equivalent to $W \in A_p$, whereas the fourth is equivalent to $W^{-p'/p} \in A_{p'}$. □

4. Doubling measures

Let W be a doubling matrix of order p, i.e., (1.5) holds for any $y \in \mathcal{H}$, $\delta > 0$ and $z \in \mathbb{R}^n$. For $p = 2$ this simplifies to

$$\int_{B_{2\delta}} W(t)\, dt \le c \int_{B_\delta} W(t)\, dt \tag{4.1}$$

for a given δ, where the inequality is understood in the sense of selfadjoint operators.

Remark 4.1. Note that $\|W^{1/p}(t)\|^p$ is independent of p. If $w_y(t) = \|W^{1/p}(t)\, y\|_{\mathcal{H}}^p$ is doubling of order p for any $y \in \mathcal{H}$, then $w(t) = \|W^{1/p}(t)\|^p$ is also a scalar-valued doubling measure.

Proof. Fix $t \in \mathbb{R}^n$. Then there exist a unitary matrix U and a diagonal matrix Λ such that $W(t) = U\,\Lambda\,U^{-1}$, and so $W^{1/p}(t) = U\,\Lambda^{1/p}\,U^{-1}$. Moreover, since the norm of a positive diagonal matrix is the largest eigenvalue, say λ_0, $\|W^{1/p}(t)\| = \lambda_0^{1/p}$ and, hence, $\|W^{1/p}(t)\|^p = \lambda_0$, regardless of what p is.

Now, since (1.5) is true with $y = e_i$ (any orthonormal basis vector of $\mathcal{H}$), by the Norm Lemma we get the second assertion:

$$\int_{B_{2\delta}} \|W^{1/p}(t)\|^p\, dt \approx \sum_{i=1}^m \int_{B_{2\delta}} \|W^{1/p}(t) e_i\|^p\, dt$$

$$\le c \sum_{i=1}^m \int_{B_\delta} \|W^{1/p}(t) e_i\|^p\, dt \approx c \int_{B_\delta} \|W^{1/p}(t)\|^p\, dt.$$

□

The doubling property of $w(t) = \|W^{1/p}(t)\|^p$ is not very helpful if one wants to understand the nature of W; it only tells us how large the weight is, not how it is distributed in different directions. Therefore, we use the definition of doubling matrix in (1.5), which involves different directions of $y \in \mathcal{H}$.

Remark 4.2. In the scalar case, (1.5) gives the standard doubling measure:

$$\int_{B_{2\delta}} w(t)|y|^p\, dt \le c \int_{B_\delta} w(t)|y|^p\, dt,$$

and if $y \neq 0$, then $w(B_{2\delta}) \le c\, w(B_\delta)$. In particular, there is no dependence on p in the scalar situation.

Similar definitions for doubling weights can be analogously given for the "dual" measure $w_y^*(t) = \|W^{-1/p}(t)y\|^{p'}$.

Lemma 4.3. *Let $x \in \mathcal{H}$ and $W \in A_p$. Then $v_x(t) := \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} = \|W^{-1/p}(t)W^{1/p}(x)\|^{p'}$ is a doubling measure, i.e., there exists a constant c such that for any $\delta > 0$,*

$$\int_{B_{2\delta}} \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} dt \leq c \int_{B_\delta} \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} dt. \tag{4.2}$$

Proof. Applying the Norm Lemma to the operator norm in the left-hand side, we obtain

$$v_x(t) \approx \sum_{i=1}^m \|W^{-1/p}(t)W^{1/p}(x)e_i\|^{p'} = \sum_{i=1}^m \|W^{-1/p}(t)y_i(x)\|^{p'} = \sum_{i=1}^m w^*_{y_i(x)}(t),$$

where $y_i(x) = W^{1/p}(x)e_i$. Then

$$v_x(B_{2\delta}) \approx \sum_{i=1}^m \int_{B_{2\delta}} w^*_{y_i(x)}(t)\,dt \leq \sum_{i=1}^m c \int_{B_\delta} w^*_{y_i(x)}(t)\,dt \leq c\, v_x(B_\delta),$$

since w_y^* is doubling ($W^{-p'/p} \in A_{p'}$). □

Remark 4.4. The doubling property (1.4) is equivalent to

$$\frac{\mu(F)}{\mu(E)} \leq c \left(\frac{|F|}{|E|}\right)^{\beta/n}, \tag{4.3}$$

where F is a ball (or a cube) and $E \subseteq F$ is a sub-ball (sub-cube) (not *any* subset of F; any subset would be equivalent to the A_∞ condition, see [14]).

Proof. Since $E \subseteq F$, there exists $j \in \mathbb{N}$ such that $2^j E \approx F$, i.e., $l(F) \approx 2^j l(E)$. Since μ is doubling, by (1.4), $\dfrac{\mu(F)}{\mu(E)} \leq c^j \approx c^{\log_2 \frac{l(F)}{l(E)}}$. Noticing that $\dfrac{|F|}{|E|} = \left[\dfrac{l(F)}{l(E)}\right]^n$, we get (4.3). □

In further estimates, it is more convenient to use (4.3) instead of (1.4).

Observe that the doubling exponent of the Lebesgue measure in $\mathbb{R}^n$ is $\beta = n$; moreover, if μ is any nonzero doubling measure in $\mathbb{R}^n$, then $\beta(\mu) \geq n$.

5. Boundedness of the inverse φ-transform

Consider $\dot{B}_p^{\alpha q}(W)$ with parameters α, p, q fixed ($\alpha \in \mathbb{R}, 1 \leq p < \infty$, $0 < q < \infty$). For $0 < \delta \leq 1$, $M > 0$ and $N \in \mathbb{Z}$ define (as in [5]) m_Q to be a *smooth* (δ, M, N)-*molecule* for Q dyadic if:

(M1) $\displaystyle\int x^\gamma m_Q(x)\,dx = 0$, for $|\gamma| \leq N$,

(M2) $\displaystyle |m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-\max(M, M-\alpha)}$,

(M3) $\displaystyle |D^\gamma m_Q(x)| \leq |Q|^{-1/2-|\gamma|/n} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-M}$ if $|\gamma| \leq [\alpha]$,

(M4) $\displaystyle |D^\gamma m_Q(x) - D^\gamma m_Q(y)| \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta \times \sup_{|z| \leq |x-y|} \left(1 + \frac{|x - z - x_Q|}{l(Q)}\right)^{-M}$ if $|\gamma| = [\alpha]$.

It is understood that (M1) is void if $N < 0$; and (M3), (M4) are void if $\alpha < 0$. Also, $[\alpha]$ stands for the greatest integer $\le \alpha$; γ is a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$, $1 \le i \le n$, and the standard notation is used.

We say $\{m_Q\}_Q$ is a *family* of *smooth molecules* for $\dot{B}_p^{\alpha q}(W)$ if each m_Q is a (δ, M, N)-molecule with

(M.i) $\alpha - [\alpha] < \delta \le 1$,
(M.ii) $M > J$, where $J = \frac{\beta}{p} + \frac{n}{p'}$ if $p > 1$ and $J = \beta$ if $p = 1$,
(M.iii) $N = \max([J - n - \alpha], -1)$.

Remark 5.1. Note that, in contrast to the case in [5], there is a dependence of the family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ on the weight W (more precisely, on the doubling exponent β).

Theorem 5.2. *Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$, and let W be a doubling matrix weight of order p. Suppose $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$. Then*

$$\left\| \sum_Q \vec{s}_Q \, m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \le c \, \| \{\vec{s}_Q\}_Q \|_{\dot{b}_p^{\alpha q}(W)}. \tag{5.1}$$

The proof uses the following estimates for Q dyadic with $l(Q) = 2^{-\mu}$, $\mu \in \mathbb{Z}$, and φ_ν, $\nu \in \mathbb{Z}$, with $\varphi \in \mathcal{A}$:

$$|\varphi_\nu * m_Q(x)| \le c \, |Q|^{-1/2} 2^{-(\mu-\nu)\sigma} \left(1 + 2^\nu |x - x_Q|\right)^{-M} \text{ for some } \sigma > J - \alpha, \tag{5.2}$$

if $\mu > \nu$, and

$$|\varphi_\nu * m_Q(x)| \le c \, |Q|^{-1/2} 2^{-(\nu-\mu)\tau} \left(1 + 2^\mu |x - x_Q|\right)^{-M} \text{ for some } \tau > \alpha, \tag{5.3}$$

if $\mu \le \nu$.

The proofs are entirely elementary, but quite tedious (see [5], Appendix B). Note that in the statement of Lemma B.1 in [5], it should say $j \le k$. For (5.2), for $N \ne -1$, apply Lemma B.1 with $j = \nu$, $k = \mu$, $L = N$, $R = M$, $S = M - \alpha$, $g = 2^{-\nu n/2} \varphi_\nu$, $h = m_Q$ with $l(Q) = 2^{-\mu}$, $x_1 = x_Q$, $J - n - \alpha - [J - n - \alpha] < \theta \le 1$. Letting $\sigma = N + n + \theta > J - \alpha$, we obtain (5.2). For $N = -1$, apply Lemma B.2 in [5] with $\sigma = n > J - \alpha$ to get (5.2). Now for (5.3), for $\alpha > 0$, apply Lemma B.1 with $k = \nu$, $j = \mu$, $L = [\alpha]$, $R = M$, $\delta = \theta$, $S = [\alpha] + n + \delta$, $x_1 = 0$, $g(x) = m_Q(x + x_Q)$, $h = 2^{-\nu n/2} \varphi_\nu$, and observe that $\varphi_\nu * m_Q(x) = 2^{\nu n/2} g * h(x - x_Q)$ to get (5.3) with $\tau = \delta + [\alpha] > \alpha$. For $\alpha < 0$, Lemma B.2 in [5] gives (5.3) with $\tau = 0 > \alpha$.

Lemma 5.3 (Squeeze Lemma)**.** *Fix a dyadic cube Q and let $w : \mathbb{R}^n \to \mathbb{R}^+$ be a scalar doubling measure with the doubling exponent β. If $L > \beta$, then for $r \ge l(Q)$,*

$$\int_{\mathbb{R}^n} w(x) \left(1 + \frac{|x - x_Q|}{r}\right)^{-L} dx \le c_\beta \left[\frac{r}{l(Q)}\right]^\beta \int_Q w(x) \, dx. \tag{5.4}$$

Proof. Decompose $\mathbb{R}^n$ into the annuli $\mathcal{R}_m$:

$$\mathbb{R}^n = \bigcup_{m=1}^\infty \{x : \; 2^{m-1} r \le |x - x_Q| < 2^m r\} \cup \{x : \; |x - x_Q| < r\} =: \bigcup_{m=0}^\infty \mathcal{R}_m.$$

Then the left-hand side of (5.4) is bounded by

$$\sum_{m=1}^\infty (1 + 2^{m-1})^{-L} w(\mathcal{R}_m) + w(\mathcal{R}_0). \tag{5.5}$$

Using the doubling property of w, we get

$$w(\mathcal{R}_m) \le w(B(x_Q, 2^m r)) \le c \left(\frac{|B(x_Q, 2^m r)|}{|\mathcal{R}_0|} \right)^{\beta/n} w(\mathcal{R}_0) = c\, 2^{m\beta} w(\mathcal{R}_0).$$

Thus, (5.5) is bounded by

$$c \sum_{m=0}^{\infty} 2^{m\beta - mL}\, w(\mathcal{R}_0) \le c_\beta\, w(\mathcal{R}_0),$$

since $L > \beta$. Note that $B(x_Q, l(Q)) \subseteq 3Q$ and so $w(B(x_Q, l(Q))) \le c_\beta\, w(Q)$. If $r > l(Q)$, then

$$w(\mathcal{R}_0) \le c \left(\frac{|\mathcal{R}_0|}{|B(x_Q, l(Q))|} \right)^{\beta/n} w(B(x_Q, l(Q))) \le c_\beta \left[\frac{r}{l(Q)} \right]^{\beta} w(Q),$$

which is (5.4). □

Lemma 5.4 (Summation Lemma)**.** *Let $\mu, \nu \in \mathbb{Z}$ and $y \in \mathbb{R}^n$. Then for $M > n$,*

$$\sum_{l(Q)=2^{-\mu}} \left(1 + \frac{|y - x_Q|}{2^{-\nu}} \right)^{-M} \le c_{n,M}\, 2^{(\mu-\nu)n}, \text{ if } \mu \ge \nu. \tag{5.6}$$

Proof. If $\mu \ge \nu$ or $2^{-\nu} \ge 2^{-\mu}$, there are $2^{(\mu-\nu)n}$ cubes of size $2^{-\mu}$ in a cube of size $2^{-\nu}$. Fix $l \in \mathbb{Z}^n$ such that $y \in Q_{\nu l}$. Then the left-hand side of (5.6) is

$$\begin{aligned}\sum_{k \in \mathbb{Z}^n} (1 + 2^\nu |y - x_{Q_{\mu k}}|)^{-M} &= \sum_{i \in \mathbb{Z}^n} \sum_{k\colon Q_{\mu k} \subseteq Q_{\nu(l+i)}} (1 + 2^\nu |y - x_{Q_{\mu k}}|)^{-M} \\ &\le \sum_{i \in \mathbb{Z}^n} (1 + |i|)^{-M} \times 2^{(\mu-\nu)n} \le c_n\, 2^{(\mu-\nu)n},\end{aligned}$$

again since $M > n$. □

Proof of Theorem 5.2. By definition,

$$\begin{aligned}\left\| \sum_Q \vec{s}_Q\, m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} &= \left\| \left\{ \left\| W^{1/p} \sum_Q \vec{s}_Q\, (\varphi_\nu * m_Q) \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\ &= \left\| \left\{ \left\| \sum_{\mu \in \mathbb{Z}} \left[\sum_{l(Q)=2^{-\mu}} (W^{1/p} \vec{s}_Q)(\varphi_\nu * m_Q) \right] \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha}.\end{aligned}$$

By Minkowski's (or the triangle) inequality, the last expression is bounded by

$$\left\| \left\{ \sum_{\mu \in \mathbb{Z}} \left\| \sum_{l(Q)=2^{-\mu}} (W^{1/p} \vec{s}_Q)(\varphi_\nu * m_Q) \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha}$$

$$\le \left\| \left\{ \sum_{\mu \in \mathbb{Z}} \left(\int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} \|W^{1/p}(x) \vec{s}_Q\|\, |(\varphi_\nu * m_Q)(x)| \right)^p dx \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha}$$

$$=: \left\| \left\{ \sum_{\mu>\nu} J_1^{1/p} + \sum_{\mu\le\nu} J_2^{1/p} \right\}_\nu \right\|_{l_q^\alpha}. \tag{5.7}$$

Using estimates (5.2) and (5.3) with $\theta_1 = -(\mu-\nu)\sigma$, $\theta_2 = -(\nu-\mu)\tau$ and $r_1 = 2^{-\nu}$, $r_2 = 2^{-\mu}$, we bound each J_i, $i = 1, 2$:

$$J_i \le c \int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} \|W^{1/p}(x)\vec{s}_Q\|_{\mathcal{H}} |Q|^{-1/2}\, 2^{\theta_i} \left(1 + \frac{|x-x_Q|}{r_i}\right)^{-M} \right)^p dx.$$

Split $M = M_1 + M_2$, where $M_1 > \beta/p$ and $M_2 > n/p'$ (if $p = 1$, let $M_2 = 0$ and interpret $n/p' = 0$). This splitting is possible since $M > J$. Then by the discrete Hölder inequality with $w_Q(x) = \|W^{1/p}(x)\vec{s}_Q\|_{\mathcal{H}}^p$, we get

$$J_i \le c_p \int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} w_Q(x)|Q|^{-p/2}\, 2^{\theta_i p} \left(1 + \frac{|x-x_Q|}{r_i}\right)^{-M_1 p} \right)$$
$$\times \left[\sum_{l(Q)=2^{-\mu}} \left(1 + \frac{|x-x_Q|}{r_i}\right)^{-M_2 p'} \right]^{p/p'} dx.$$

By the Summation Lemma 5.4 (with $\nu = \mu$ in (5.6)), we have

$$J_2 \le c_{p,n} 2^{\theta_2 p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \int_{\mathbb{R}^n} w_Q(x)(1 + 2^\mu|x-x_Q|)^{-M_1 p}\, dx,$$

since $M_2 > n/p'$. Applying the Squeeze Lemma 5.3 with $r = 2^{-\mu} = l(Q)$ and $L = M_1 p$ (and so $L > \beta$), we get

$$J_2 \le c_{p,n,\beta}\, 2^{-(\nu-\mu)\tau p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} w_Q(Q).$$

By the Summation Lemma 5.4 (with $\mu > \nu$ in (5.6)), we have

$$J_1 \le c_{p,n} 2^{(\nu-\mu)(\sigma-n/p')p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \int_{\mathbb{R}^n} w_Q(x)(1 + 2^\nu|x-x_Q|)^{-M_1 p}\, dx,$$

again since $M_2 > n/p'$. Applying the Squeeze Lemma 5.3 again with $r = 2^{-\nu} > 2^{-\mu} = l(Q)$ and $L = M_1 p$, we get

$$J_1 \le c_{p,n,\beta}\, 2^{(\nu-\mu)(\sigma-n/p'-\beta/p)\,p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} w_Q(Q).$$

Observe that the last sum is equal to $\left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2}\vec{s}_Q\chi_Q \right\|_{L^p(W)}^p$. Combining the estimates for J_1 and J_2 (recall that $J = \frac{n}{p'} + \frac{\beta}{p}$), we have

$$2^{\nu\alpha} \left(\sum_{\mu>\nu} J_1^{1/p} + \sum_{\mu\le\nu} J_2^{1/p} \right) \le c_{p,n,\beta} \sum_{\mu\in\mathbb{Z}} 2^{(\nu-\mu)\alpha} \Big(2^{(\nu-\mu)(\sigma-J)} \chi_{\{\nu-\mu<0\}}$$

$$+ 2^{-(\nu-\mu)\tau} \chi_{\{\nu-\mu\ge0\}} \Big) \times 2^{\mu\alpha} \left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2}\vec{s}_Q\chi_Q \right\|_{L^p(W)}. \tag{5.8}$$

Denote

$$a_i = 2^{i\alpha}\left(2^{i(\sigma-J)}\chi_{\{i<0\}} + 2^{-i\tau}\chi_{\{i\geq 0\}}\right)$$

and

$$b_\mu = 2^{\mu\alpha}\left\|\sum_{l(Q)=2^{-\mu}} |Q|^{-1/2}\vec{s_Q}\chi_Q\right\|_{L^p(W)}.$$

Then the right side of (5.8) is nothing else but $c\sum_{\mu\in\mathbb{Z}} a_{\nu-\mu}\, b_\mu = c\,(a*b)(\nu)$. Substituting this into (5.7), we get

$$\left\|\sum_Q \vec{s}_Q\, m_Q\right\|_{\dot{B}_p^{\alpha q}(W)} \leq \left\|\left\{\sum_{\mu\in\mathbb{Z}}\sum_{i=1,2} J_i^{1/p}\right\}_\nu\right\|_{l_q^\alpha} \leq c_{p,n,\beta}\,\|a*b\|_{l^q}. \tag{5.9}$$

Observe that

$$\|a*b\|_{l^q} \leq \|a\|_{l^1}\|b\|_{l^q} \text{ for } q\geq 1 \tag{5.10}$$

and

$$\|a*b\|_{l^q} \leq \|a\|_{l^q}\|b\|_{l^q} \text{ for } q<1 \tag{5.11}$$

(to get the last inequality, apply the q-triangle inequality followed by $\|a*b\|_{l^1} \leq \|a\|_{l^1}\|b\|_{l^1}$). For any $0<q<\infty$, $\|a\|_{l^q}^q = \sum_{i<0} 2^{i(\sigma+\alpha-J)q} + \sum_{i\geq 0} 2^{-i(\tau-\alpha)q}$. Both sums converge, since $\tau>\alpha$ and $\sigma+\alpha>J$ by (5.2) and (5.3). Hence, $\|a\|_{l^q}\leq c_q$ for any $q>0$. (In fact, here we only need $0<q\leq 1$.) Combining all the estimates together into (5.9), we obtain

$$\begin{aligned}\left\|\sum_Q \vec{s}_Q\, m_Q\right\|_{\dot{B}_p^{\alpha q}(W)} &\leq c\,\|b\|_{l^q} = c\left\|\left\{2^{\nu\alpha}\left\|\sum_{l(Q)=2^{-\nu}} |Q|^{-1/2}\vec{s}_Q\chi_Q\right\|_{L^p(W)}\right\}_\nu\right\|_{l^q}\\ &= c\,\|\{\vec{s}_Q\}\|_{\dot{b}_p^{\alpha q}(W)},\end{aligned}$$

where $c = c_{n,p,q,\beta}$. □

Remark 5.5. Since $\psi\in\mathcal{A}$, observe the following properties of ψ_Q:

1. $0\notin \operatorname{supp}\hat{\psi}_Q$ for any dyadic Q, and, therefore, $\int x^\gamma\psi_Q(x)\,dx = 0$ for any multi-index γ;
2. $|D^\gamma\psi_Q| \leq c_{\gamma,L}|Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}}\left(1+\dfrac{|x-x_Q|}{l(Q)}\right)^{-L-|\gamma|}$ for each $L\in\mathbb{N}\cup\{0\}$ and γ as before.

Hence, $\{\psi_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, and for $\vec{f} = \sum_Q \vec{s}_Q\,\psi_Q$, we obtain the boundedness of the inverse φ-transform T_ψ:

Corollary 5.6. *Let W be a doubling matrix of order p, and consider the sequence $\vec{s} = \{\vec{s}_Q\}_Q \in \dot{b}_p^{\alpha q}(W)$. Then for all $1\leq p<\infty$, $0<q\leq\infty$ and $\alpha\in\mathbb{R}$,*

$$\|T_\psi\vec{s}\|_{\dot{B}_p^{\alpha q}(W)} = \left\|\sum_Q \vec{s}_Q\,\psi_Q\right\|_{\dot{B}_p^{\alpha q}(W)} \leq c\,\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \tag{5.12}$$

In particular, given $\vec{f} \in \dot{B}_p^{\alpha q}(W)$*, consider* $\vec{s} = S_\varphi \vec{f}$*. Then by* (2.2),

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} = \left\|\sum_Q \vec{s}_Q\, \psi_Q\right\|_{\dot{B}_p^{\alpha q}(W)} \leq c\, \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = c\, \left\|S_\varphi \vec{f}\right\|_{\dot{b}_p^{\alpha q}(W)}.$$

6. Boundedness of the φ-transform

Definition 6.1. For $\nu \in \mathbb{Z}$, let $E_\nu = \{\vec{f} : f_i \in \mathcal{S}' \text{ and } \operatorname{supp} \hat{f}_i \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{\nu+1}\}, i = 1, ..., m\}$. Then we say that E_ν consists of vector functions of exponential type $2^{\nu+1}$.

Consider the following lemma on the decomposition of an exponential type function (for the proof the reader is referred to [3], p. 55):

Lemma 6.2. *Suppose* $g \in \mathcal{S}'(\mathbb{R}^n), h \in \mathcal{S}(\mathbb{R}^n)$ *and* $\operatorname{supp} \hat{g}, \operatorname{supp} \hat{h} \subseteq \{|\xi| < 2^\nu \pi\}$ *for some* $\nu \in \mathbb{Z}$*. Then*

$$(g * h)(x) = \sum_{k \in \mathbb{Z}^n} 2^{-\nu n} g(2^{-\nu} k)\; h(x - 2^{-\nu} k). \tag{6.1}$$

Now we will develop two "maximal operator" type inequalities:

Lemma 6.3. *Let* $1 < p < \infty$*,* $W \in A_p$ *and* $\vec{g} \in E_0$*. Then*

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\, \vec{g}(k)\,\|^p\, dx \leq c_{p,n}\, \|\vec{g}\,\|^p_{L^p(W)}. \tag{6.2}$$

Remark 6.4. Note that in terms of reducing operators, (6.2) is equivalent to

$$\|\{A_{Q_{0k}}\, \vec{g}(k)\}_{k \in \mathbb{Z}^n}\|_{l^p} = \left(\sum_{k \in \mathbb{Z}^n} \|A_{Q_{0k}}\, \vec{g}(k)\,\|^p\right)^{1/p} \leq c_{p,n}\, \|\vec{g}\,\|_{L^p(W)}. \tag{6.3}$$

Proof. Choose a scalar-valued function $\gamma \in \mathcal{S}$ with $\hat{\gamma} = 1$ for $|\xi| \leq 2$ and $\operatorname{supp} \hat{\gamma} \subseteq \{|\xi| < \pi\}$. Then for $\vec{g} \in E_0$, we have $\vec{g} = \gamma * \vec{g}$, and the left-hand side of (6.2) is

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left\| W^{1/p}(x) \int_{\mathbb{R}^n} \vec{g}(y) \gamma(k - y)\, dy \right\|^p dx$$

$$\leq c_M \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left(\int_{\mathbb{R}^n} \frac{\|W^{1/p}(x) \vec{g}(y)\|}{(1 + |k - y|)^M}\, dy\right)^p dx,$$

for some $M > n + \beta p/p'$, where β is the doubling exponent of W, since $\gamma \in \mathcal{S}$. Since $\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^n} Q_{0m}$ and $m_i \leq y_i < m_i + 1$, $i = 1, ..., n$, on each Q_{0m}, the last sum is bounded by

$$c \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left(\sum_{m \in \mathbb{Z}^n} \frac{\int_{Q_{0m}} \|W^{1/p}(x) \vec{g}(y)\|\, dy}{(1 + |k - m|)^M}\right)^p dx.$$

Writing $M = M/p + M/p'$ and using the discrete Hölder inequality (note that $M > n$), we bound the last expression by

$$c \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \sum_{m \in \mathbb{Z}^n} \frac{\left(\int_{Q_{0m}} \|W^{1/p}(x) \vec{g}(y)\|\, dy\right)^p}{(1 + |k - m|)^M}\, dx. \tag{6.4}$$

Observe that

$$\left(\int_{Q_{0m}} \|W^{1/p}(x)\vec{g}(y)\|\,dy\right)^p \le \left(\int_{Q_{0m}} \|W^{1/p}(x)W^{-1/p}(y)\|\,\|W^{1/p}(y)\vec{g}(y)\|\,dy\right)^p$$

$$\le \left(\int_{Q_{0m}} \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}\,dy\right)^{p/p'} \left(\int_{Q_{0m}} \|W^{1/p}(y)\vec{g}(y)\|^p\,dy\right),$$

again by Hölder's inequality. By Lemma 4.3, $v_x(y) = \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}$ is a doubling measure with the doubling exponent β:

$$v_x(Q_{0m}) \le v_x(B(m, |k-m| + \sqrt{n})) \le c\,(1+|k-m|)^\beta v_x(Q_{0k}).$$

Thus, (6.4) is bounded by

$$(6.5)\quad c \sum_{k,m\in\mathbb{Z}^n} (1+|k-m|)^{\beta\frac{p}{p'}-M} \left[\int_{Q_{0k}} \left(\int_{Q_{0k}} \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}\,dy\right)^{\frac{p}{p'}} dx\right]$$

$$\times \int_{Q_{0m}} \|W^{1/p}(y)\vec{g}(y)\|^p\,dy.$$

By Lemma 1.3, the expression in the square brackets of (6.5) is bounded by a constant independent of k. Since $M > \beta p/p' + n$, the sum on k converges and, therefore, (6.5) is estimated above by

$$c \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \|W^{1/p}(y)\vec{g}(y)\|^p\,dy = c\int_{\mathbb{R}^n} \|W^{1/p}(y)\vec{g}(y)\|^p\,dy = c\,\|\vec{g}\,\|^p_{L^p(W)}.$$

□

Lemma 6.5. *Let W be a doubling matrix of order p with doubling exponent β such that $p > \beta$, and let $\vec{g} \in E_0$. Then* (6.2) *holds. Furthermore, if W is a diagonal matrix, then* (6.2) *holds for any $1 \le p < \infty$.*

Proof. First, assume $(\vec{g})_i \in \mathcal{S}$ with supp $\hat{\vec{g}}_i \subseteq \{|\xi| < \pi\}$, $\quad i = 1, ..., m$. We want to show that for such $\vec{g}$, the sum on the left-hand side of (6.2) is finite. Choosing $r > \beta + n$, we have

$$\sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\vec{g}(k)\|^p\,dx \le \sum_{k\in\mathbb{Z}^n} \frac{c}{(1+|k|)^r} \int_{Q_{0k}} \|W^{1/p}(x)\|^p\,dx.$$

Since $w(x) = \|W^{1/p}(x)\|^p$ is a scalar doubling measure, $w(Q_{0k}) \le c\,(1+|k|)^\beta w(Q_{00})$. Hence,

$$\sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\vec{g}(k)\|^p\,dx \le \sum_{k\in\mathbb{Z}^n} \frac{c\,w(Q_{00})}{(1+|k|)^{r-\beta}} \le c\,w(Q_{00}) < \infty,$$

since $r - \beta > n$.

Now we will prove (6.2) for $\vec{g}$ with $(\vec{g})_i \in S$ and supp $\hat{\vec{g}}_i \subseteq \{|\xi| \le 3\}$, and then generalize it to $(\vec{g})_i \in S'$. Let $0 < \delta < 1$. Then $B_\delta(k) \subseteq 3Q_{0k}$. Using the doubling property of $w_k(x) = \|W^{1/p}(x)\vec{g}(k)\|^p$, we "squeeze" each Q_{0k} into $B_\delta(k)$:

$$w_k(Q_{0k}) \le w_k(3Q_{0k}) \le c\left[\frac{|3Q_{0k}|}{|B_\delta(k)|}\right]^{\beta/n} w_k(B_\delta(k)) \le c_\beta\,\delta^{-\beta} w_k(B_\delta(k)).$$

Hence, the left-hand side of (6.2) is bounded by

$$c_\beta\, \delta^{-\beta} \sum_{k\in\mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(k)\|^p\, dx. \tag{6.6}$$

To estimate the integral, we will use the trivial identity $\vec{g}(k) = \vec{g}(x) + [\vec{g}(k) - \vec{g}(x)]$ for $x \in B_\delta(k)$. Apply the decomposition from Lemma 6.2 (γ is the same as in the previous lemma):

$$\vec{g}(k) = \sum_{m\in\mathbb{Z}^n} \vec{g}(m)\gamma(k-m) \quad \text{and} \quad \vec{g}(x) = \sum_{m\in\mathbb{Z}^n} \vec{g}(m)\gamma(x-m).$$

Using the Mean Value Theorem for $[\gamma(k-m) - \gamma(x-m)]$ and the properties of $\gamma \in \mathcal{S}$ (note that $|x-k| < \delta$), we have

$$\|W^{1/p}(x)\vec{g}(k)\|^p \le c_p\|W^{1/p}(x)\vec{g}(x)\|^p + c_{p,M}\, \delta^p \sum_{m\in\mathbb{Z}^n} \frac{\|W^{1/p}(x)\,\vec{g}(m)\,\|^p}{(1+|k-m|)^M}, \tag{6.7}$$

for some $M > \beta + n$. Integrating (6.7) over $B_\delta(k)$, we get

$$\int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(k)\|^p\, dx \le c_p \int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(x)\|^p\, dx$$

$$+\; c\,\delta^p \sum_{m\in\mathbb{Z}^n} \frac{\int_{B_\delta(k)} \|W^{1/p}(x)\,\vec{g}(m)\,\|^p\, dx}{(1+|k-m|)^M}. \tag{6.8}$$

Apply the doubling property of $w_m(x) = \|W^{1/p}(x)\,\vec{g}(m)\|^p$ again:

$$w_m(B_\delta(k)) \le w_m(B(m, |k-m|+\delta)) \le c\, \left[\frac{(\delta + |k-m|)^n}{\delta^n}\right]^{\beta/n} w_m(B_\delta(m))$$

$$= c\,\delta^{-\beta}(1+|k-m|)^\beta w_m(B_\delta(m)).$$

Substituting this estimate into (6.8) and summing over $k \in \mathbb{Z}^n$, we have

$$\sum_{k\in\mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(k)\|^p\, dx \le c_p \sum_{k\in\mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(x)\|^p\, dx$$

$$+\; c\,\delta^{p-\beta} \sum_{m\in\mathbb{Z}^n} \int_{B_\delta(m)} \|W^{1/p}(x)\,\vec{g}(m)\,\|^p\, dx \left(\sum_{k\in\mathbb{Z}^n} (1+|k-m|)^{\beta-M}\right),$$

where the last sum converges since $M > \beta + n$. If $p > \beta$, by choosing $0 < \delta < 1/2$ such that $1 - c\,\delta^{p-\beta} > 0$, we subtract the last term from both sides (note that it is finite because of our estimates above for $\vec{g}_i \in \mathcal{S}$), substitute it into (6.6) and get the estimate of the left-hand side of (6.2) (note that $\sum_{k\in\mathbb{Z}^n} \int_{B_\delta(k)} \dots \le \int_{\mathbb{R}^n} \dots$):

$$\sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\vec{g}(k)\|^p\, dx \le \frac{c_\beta\, \delta^{-\beta}\, c_p}{(1 - c\,\delta^{p-\beta})} \sum_{k\in\mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x)\vec{g}(x)\|^p\, dx$$

$$\le c_{n,\beta,p} \int_{\mathbb{R}^n} \|W^{1/p}(x)\vec{g}(x)\|^p\, dx = c_{n,\beta,p} \|\vec{g}\,\|^p_{L^p(W)}. \tag{6.9}$$

Now let $(\vec{g})_i \in S'$, $i = 1, ..., m$. Since $\vec{g} \in E_0$, it follows that $(\vec{g})_i \in C^\infty$, and $\vec{g}$ and all its derivatives are slowly increasing. Pick a scalar-valued $\gamma \in \mathcal{S}$ such that $\gamma(0) = 1$ and $\operatorname{supp}\hat{\gamma} \subseteq B(0,1)$. Then for $0 < \epsilon < 1$, the function $\vec{g}^{\,\epsilon}(x) := \vec{g}(x)\gamma(\epsilon x)$ has its components in $\mathcal{S}$. Observe that $\hat{\vec{g}}^{\epsilon} = \hat{\vec{g}} * [\gamma(\epsilon x)]\hat{}$, with $[\gamma(\epsilon x)]\hat{}(\xi) = (1/\varepsilon)\hat{\gamma}(\xi/\varepsilon)$, and, therefore,

$$\operatorname{supp}\hat{\vec{g}}^{\epsilon} \subseteq \operatorname{supp}\hat{\vec{g}} + \operatorname{supp}(1/\varepsilon)\hat{\gamma}(\xi/\varepsilon) \subseteq \{\xi : |\xi| < 3\}.$$

We can apply the result (6.9) to $\vec{g}^{\epsilon}$:

$$\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}} \|W^{1/p}(x)\vec{g}^{\epsilon}(k)\|^p\,dx \le c\,\|\vec{g}^{\epsilon}\|^p_{L^p(W)},$$

or

$$\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}} \|W^{1/p}(x)\vec{g}(k)\gamma(\epsilon k)\|^p\,dx \le c\int_{\mathbb{R}^n} \|W^{1/p}(x)\vec{g}(x)\|^p|\gamma(\epsilon x)|^p\,dx.$$

Taking $\liminf$ as $\epsilon\to 0$ of both sides and using Fatou's Lemma on the left-hand side (with a discrete measure for the sum) and the Dominated Convergence Theorem on the right-hand side, we obtain

$$\sum_{k\in\mathbb{Z}^n}\liminf_{\epsilon\to 0}|\gamma(\epsilon k)|^p \int_{Q_{0k}} \|W^{1/p}(x)\vec{g}(k)\|^p\,dx$$
$$\le c\int_{\mathbb{R}^n}\|W^{1/p}(x)\vec{g}(x)\|^p \lim_{\epsilon\to 0}|\gamma(\epsilon x)|^p\,dx.$$

Since $\gamma(\epsilon x) \underset{\epsilon\to 0}{\longrightarrow} \gamma(0)$, we obtain (6.2) for all $\vec{g}\in E_0$.

To get the second assertion of the lemma, we consider the scalar case with w a scalar doubling measure. Then (6.8) becomes

$$w(B_\delta(k))|g(k)|^p \le c_p\int_{B_\delta(k)} w(x)|g(x)|^p\,dx + c_p\,\delta^p w(B_\delta(k))\sum_{m\in\mathbb{Z}^n}\frac{|g(m)|^p}{(1+|k-m|)^M}, \tag{6.10}$$

or

$$|g(k)|^p \le c_p\frac{1}{w(B_\delta(k))}\int_{B_\delta(k)} w(x)|g(x)|^p\,dx + c\,\delta^p\sum_{m\in\mathbb{Z}^n}\frac{|g(m)|^p}{(1+|k-m|)^M}.$$

We want to estimate the last sum on m. Fix $l\in\mathbb{Z}^n$. Dividing everything by $(1+|k-l|)^M$ and summing on $k\in\mathbb{Z}^n$, we get

$$\sum_{k\in\mathbb{Z}^n}\frac{|g(k)|^p}{(1+|k-l|)^M} \le c_p\sum_{k\in\mathbb{Z}^n}\frac{\int_{B_\delta(k)} w(x)|g(x)|^p\,dx}{(1+|k-l|)^M w(B_\delta(k))}$$
$$+\,c\,\delta^p\sum_{k\in\mathbb{Z}^n}\frac{1}{(1+|k-l|)^M}\sum_{m\in\mathbb{Z}^n}\frac{|g(m)|^p}{(1+|k-m|)^M}.$$

Note that in the last term

$$\sum_{k\in\mathbb{Z}^n}\frac{1}{(1+|k-l|)^M(1+|k-m|)^M} \le \frac{c}{(1+|l-m|)^M},$$

since $M>n$. Therefore,

$$\sum_{k\in\mathbb{Z}^n}\frac{|g(k)|^p}{(1+|k-l|)^M} \le c_p\sum_{k\in\mathbb{Z}^n}\frac{\int_{B_\delta(k)} w(x)|g(x)|^p\,dx}{(1+|k-l|)^M w(B_\delta(k))} + c\,\delta^p\sum_{m\in\mathbb{Z}^n}\frac{|g(m)|^p}{(1+|l-m|)^M}.$$

Choose $0 < \delta < 1/2$ such that $1 - c\,\delta^p > 0$. Then

$$\sum_{m\in\mathbb{Z}^n} \frac{|g(m)|^p}{(1+|l-m|)^M} \le \frac{c_p}{1-c\,\delta^p} \sum_{m\in\mathbb{Z}^n} \frac{\int_{B_\delta(m)} w(x)|g(x)|^p\,dx}{(1+|l-m|)^M w(B_\delta(m))}.$$

Substituting this into (6.10) and summing on $k \in \mathbb{Z}^n$ (again using $\sum_{k\in\mathbb{Z}^n}\int_{B_\delta(k)} \dots \le \int_{\mathbb{R}^n} \dots$), we obtain

$$\sum_{k\in\mathbb{Z}^n} w(B_\delta(k))|g(k)|^p$$
$$\le c_p\,\|g\|^p_{L^p(w)} + \ c\,\delta^p \sum_{k\in\mathbb{Z}^n} w(B_\delta(k)) \sum_{m\in\mathbb{Z}^n} \frac{\int_{B_\delta(m)} w(x)|g(x)|^p\,dx}{(1+|k-m|)^M w(B_\delta(m))}.$$

Use the doubling property of w to shift $B_\delta(k)$ to $B_\delta(m)$. Since δ is fixed, $w(B_\delta(k)) \le c_{\delta,n}\,(1+|k-m|)^\beta w(B_\delta(m))$, and thus, the last term is dominated by

$$c\,\delta^p \sum_{m\in\mathbb{Z}^n} \int_{B_\delta(m)} w(x)|g(x)|^p\,dx \times \left(\sum_{k\in\mathbb{Z}^n} (1+|k-m|)^{\beta-M}\right), \tag{6.11}$$

where the sum on k converges, since $M > \beta + n$. Thus, (6.11) is estimated by $c_{p,n,\beta}\,\|g\|^p_{L^p(w)}$. Hence,

$$\sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} w(x)|g(k)|^p\,dx \le c_{p,n,\beta} \sum_{k\in\mathbb{Z}^n} w(B_\delta(k))|g(k)|^p \le c_{p,n,\beta}\,\|g\|^p_{L^p(w)}.$$

Now if W is a diagonal matrix, then

$$\|W^{1/p}(x)\,\vec{u}\,\|^p \approx \sum_{i=1}^m w_{ii}(x)\,|\vec{u}_i|^p,$$

and thus, applying the scalar case, we get

$$\sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\,\vec{g}(k)\|^p\,dx \approx \sum_{i=1}^m \sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} w_{ii}(x)|\vec{g}_i(k)|^p\,dx$$
$$\le \sum_{i=1}^m c\,\|\vec{g}_i\|^p_{L^p(w_{ii})} \approx c_{p,n,\beta,m}\,\|\vec{g}\|^p_{L^p(W)}.$$

□

Theorem 6.6. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p < \infty$, and let W satisfy any of (A1)-(A3). Then*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}^{\alpha q}_p(W)} \le c\,\|\vec{f}\,\|_{\dot{B}^{\alpha q}_p(W)}, \tag{6.12}$$

where $\vec{s}_Q = S_\varphi \vec{f} = \left\langle \vec{f}, \varphi_Q \right\rangle$ for a given $\vec{f}$.

Proof. By definition,

(6.13)

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}^{\alpha q}_p(W)} = \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-1/2} \left\| W^{1/p}\cdot\vec{s}_Q \right\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l^\alpha_q} =: \|\{J_\nu\}_\nu\|_{l^\alpha_q}\,.$$

Fix $\nu \in \mathbb{Z}$. Then $Q = Q_{\nu k} = \prod_{i=1}^{n} \left[\frac{k_i}{2^\nu}, \frac{k_i+1}{2^\nu}\right]$, $k \in \mathbb{Z}^n$, $|Q| = 2^{-\nu n}$, $\vec{s}_Q = |Q|^{1/2}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)$ and

$$\begin{aligned} J_\nu^p &= \sum_{l(Q)=2^{-\nu}} |Q|^{-p/2} \int_Q \|W^{1/p}(t)\vec{s}_Q\|^p \, dt \\ &= \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \|W^{1/p}(t)(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)\|^p \, dt. \end{aligned}$$

Let $\vec{f}_\nu(x) = \vec{f}(2^{-\nu}x)$. Then $(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k) = (\tilde{\varphi} * \vec{f}_\nu)(k)$. We substitute this in the last integral and note that the change of variables $y = 2^\nu t$ (with $W_\nu(t) := W(2^{-\nu}t)$) will yield

$$J_\nu^p = 2^{-\nu n} \sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \|W_\nu^{1/p}(t)(\tilde{\varphi} * \vec{f}_\nu)(k)\|^p \, dt. \tag{6.14}$$

Observe that $(\tilde{\varphi} * \vec{f}_\nu)_i \in \mathcal{S}'$, $i = 1, ..., m$, and $\tilde{\varphi} * \vec{f}_\nu \in E_0$, since supp $\hat{\tilde{\varphi}} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$. Using either Lemma 6.3 or Lemma 6.5 with $\vec{g} = \tilde{\varphi} * \vec{f}_\nu$ and W_ν instead of W (both the A_p condition and the doubling condition are invariant with respect to dilation), we obtain

$$J_\nu^p \le c\, 2^{-\nu n} \int_{\mathbb{R}^n} \|W_\nu^{1/p}(t)(\tilde{\varphi} * \vec{f}_\nu)(t)\|^p \, dt.$$

Changing variables, we get

$$J_\nu^p \le c \int_{\mathbb{R}^n} \|W^{1/p}(t)(\tilde{\varphi}_\nu * \vec{f})(t)\|^p \, dt = c\, \|(\tilde{\varphi}_\nu * \vec{f})\|^p_{L^p(W)}.$$

Combining the estimates of J_ν for all ν into (6.13), we get

$$\begin{aligned} \left\| \left\{ \left\langle \vec{f}, \varphi_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} &= \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \|\{J_\nu\}_\nu\|_{l_q^\alpha} \\ &\le c \left\| \left\{ \left\| \left(\tilde{\varphi}_\nu * \vec{f}\right) \right\|_{L^p(W)} \right\}_\nu \right\|_{l_q^\alpha} = c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W,\tilde{\varphi})}, \end{aligned} \tag{6.15}$$

where $c = c(p, \beta, n)$.

To finish the proof of the theorem, we have to establish the equivalence between $\dot{B}_p^{\alpha q}(W, \varphi)$ and $\dot{B}_p^{\alpha q}(W, \tilde{\varphi})$. As we mentioned in Section 2, $\tilde{\varphi} \in \mathcal{A}$, and so the pair $(\tilde{\varphi}, \tilde{\psi})$ satisfies (2.1), since $\hat{\tilde{\varphi}} = \bar{\hat{\varphi}}$ and $\hat{\tilde{\psi}} = \bar{\hat{\psi}}$. By (2.2), $\vec{f} = \sum_Q \left\langle \vec{f}, \tilde{\varphi}_Q \right\rangle \tilde{\psi}_Q$. Since $\{\tilde{\psi}_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ (see Remark 5.5), by Theorem 5.2 we have

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W,\tilde{\varphi})} \le c \left\| \left\{ \left\langle \vec{f}, \tilde{\varphi}_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)}. \tag{6.16}$$

Applying (6.15) to the right-hand side of the last inequality, we bound it by

$$c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W,\tilde{\tilde{\varphi}})} = c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W,\varphi)}. \tag{6.17}$$

Finally, combining (6.15) with (6.16) and (6.17), we obtain

$$\left\| \left\{ \left\langle \vec{f}, \varphi_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \equiv \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W,\varphi)}.$$

□

Remark 6.7. The fact that φ and $\tilde{\varphi}$ were interchanged in the last step of the previous theorem can be generalized into Theorem 1.8 about the independence of the space $\dot{B}_p^{\alpha q}(W)$ from the choice of φ:

Proof of Theorem 1.8. Let $\{\varphi^{(1)}, \psi^{(1)}\}$ and $\{\varphi^{(2)}, \psi^{(2)}\}$ be two different sets of mutually admissible kernels. Decompose $\vec{f}$ in the second system:

$$\vec{f} = \sum_Q \left\langle \vec{f}, \varphi_Q^{(2)} \right\rangle \psi_Q^{(2)} = \sum_Q \vec{s}_Q^{\,(2)} \psi_Q^{(2)}.$$

Observe that $\psi_Q^{(2)}$ is a molecule for Q and, therefore, by Theorem 5.2,

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W,\varphi^{(1)})} \leq c \, \|\{\vec{s}_Q^{\,(2)}\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \, \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W,\varphi^{(2)})},$$

where the last inequality holds by Theorem 6.6. Interchanging $\varphi^{(1)}$ with $\varphi^{(2)}$, we get the norm equivalence between $\dot{B}_p^{\alpha q}(W, \varphi^{(1)})$ and $\dot{B}_p^{\alpha q}(W, \varphi^{(2)})$. In other words, the space $\dot{B}_p^{\alpha q}(W)$ is independent of the choice of φ under any of the three assumptions on W. □

Remark 6.8. Combining boundedness of the φ-transform (Theorem 6.6) and that of the inverse φ-transform (Corollary 5.6), we get the norm equivalence claimed in Theorems 1.4 and 1.6.

7. Connection with reducing operators

Now we connect the weighted sequence Besov space with its reducing operator equivalent. Recall that for each matrix weight W, we can find a sequence of reducing operators $\{A_Q\}_Q$ such that

$$\rho_{p,Q}(\vec{u}) = \left(\frac{1}{|Q|} \int \left\| W^{1/p}(t) \cdot \vec{u} \right\|_{\mathcal{H}}^p \chi_Q(t)\, dt \right)^{1/p} \approx \|A_Q \vec{u}\|_{\mathcal{H}}, \text{ for all } \vec{u} \in \mathcal{H}. \tag{7.1}$$

Lemma 7.1. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let $\{A_Q\}_Q$ be reducing operators for W. Then*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}. \tag{7.2}$$

Proof. Using (7.1), we get the equivalence

$$
\begin{aligned}
\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} &= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \|W^{1/p}\cdot \vec{s}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\
&= \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{p}{2}} \left[\rho_{p,Q}(\vec{s}_Q)\right]^p |Q| \right)^{\frac{1}{p}} \right\}_\nu \right\|_{l_q^\alpha} \\
&\approx \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{p}{2}} \|A_Q \vec{s}_Q\|_{\mathcal{H}}^p \int \chi_Q(t)\,dt \right)^{\frac{1}{p}} \right\}_\nu \right\|_{l_q^\alpha} \\
&= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \|A_Q \vec{s}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\
&= \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.
\end{aligned}
$$

□

Finally, combining Theorems 1.4 and 1.6 with (7.2), we get Theorem 1.9.

Corollary 7.2. *The space $\dot{B}_p^{\alpha q}(W)$ is complete when $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p < \infty$ and W satisfies any of (A1)-(A3).*

Proof. If $\left\{\vec{f}_n\right\}_{n\in\mathbb{N}}$ is Cauchy in $\dot{B}_p^{\alpha q}(W)$, then $\left\{\left\{\vec{s}_Q\left(\vec{f}_n\right)\right\}_Q\right\}_{n\in\mathbb{N}}$ is Cauchy in $\dot{b}_p^{\alpha q}(\{A_Q\})$ by Theorem 6.6 and Lemma 7.1 (or just Theorem 1.9). This implies that

$$
\begin{aligned}
&\left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \left\| A_Q \left[\vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q\left(\vec{f}_m\right) \right] \right\|_{\mathcal{H}} \chi_Q \right\|_{L^p}^p \\
&= 2^{\nu n(p/2-1)} \sum_{l(Q)=2^{-\nu}} \left\| A_Q \left[\vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q\left(\vec{f}_m\right) \right] \right\|_{\mathcal{H}}^p \underset{n,m\to\infty}{\longrightarrow} 0, \text{ for each } \nu \in \mathbb{Z}.
\end{aligned}
$$

Hence, $\left\| A_Q \left[\vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q\left(\vec{f}_m\right) \right] \right\|_{\mathcal{H}} \underset{n,m\to\infty}{\longrightarrow} 0$ for each Q. Since the A_Q's are invertible, $\left\{\vec{s}_Q\left(\vec{f}_n\right)\right\}_{n\in\mathbb{N}}$ is a vector-valued Cauchy sequence in $\mathcal{H}$ for each Q. Therefore, we can define $\vec{s}_Q = \lim_{n\to\infty} \vec{s}_Q(\vec{f}_n)$. Set $\vec{f} = \sum_Q \vec{s}_Q\,\psi_Q$. Observe that

$$
\begin{aligned}
\left\|\vec{f}_n - \vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} &= \left\| \sum_Q \left[\vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q \right] \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \\
&\le c \left\| \left\{ \vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \\
&\le c \liminf_{m\to\infty} \left\| \left\{ \vec{s}_Q\left(\vec{f}_n\right) - \vec{s}_Q\left(\vec{f}_m\right) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

by Corollary 5.6 and Lemma 7.1, the discrete version of Fatou's Lemma and the fact that $\{\{\vec{s}_Q(\vec{f}_n)\}_Q\}_{n\in\mathbb{N}}$ is Cauchy in $\dot{b}_p^{\alpha q}(\{A_Q\})$. Furthermore, $\vec{f} = (\vec{f} - \vec{f}_n) + \vec{f}_n \in \dot{B}_p^{\alpha q}(W)$. Thus, $\dot{B}_p^{\alpha q}(W)$ is complete. □

Recall the A_p condition in terms of reducing operators: $\|A_Q A_Q^{\#}\| \le c$ for any cube $Q \in \mathbb{R}^n$; in other words, $\|A_Q\, y\| \le c\,\|(A_Q^{\#})^{-1}\, y\|$ holds for any $y \in \mathcal{H}$. Note that the inverse inequality $\|(A_Q\, A_Q^{\#})^{-1}\| \le c$ (or, equivalently, $\|(A_Q^{\#})^{-1}\, y\| \le c\,\|A_Q\, y\|$ for any $y \in \mathcal{H}$) holds automatically (a straightforward application of Hölder's inequality). This implies the following imbeddings of the sequence Besov spaces:

Corollary 7.3. *For $\alpha \in \mathbb{R}$, $1 < p < \infty$, $0 < q \le \infty$, and W a matrix weight with corresponding reducing operators A_Q and $A_Q^{\#}$,*

1. $\dot{b}_p^{\alpha q}(\{A_Q\}) \subseteq \dot{b}_p^{\alpha q}(\{(A_Q^{\#})^{-1}\})$ *always, and*
2. $\dot{b}_p^{\alpha q}(\{(A_Q^{\#})^{-1}\}) \subseteq \dot{b}_p^{\alpha q}(\{A_Q\})$ *if $W \in A_p$.*

8. Almost diagonal operators

Consider $\dot{b}_p^{\alpha q}(W)$ with parameters α, p, q fixed ($\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$), and W a doubling matrix of order p with doubling exponent β. If $p = 1$, then $1/p' = 0$.

Definition 8.1. A matrix $A = (a_{QP})_{Q,P\,\text{dyadic}}$ is *almost diagonal*, $A \in \mathbf{ad}_p^{\alpha q}(\beta)$, if there exist $M > J = \frac{n}{p'} + \frac{\beta}{p}$ and $c > 0$ such that for all Q, P,

$$(8.1)\qquad |a_{QP}| \le c\,\min\left(\left[\frac{l(Q)}{l(P)}\right]^{\alpha_1}, \left[\frac{l(P)}{l(Q)}\right]^{\alpha_2}\right)\left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))}\right)^{-M},$$

with $\alpha_1 > \alpha + \frac{n}{2}$ and $\alpha_2 > J - (\alpha + \frac{n}{2})$.

Remark 8.2. This definition differs from the definition of almost diagonality in [3], since both α_2 and M depend on the doubling exponent β.

To simplify notation for the matrix A above, we will only write (a_{QP}) without specifying indices Q, P.

Example 8.3 (An almost diagonal matrix). Let $\varphi \in \mathcal{A}$. If $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, then

$$(8.2)\qquad (a_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta),$$

where $a_{QP} = \langle m_P, \varphi_Q\rangle$, by (5.2) and (5.3), since $\langle m_P, \varphi_Q\rangle = |Q|^{1/2}(\tilde{\varphi}_\nu * m_P)(x_Q)$ if $l(Q) = 2^{-\nu}$.

Now we show that almost diagonal matrices are bounded on $\dot{b}_p^{\alpha q}(W)$, i.e., Theorem 1.10. First we need the following approximation lemma, whose proof is trivial:

Lemma 8.4. *Let P, Q be dyadic cubes and $t \in Q$. Then*

$$1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))} \underset{(n)}{\approx} 1 + \frac{|t - x_P|}{\max(l(Q), l(P))}. \tag{8.3}$$

Proof of Theorem 1.10. Let $A = (a_{QP})$ with $A \in \mathbf{ad}_p^{\alpha q}(\beta)$. We want to show that

$$\left\| \left\{ \sum_P a_{QP} \vec{s}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \leq c_{n,p,q,\beta} \, \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \tag{8.4}$$

By definition,

$$\begin{aligned} &\left\| \left\{ \sum_P a_{QP} \vec{s}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \\ &\quad \leq \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-p/2} \int_Q \left(\sum_P |a_{QP}| \|W^{1/p}(t) \vec{s}_P\| \right)^p dt \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha} \\ &\quad =: \left\| \left\{ 2^{\nu\alpha} \, 2^{\nu n/2} \left(\sum_{l(Q)=2^{-\nu}} J_Q \right)^{1/p} \right\}_\nu \right\|_{l^q}. \end{aligned} \tag{8.5}$$

Substituting the estimate (8.1) for a_{QP} in J_Q, we get

$$J_Q \leq c_{p,M} \int_Q \left(\sum_{j \geq 0} 2^{-j\alpha_2} \sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| \left(1 + 2^\nu |x_Q - x_P|\right)^{-M} \right)^p dt$$

$$+ \, c_{p,M} \int_Q \left(\sum_{j<0} 2^{j\alpha_1} \sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| \left(1 + 2^{(\nu+j)} |x_Q - x_P|\right)^{-M} \right)^p dt.$$

Pick $\epsilon > 0$ sufficiently small such that (i) $\alpha_1 - \epsilon > \alpha + n/2$, (ii) $\alpha_2 - \epsilon > J - \alpha - n/2$ and (iii) $M > \beta/p + (n+\epsilon)/p'$. Apply the discrete Hölder inequality twice, first with $\alpha_i = \epsilon + (\alpha_i - \epsilon)$ for the sum on j (note that $\alpha_1, \alpha_2 > 0$) and second with

$M = \frac{n+\epsilon}{p'} + \left(M - \frac{n+\epsilon}{p'}\right)$ for the sum on P:

$$\begin{aligned}
J_Q \le{}& c_{p,M} \int_Q \left(\sum_{j\ge 0} 2^{-j\epsilon p'}\right)^{p/p'} \Bigg[\sum_{j\ge 0} 2^{-j(\alpha_2-\epsilon)p} \\
&\times \left(\sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t)\vec{s}_P\| \,(1+2^{\nu}|x_Q-x_P|)^{-M}\right)^{p}\Bigg]\, dt \\
&+ c_{p,M} \int_Q \left(\sum_{j<0} 2^{j\epsilon p'}\right)^{p/p'} \Bigg[\sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \\
&\times \left(\sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t)\vec{s}_P\| \left(1+2^{(\nu+j)}|x_Q-x_P|\right)^{-M}\right)^{p}\Bigg]\, dt \\
\le{}& c_{p,M,\epsilon} \sum_{j\ge 0} 2^{-j(\alpha_2-\epsilon)p} \left[\sum_{l(P)=2^{-(\nu+j)}} (1+2^{\nu}|x_Q-x_P|)^{-n-\epsilon}\right]^{p/p'} \\
&\times \sum_{l(P)=2^{-(\nu+j)}} \int_Q \|W^{1/p}(t)\vec{s}_P\|^p \,(1+2^{\nu}|x_Q-x_P|)^{-(M-\frac{n+\epsilon}{p'})p}\, dt \\
&+ c_{p,M,\epsilon} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \left[\sum_{l(P)=2^{-(\nu+j)}} \left(1+2^{(\nu+j)}|x_Q-x_P|\right)^{-n-\epsilon}\right]^{p/p'} \\
&\times \sum_{l(P)=2^{-(\nu+j)}} \int_Q \|W^{1/p}(t)\vec{s}_P\|^p \left(1+2^{(\nu+j)}|x_Q-x_P|\right)^{-(M-\frac{n+\epsilon}{p'})p}\, dt.
\end{aligned}$$

Use the Summation Lemma 5.4 to estimate the square brackets and denote $w_P(t) = \|W^{1/p}(t)\vec{s}_P\|^p$. By Lemma 8.4, x_Q can be replaced by any $t \in Q$, and so we get

$$\begin{aligned}
J_Q \le{}& c_{p,M} \sum_{j\ge 0} 2^{-j(\alpha_2-\epsilon)p + jnp/p'} \\
&\times \sum_{l(P)=2^{-(\nu+j)}} \int_Q w_P(t) \left(1+\frac{|t-x_P|}{l(Q)}\right)^{-(M-\frac{n+\epsilon}{p'})p} dt \\
&+ c_{p,M} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \sum_{l(P)=2^{-(\nu+j)}} \int_Q w_P(t) \left(1+\frac{|t-x_P|}{l(P)}\right)^{-(M-\frac{n+\epsilon}{p'})p} dt.
\end{aligned}$$

Summing on Q and applying the Squeeze Lemma 5.3 (recall $M > \beta/p + (n+\epsilon)/p'$), we get

$$\begin{aligned}
\sum_{l(Q)=2^{-\nu}} J_Q &\le c_{p,n} \sum_{j\ge 0} 2^{-j(\alpha_2-\epsilon)p+jnp/p'} \\
&\times \sum_{l(P)=2^{-(\nu+j)}} \sum_{l(Q)=2^{-\nu}} \int_Q w_P(t)\ (1+2^{\nu}|t-x_P|)^{-(M-\frac{n+\epsilon}{p'})p}\ dt \\
&+ c_{p,n} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \sum_{l(P)=2^{-(\nu+j)}} \sum_{l(Q)=2^{-\nu}} \int_Q w_P(t) \left(1+2^{\nu+j}|t-x_P|\right)^{-(M-\frac{n+\epsilon}{p'})p} dt \\
&\le c_{p,n,\beta} \sum_{j\in\mathbb{Z}} \left(2^{-j(\alpha_2-\epsilon)p+jnp/p'+j\beta}\chi_{\{j\ge 0\}} + 2^{j(\alpha_1-\epsilon)p}\chi_{\{j<0\}}\right) \sum_{l(P)=2^{-(\nu+j)}} w_P(P).
\end{aligned}$$

Observe that $2^{\nu np/2} = |P|^{-p/2}2^{-jnp/2}$ for $l(P) = 2^{-(\nu+j)}$, and

$$\sum_{l(P)=2^{-(\nu+j)}} |P|^{-p/2} w_P(P) = \left\| \sum_{l(P)=2^{-(\nu+j)}} |P|^{-1/2}\vec{s}_P\,\chi_P \right\|^p_{L^p(W)}.$$

Then, using $1<p<\infty$ to take the power $1/p$ inside the sum on j, we get

$$\begin{aligned}
2^{\nu\alpha}\,2^{\nu n/2}\left(\sum_{l(Q)=2^{-\nu}} J_Q\right)^{1/p} &\le c \sum_{j\in\mathbb{Z}} \Big[2^{-j\alpha}2^{-jn/2}\Big(2^{-j(\alpha_2-\epsilon)p+jnp/p'+j\beta}\chi_{\{j\ge 0\}} \\
&+ 2^{j(\alpha_1-\epsilon)p}\chi_{\{j<0\}}\Big)^{1/p}\Big] \times 2^{(\nu+j)\alpha} \left\| \sum_{l(P)=2^{-(\nu+j)}} |P|^{-1/2}\vec{s}_P\chi_P \right\|_{L^p(W)} \\
&=: c\sum_{j\in\mathbb{Z}} a_{-j}\times b_{\nu+j} = c\,(a*b)(\nu).
\end{aligned}$$

Use (5.10) and (5.11) to estimate the norm of the convolution $\|a*b\|_{l^q}$. Then for $q\le 1$,

$$\|a\|^q_{l^q} = \sum_{j\le 0} 2^{j(\alpha+n/2+(\alpha_2-\epsilon)-J)q} + \sum_{j>0} 2^{j(\alpha+n/2-(\alpha_1-\epsilon))q} \le c_q,$$

since $\alpha_1-\epsilon > \alpha+n/2$ and $\alpha_2-\epsilon > J-(\alpha+n/2)$. Using the $\|a\|_{l^1}$ estimate for $q\ge 1$, and the $\|a\|_{l^q}$ estimate for $q<1$, and substituting into (8), we obtain

$$\begin{aligned}
\left\| \left\{ \sum_P a_{QP}\vec{s}_P \right\}_Q \right\|_{\dot{b}^{\alpha q}_p(W)} &\le c\,\|b\|_{l^q} \\
&= c\left\| \left\{ 2^{\mu\alpha} \left\| \sum_{l(P)=2^{-\mu}} |P|^{-1/2}\vec{s}_P\chi_P \right\|_{L^p(W)} \right\}_\mu \right\|_{l^q} \\
&= c\,\|\{\vec{s}_P\}_P\|_{\dot{b}^{\alpha q}_p(W)},
\end{aligned}$$

where $c = c_{n,p,q,\beta}$. □

Now we will show that the class of almost diagonal matrices is closed under composition. For $\epsilon > 0, \delta > 0, J = \frac{n}{p'} + \frac{\beta}{p}$ and P, Q dyadic, denote

$$w_{QP}(\delta,\epsilon) = \left[\frac{l(Q)}{l(P)}\right]^{\alpha+\frac{n}{2}} \min\left(\left[\frac{l(Q)}{l(P)}\right]^{\frac{\epsilon}{2}}, \left[\frac{l(P)}{l(Q)}\right]^{\frac{\epsilon}{2}+J}\right)\left(1+\frac{|x_Q-x_P|}{\max(l(Q),l(P))}\right)^{-J-\delta}.$$

Theorem 8.5. *Let $A, B \in \mathbf{ad}_p^{\alpha q}(\beta)$. Then $A \circ B \in \mathbf{ad}_p^{\alpha q}(\beta)$.*

We need the following lemma, which is a modification of Theorem D.2 in [5] adjusted to the weighted **ad** condition:

Lemma 8.6. *Let $\delta, \gamma_1, \gamma_2 > 0$, $\gamma_1 \neq \gamma_2$, and $2\delta < \gamma_1 + \gamma_2$. Then there exists a constant $c = c_{n,\delta,\gamma_1,\gamma_2,J}$ such that*

$$\sum_R w_{QR}(\delta,\gamma_1)\, w_{RP}(\delta,\gamma_2) \le c\, w_{QP}(\delta, \min(\gamma_1,\gamma_2)). \tag{8.6}$$

Proof of Theorem 8.5. Since $A = (a_{QP}), B = (b_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$, for each $i = A, B$ there exist $0 < \epsilon_i < \min(\alpha_1 - (\alpha + n/2), \alpha_2 - J + \alpha + n/2)$ and $0 < \delta < M - J$ such that $|a_{QP}| \le c\, w_{QP}(\delta, \epsilon_A)$ and $|b_{QP}| \le c\, w_{QP}(\delta,\epsilon_B)$. Without loss of generality, we may assume $\epsilon_A < \epsilon_B$ and $\delta < \dfrac{\epsilon_A + \epsilon_B}{2}$. Then

$$|(AB)_{QP}| \le |\sum_R a_{QR}\, b_{RP}| \le c \sum_R w_{QR}(\delta,\epsilon_A)\, w_{RP}(\delta,\epsilon_B) \le c\, w_{QP}(\delta,\epsilon_A),$$

by Lemma 8.6, which means that $A \circ B \in \mathbf{ad}_p^{\alpha q}(\beta)$. □

Definition 8.7. Let T be a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}'$. We say that T is an *almost diagonal operator* for $\dot{B}_p^{\alpha q}(W)$, and write $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, if for some pair of mutually admissible kernels (φ, ψ), the matrix $(a_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$, where $a_{QP} = \langle T\psi_P, \varphi_Q\rangle$.

Remark 8.8. The definition of $T \in \mathbf{AD}_p^{\alpha q}(\beta)$ is independent of the choice of the pair (φ, ψ).

Proof. Define $\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \notin \operatorname{supp} \hat{f}\}$. Observe that $\psi \in \mathcal{A}$ implies $\psi, \psi_\nu, \psi_Q \in \mathcal{S}_0$ for $\nu \in \mathbb{Z}$ and Q dyadic. Moreover, if $g \in \mathcal{S}_0$, then both

$$g_N := \sum_{\nu=-N}^{N} \sum_{k\in\mathbb{Z}^n} s_{Q_{\nu k}}(g)\psi_{Q_{\nu k}} \underset{N\to\infty}{\longrightarrow} g$$

and $\sum_{|k|\le M} s_{Q_{\nu k}}(g)\psi_{Q_{\nu k}} \underset{M\to\infty}{\longrightarrow} \sum_{k\in\mathbb{Z}^n} s_{Q_{\nu k}}(g)\psi_{Q_{\nu k}}$ in the $\mathcal{S}$-topology (see [13, Appendix]). Since T is continuous from $\mathcal{S}$ into $\mathcal{S}'$, we have $Tg = \sum_Q s_Q(g)\, T\psi_Q$. Now, suppose $(\langle T\psi_P, \varphi_Q\rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$ for some fixed pair (φ,ψ) of mutually admissible kernels. Take any other such pair $(\tilde{\varphi}, \tilde{\psi})$. Then $\tilde{\psi}_P = \sum_L \left\langle \tilde{\psi}_P, \varphi_L\right\rangle \psi_L$ and $\tilde{\varphi}_Q = \sum_R \langle\tilde{\varphi}_Q, \psi_R\rangle\, \varphi_R$, which gives

$$\left\langle T\tilde{\psi}_P, \tilde{\varphi}_Q\right\rangle = \sum_{R,L} \left\langle \tilde{\psi}_P, \varphi_L\right\rangle \langle T\psi_L, \varphi_R\rangle\, \overline{\langle\tilde{\varphi}_Q, \psi_R\rangle}.$$

Since both $\{\psi_R\}_R$ and $\{\varphi_L\}_L$ constitute families of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, by (8.2) the matrices $\left(\left\langle \tilde{\psi}_P, \varphi_L \right\rangle_{LP}\right)$, $(\langle \tilde{\varphi}_Q, \psi_R\rangle_{QR}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. By Theorem 8.5, $\left(\left\langle T\tilde{\psi}_P, \tilde{\varphi}_Q \right\rangle_{QP}\right) \in \mathbf{ad}_p^{\alpha q}(\beta)$. ☐

A straightforward consequence of Theorem 1.10 is the following statement:

Corollary 8.9. *Let $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q < \infty$. Then T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$ if W satisfies any of (A1)-(A3).*

Proof. First, consider $\vec{f}$ with $\left(\vec{f}\right)_i \in \mathcal{S}_0$. Let (φ, ψ) be a pair of mutually admissible kernels. Denote $\vec{t}_Q = \sum_P \langle T\psi_P, \varphi_Q\rangle\, \vec{s}_P(\vec{f}\,)$ and observe that $(\langle T\psi_P, \varphi_Q\rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. Using the φ-transform decomposition $\vec{f} = \sum_P \vec{s}_P(\vec{f}\,)\,\psi_P$ and taking T inside the sum as in the previous remark, we get

$$\begin{aligned}
\|T\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} &= \Big\|\sum_P \vec{s}_P(\vec{f})T\psi_P\Big\|_{\dot{B}_p^{\alpha q}(W)} \\
&= \Big\|\sum_Q\Big(\sum_P \langle T\psi_P, \varphi_Q\rangle\, \vec{s}_P(\vec{f}\,)\Big)\psi_Q\Big\|_{\dot{B}_p^{\alpha q}(W)} \\
&= \|\sum_Q \vec{t}_Q\psi_Q\|_{\dot{B}_p^{\alpha q}(W)} \le c\,\|\{\vec{t}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \\
&\le c\,\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \le c\,\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)},
\end{aligned}$$

by Corollary 5.6, Theorem 1.10 and Theorem 6.6.

Note that $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(W)$ (since $q < \infty$) and W satisfies any of (A1)-(A3) (this follows from Corollary 5.6, Theorem 6.6 and the fact that the tail of a convergent series goes to zero, see [13, Appendix]). Thus, T extends to all of $\dot{B}_p^{\alpha q}(W)$. ☐

Note that if $q = \infty$, then T extends to a bounded operator on the closure of $\mathcal{S}_0$ in $\dot{B}_p^{\alpha\infty}(W)$.

Remark 8.10. Let $\{m_Q\}_Q$ be a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$. Apply the φ-transform to $\sum_P \vec{s}_P\, m_P$:

$$\vec{t}_Q := S_\varphi\left(\sum_P \vec{s}_P\, m_P\right) = \left\langle \sum_P \vec{s}_P\, m_P, \varphi_Q\right\rangle = \sum_P \langle m_P, \varphi_Q\rangle\ \vec{s}_P.$$

Then $\left(\langle m_P, \varphi_Q\rangle_{QP}\right)$ forms an almost diagonal matrix by (8.2), and therefore, by Theorem 1.10,

$$\|\{\vec{t}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \equiv \|S_\varphi(\sum_P \vec{s}_P\, m_P)\|_{\dot{b}_p^{\alpha q}(W)} \le c\,\|\{\vec{s}_P\}_P\|_{\dot{b}_p^{\alpha q}(W)}$$

if W is doubling.

Corollary 8.11. *Let $T, S \in \mathbf{AD}_p^{\alpha q}(\beta)$. Then $T \circ S \in \mathbf{AD}_p^{\alpha q}(\beta)$.*

Proof. Since $T, S \in \mathbf{AD}_p^{\alpha q}(\beta)$, it follows that $(t_{QP}) := (\langle T\psi_P, \varphi_Q\rangle_{QP})$ is in $\mathbf{ad}_p^{\alpha q}(\beta)$, and so is $(s_{QP}) := (\langle S\psi_P, \varphi_Q\rangle_{QP})$. Thus, for Q, P dyadic we have $S\psi_P = \sum_R \langle S\psi_P, \varphi_R\rangle \psi_R$, and so

$$\langle T \circ S\psi_P, \varphi_Q\rangle = \sum_R \langle S\psi_P, \varphi_R\rangle \langle T\psi_R, \varphi_Q\rangle = \sum_R t_{QR}\, s_{RP} \in \mathbf{ad}_p^{\alpha q}(\beta),$$

by Theorem 8.5 (composition of almost diagonal matrices). □

9. Calderón-Zygmund operators

In this section we show that Calderón-Zygmund operators (CZOs) are bounded on $\dot{B}_p^{\alpha q}(W)$ for certain parameters α, p, q, β. First we recall the definition of smooth atoms and the fact that a CZO maps smooth atoms into smooth molecules. Then we use a general criterion for boundedness of operators: if an operator T maps smooth atoms into molecules, then its matrix $(\langle T\psi_P, \varphi_Q\rangle_{QP})$ forms an almost diagonal operator on $\dot{b}_p^{\alpha q}(W)$, and therefore, T is bounded on $\dot{B}_p^{\alpha q}(W)$.

Definition 9.1. Let $N \in \mathbb{N} \cup \{0\}$. A function $a_Q \in \mathcal{D}(\mathbb{R}^n)$ is a *smooth N-atom* for Q if

1. supp $a_Q \subseteq 3Q$,
2. $\int x^\gamma a_Q(x)\, dx = 0$ for $|\gamma| \le N$, and
3. $|D^\gamma a_Q(x)| \le c_\gamma\, l(Q)^{-|\gamma|-n/2}$ for all $|\gamma| \ge 0$.

Let $0 < \delta \le 1$, $M > 0$, $N \in \mathbb{N} \cup \{0, -1\}$, $N_0 \in \mathbb{N} \cup \{0\}$.

Lemma 9.2 (Boundedness criterion)**.** *Suppose a continuous linear operator $T : \mathcal{S} \to \mathcal{S}'$ maps any smooth N_0-atom into a fixed multiple of a smooth (δ, M, N)-molecule for $\dot{B}_p^{\alpha q}(W)$, $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$ with δ, M, N satisfying (M.i), (M.ii) and (M.iii). Suppose W satisfies any of (A1)-(A3). Then $T \in \mathbf{AD}_p^{\alpha q}(\beta)$ and, if $q < \infty$, T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$.*

Proof. By Corollary 8.9, it suffices to show that $(\langle T\psi_P, \varphi_Q\rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$ for some $\varphi, \psi \in \mathcal{A}$ satisfying (2.1). Observe that if $\psi \in \mathcal{A}$, then there exists $\theta \in \mathcal{S}$ with supp $\theta \subseteq B_1(0)$, $\int x^\gamma \theta(x)\, dx = 0$, if $|\gamma| \le N_0$, and $\sum_{\nu\in\mathbb{Z}} \hat{\theta}(2^{-\nu}\xi)\hat{\varphi}(2^{-\nu}\xi) = 1$ for $\xi \ne 0$ ([3], Lemma 5.12). Using $\psi_P = \sum_{\nu\in\mathbb{Z}} \theta_\nu * \varphi_\nu * \psi_P$ as in the atomic decomposition theorem ([3], Thm. 5.11), we have

$$\psi_P(x) = \sum_Q t_{QP} a_Q^{(P)}(x) \tag{9.1}$$

with $t_{QP} = |Q|^{1/2} \sup_{y\in Q} |(\varphi_\nu * \psi_P)(y)|$ for $l(Q) = 2^{-\nu}$, and each $a_Q^{(P)}$ is an N_0-atom defined by

$$a_Q^{(P)}(x) = \frac{1}{t_{QP}} \int_Q \theta_\nu(x-y)\, (\varphi_\nu * \psi_P)(y)\, dy \quad \text{if } t_{QP} \ne 0 \tag{9.2}$$

and $a_Q^{(P)} = 0$ if $t_{QP} = 0$. Using (5.2)-(5.3) (valid because $\{\psi_P\}_P$ is a family of molecules for $\dot{B}_p^{\alpha q}(W)$), we get

$$|(\varphi_\nu * \psi_P)(y)| \le c\,|P|^{-1/2} \min\left(\left[\frac{l(Q)}{l(P)}\right]^\tau, \left[\frac{l(P)}{l(Q)}\right]^\sigma\right)\left(1 + \frac{|y - x_P|}{\max(l(Q), l(P))}\right)^{-M},$$

for some $\tau > \alpha$ and $\sigma > J - \alpha$. In fact, $\varphi_\nu * \psi_P = 0$ if $|\mu - \nu| > 1$ ($2^{-\mu} = l(P)$), since $\varphi, \psi \in \mathcal{A}$, but all we require is the previous estimate. Since $y \in Q$, y can be replaced by x_Q in the last expression by Lemma 8.4, and so

$$|t_{QP}| \le c \left(\frac{|Q|}{|P|}\right)^{1/2} \min\left(\left[\frac{l(Q)}{l(P)}\right]^\tau, \left[\frac{l(P)}{l(Q)}\right]^\sigma\right)\left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))}\right)^{-M},$$

which is exactly (8.1). Thus $(t_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. Using (9.1), we obtain

$$\langle T\psi_P, \varphi_Q\rangle = \left\langle \sum_R t_{RP} T a_R^{(P)}, \varphi_Q \right\rangle = \sum_R t_{RP} \left\langle T a_R^{(P)}, \varphi_Q \right\rangle.$$

Since T maps any N_0-atom $a_R^{(P)}$ into a fixed multiple of a smooth (δ, M, N)-molecule m_R: $Ta_R^{(P)} = c\, m_R$ and c depends neither on R nor on Q, we get

$$\left\langle T a_R^{(P)}, \varphi_Q \right\rangle = c\, \langle m_R, \varphi_Q\rangle =: c\, \tilde{t}_{QR},$$

and by (8.2), since m_R is a smooth (δ, M, N)-molecule for $\dot{B}_p^{\alpha q}(W)$, $(\tilde{t}_{QR}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. Hence,

$$\left(\langle T\psi_P, \varphi_Q\rangle_{QP}\right) = \left(c \sum_R \tilde{t}_{QR}\, t_{RP}\right) \in \mathbf{ad}_p^{\alpha q}(\beta),$$

since the composition of two almost diagonal operators is again almost diagonal by Theorem 8.11. □

Let T be a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, and let $K = K(x, y)$ be its distributional kernel defined on $\mathbb{R}^{2n} \setminus \Delta$, where $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ (for definitions refer to [3], Chapter 8). Then $T \in CZO(\epsilon)$, $0 < \epsilon \le 1$, if K has the following properties:

(I) $|K(x, y)| \le \dfrac{c}{|x - y|^n}$,

(II_ϵ) $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le c\, \dfrac{|x - x'|^\epsilon}{|x - y|^{n+\epsilon}}$ if $2|x - x'| \le |x - y|$.

To show that a CZO maps atoms into molecules we start with the following result from [3]:

Theorem 9.3 ([3], Thm. 8.13)**.** *Let $0 < \epsilon \le 1$ and $0 < \alpha < 1$. If $T \in CZO(\epsilon) \cap WBP$ and $T1 = 0$, then T maps any smooth 0-atom a_Q into a fixed multiple of a smooth $(\epsilon, n + \epsilon, -1)$-molecule m_Q.*

Thus, if a_Q is a smooth 0-atom for Q, then $Ta_Q = c\, m_Q$, where m_Q satisfies

1. $|m_Q(x)| \le |Q|^{-1/2} \left(1 + \dfrac{|x - x_Q|}{l(Q)}\right)^{-(n+\epsilon)}$,
2. $|m_Q(x) - m_Q(y)| \le |Q|^{-\frac{1}{2}} \left(\dfrac{|x - y|}{l(Q)}\right)^\epsilon \sup\limits_{|z| \le |x-y|} \left(1 + \dfrac{|x - z - x_Q|}{l(Q)}\right)^{-(n+\epsilon)}$,

and c is uniform for all Q. Moreover, an $(\epsilon, n+\epsilon, -1)$-molecule is a smooth molecule for $\dot{B}_p^{\alpha q}(W)$ (see Section 5) if $1 < p < \infty$, $0 < q \le \infty$, $0 < \alpha < \epsilon$ and $\beta < n + p\alpha$:

(i) $\delta = \epsilon$ and $0 < \alpha < \epsilon \le 1$,

(ii) $J = \frac{n}{p'} + \frac{\beta}{p} < n + \alpha < n + \epsilon = M$,

(iii) $J - n - \alpha = \frac{\beta - n}{p} - \alpha < 0 \implies N = \max([J - n - \alpha], -1) = -1$.

The next theorem follows by combining the two statements mentioned above and gives the boundedness of certain Calderón-Zygmund operators on $\dot{B}_p^{\alpha q}(W)$ with some restriction on the weight W:

Theorem 9.4. *Suppose* $0 < \epsilon \le 1$, $0 < \alpha < \epsilon$, $1 \le p < \infty$, $0 < q < \infty$, *and* W *satisfies any of (A1)-(A3). Assume* $\beta < n + p\alpha$. *If* $T \in CZO(\epsilon) \cap WBP$ *and* $T1 = 0$, *then* T *extends to a bounded operator on* $\dot{B}_p^{\alpha q}(W)$.

Remark 9.5. If also $T^*1 = 0$ in Theorem 9.3, then $\alpha = 0$ can be included into the range, since $\int Ta(x)\,dx = \langle Ta, 1\rangle = \langle a, T^*1\rangle = 0$ and so T maps any smooth 0-atom into a smooth $(\epsilon, n+\epsilon, 0)$-molecule (see also [3], Cor. 8.21).

Corollary 9.6. *Let* $1 \le p < \infty$, $0 < q < \infty$, $0 < \epsilon \le 1$ *and* $0 \le \alpha < \epsilon$. *Assume* $\beta < n + p\epsilon$. *If* $T \in CZO(\epsilon) \cap WBP$ *and* $T1 = T^*1 = 0$, *then* T *extends to a bounded operator on* $\dot{B}_p^{\alpha q}(W)$, *in particular, for* $\alpha = 0$.

Proof. Since $N = 0$, the bound on β from the previous theorem can be relaxed to $\beta < n + p\epsilon$. □

Remark 9.7. The condition $T^*(y^\gamma) = 0$ for $|\gamma| \le N, N \ge 1$, produces more vanishing moments of a molecule Ta; so it is not difficult to satisfy (M.iii). But (M.ii) $M = n + \epsilon > J = n + \frac{\beta - n}{p} \iff \beta < n + p\epsilon$ creates a major restriction on the doubling exponent of W. Note that in this case, we get that T maps any smooth 0-atom into a smooth $(\epsilon, n+\epsilon, N)$-molecule, but this molecule is not a smooth molecule for $\dot{B}_p^{\alpha q}(W)$. From now on $N \ge 0$, since the case $N = -1$ is completely covered by Theorem 9.4.

Now we want to show that the restriction on the weight W (to be more precise, the restriction on the doubling exponent β) can be removed in some cases by requiring more smoothnes than (II_ϵ) on the kernel K.

We say that $T \in CZO(N+\epsilon)$, $N \in \mathbb{N} \cup \{0\}$, $0 < \epsilon \le 1$, if T is a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ and K, its distributional kernel defined on $\mathbb{R}^{2n} \setminus \Delta$, has the following properties:

(I) $|K(x,y)| \le \dfrac{c}{|x-y|^n}$,

(II_N) $|D^\gamma_{(2)} K(x,y)| \le \dfrac{c}{|x-y|^{n+|\gamma|}}$, for $|\gamma| \le N$, and

$(\mathrm{II}_{N+\epsilon})$ The inequality

$$|D^\gamma_{(2)} K(x,y) - D^\gamma_{(2)} K(x',y)| + |D^\gamma_{(2)} K(y,x) - D^\gamma_{(2)} K(y,x')| \le c\, \frac{|x-x'|^\epsilon}{|x-y|^{n+|\gamma|+\epsilon}}$$

holds for $2\,|x-x'| \le |x-y|$ and $|\gamma| = N$,

where the subscript 2 in $D^\gamma_{(2)}$ refers to differentiation with respect to the second argument of $K(x,y)$.

Note that $CZO(\epsilon) \supseteq CZO(N+\epsilon)$, for $N \ge 0$.

Theorem 9.8. *Let $0 \le \alpha < 1$, $0 < \epsilon \le 1$, $N_0 \in \mathbb{N} \cup \{0\}$. Suppose $T \in CZO(N_0 + \epsilon) \cap WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \le N_0$. Then T maps any N_0-atom a_Q into a fixed multiple of a smooth $(\epsilon, N_0 + n + \epsilon, N_0)$-molecule.*

More precisely, we will show that $Ta_Q = c\, m_Q$ with c independent of Q and

(i) $\displaystyle\int x^\gamma T a_Q(x)\, dx = 0$, for $|\gamma| \le N_0$,

(ii) $\displaystyle |Ta_Q(x)| \le c\, |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-(N_0+n+\epsilon)}$,

(iii) $\displaystyle |Ta_Q(x) - Ta_Q(y)| \le c\, |Q|^{-\frac{1}{2}} \left[\frac{|x-y|}{l(Q)}\right]^\epsilon$
$\displaystyle \times \sup_{|z| \le |x-y|} \left(1 + \frac{|z - (x - x_Q)|}{l(Q)}\right)^{-(N_0+n+\epsilon)}$.

Before we start the proof, we quote the following estimate:

Lemma 9.9 ([10])**.** *Let $T : \mathcal{D} \to \mathcal{D}'$ be a continuous linear operator with $T \in CZO(\epsilon) \cap WBP$, $0 < \epsilon \le 1$, and $T1 = 0$. Then T maps $\mathcal{D}$ into L^∞, and there exists a constant c such that for any fixed $z \in \mathbb{R}^n$, $t > 0$, $\varphi \in \mathcal{D}$ with* $\operatorname{supp} \varphi \in B_t(z)$,

$$\|T\varphi\|_{L^\infty} \le c\, (\|\varphi\|_{L^\infty} + t\, \|\bigtriangledown \varphi\|_{L^\infty}).$$

Proof of Theorem 9.8. For simplicity, we give the proofs of (i), (ii) and (iii) for $Q = Q_{00}$. The same methods apply to the general cube because of the dilation-translation nature of the estimates. Thus, consider the unit atom $a = a_{Q_{00}}$ with $x_{Q_{00}} = 0$ and $l(Q_{00}) = 1$. First, property (i) immediately follows from the fact that $T^*(y^\gamma) = 0$ for $|\gamma| \le N_0$. To get (ii) we consider two cases: $|x| \le 6\sqrt{n}$ and $|x| > 6\sqrt{n}$. For $|x| \le 6\sqrt{n}$, use Lemma 9.9 to obtain

$$|T\, a(x)| \le \|T\, a\|_{L^\infty} \le c\, (\|a\|_{L^\infty} + \|\bigtriangledown a\|_{L^\infty}) \le c.$$

If $|x| > 6\sqrt{n}$, we get

$$
\begin{aligned}
|Ta(x)| &= \left| \int K(x,y)\, a(y)\, dy \right| \\
&= \left| \int_{3Q_{00}} \left[K(x,y) - \sum_{|\gamma| \le N_0} \frac{D^\gamma_{(y)} K(x,0)}{\gamma!}\, y^\gamma \right] a(y)\, dy \right|,
\end{aligned} \tag{9.3}
$$

since a_Q is an N_0-atom, and thus, has N_0 vanishing moments $\int y^\gamma a_Q(y)\, dy = 0$ for $|\gamma| \le N_0$. Then (9.3) is bounded by

$$\int_{3Q_{00}} \sum_{|\gamma| = N_0} \left| \left[D^\gamma_{(y)} K(x, \theta(y)) - D^\gamma_{(y)} K(x, 0) \right] \frac{y^\gamma}{\gamma!} \right| |a(y)|\, dy.$$

Note that if $y \in \operatorname{supp}\, a$, then $2\,|\theta(y)| \le 2\,|y| \le 2 \cdot 3\sqrt{n} < |x|$, and, using the property $(\mathrm{II}_{N+\epsilon})$ of the kernel K to estimate the difference, we get

$$\sum_{|\gamma| = N_0} |D^\gamma_{(y)} K(x, \theta(y)) - D^\gamma_{(y)} K(x,0)| \le c\, \frac{|\theta(y)|^\epsilon}{|x|^{n+|\gamma|+\epsilon}} \le c\, \frac{|y|^\epsilon}{|x|^{n+N_0+\epsilon}}.$$

Thus,

$$|Ta(x)| \le \frac{c_{n,N_0}}{|x|^{n+N_0+\epsilon}} \int_{3Q_{00}} |y|^{N_0+\epsilon}\, |a(y)|\, dy \le \frac{c}{|x|^{n+N_0+\epsilon}}.$$

In order to show (iii), we prove that

$$|Ta(x) - Ta(x')| \le c\,|x - x'|^\epsilon \left(\frac{1}{(1+|x|)^{n+N_0+\epsilon}} + \frac{1}{(1+|x'|)^{n+N_0+\epsilon}} \right). \tag{9.4}$$

In the case $|x-x'| \ge 1$, the estimate (9.4) follows trivially from (ii) and the triangle inequality. For $|x-x'| < 1$ and $|x| > 10\sqrt{n}$, we can use vanishing moments of $a(x)$ and the integral form of the remainder to get

$$|Ta(x) - Ta(x')| = \left| \int (K(x,y) - K(x',y))\, a(y)\, dy \right| = \left| \int_{3Q_{00}} \Big[K(x,y) \right.$$

$$\left. - \sum_{|\gamma| \le N_0 - 1} \frac{D^\gamma_{(y)} K(x,0)}{\gamma!} y^\gamma - K(x',y) + \sum_{|\gamma| \le N_0-1} \frac{D^\gamma_{(y)} K(x',0)}{\gamma!} y^\gamma \Big] a(y)\, dy \right|$$

$$\le \int_{3Q_{00}} \int_0^1 \frac{(1-s)^{N_0-1}}{(N_0-1)!} \sum_{|\gamma| = N_0} \left| D^\gamma_{(y)} K(x, sy) - D^\gamma_{(y)} K(x', sy) \right| \frac{|y|^\gamma}{\gamma!}\, |a(y)|\, ds\, dy.$$

If $|x| \ge 10\sqrt{n}$ and $y \in \operatorname{supp} a$, then $|x - sy| \ge |x| - s|y| \ge 10\sqrt{n} - 3\sqrt{n} \ge 2\,|x-x'|$ and also $|x - sy| \ge |x| - s|y| \ge |x| - 3\sqrt{n} \ge |x| - \frac{|x|}{2} \ge \frac{|x|}{2}$. By ($\mathrm{II}_{N+\epsilon}$), the last integral is bounded by

$$c \int_{3Q_{00}} \int_0^1 \frac{|x-x'|^\epsilon}{|x-sy|^{n+N_0+\epsilon}}\, |y|^{N_0}\, ds\, dy \le c\, \frac{|x-x'|^\epsilon}{|x|^{n+N_0+\epsilon}}.$$

In case $|x-x'| < 1$ and $|x| \le 10\sqrt{n}$, an exact repetition of the argument on p. 85 of [3] or part (c) on p. 62 of [6] shows that

$$|Ta(x) - Ta(x')| \le c\,|x-x'|^\epsilon$$

by using the decay property (I) and the Lipschitz condition ($\mathrm{II}_{0+\epsilon}$) of the kernel K, which holds for any $CZO(N_0+\epsilon), N_0 \ge 0$. This completes the proof. □

Corollary 9.10. *Let $1 \le p < \infty$, $0 < q < \infty$, and let W satisfy any of (A1)-(A3). Suppose $0 \le \alpha \le \frac{\beta-n}{p} - [\frac{\beta-n}{p}]$, where β is the doubling exponent of W. Let $N = [\frac{\beta-n}{p} - \alpha]$ and $\frac{\beta-n}{p} - [\frac{\beta-n}{p}] < \epsilon \le 1$. If $T \in CZO(N+\epsilon) \cap WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \le N$, then T extends to a bounded operator on $\dot{B}^{\alpha q}_p(W)$.*

Proof. By the previous theorem, T maps any smooth N-atom into a smooth $(\epsilon, N+n+\epsilon, N)$-molecule. This molecule is a smooth molecule for $\dot{B}^{\alpha q}_p(W)$ if (i) $\alpha < \epsilon \le 1$, (ii) $M = N + n + \epsilon > J = n + \frac{\beta-n}{p} \iff [\frac{\beta-n}{p} - \alpha] = [\frac{\beta-n}{p}] > \frac{\beta-n}{p} - \epsilon$, and (iii) $N = \max([J-n-\alpha], -1) = [\frac{\beta-n}{p} - \alpha]$, which are all true. By the boundedness criterion (Lemma 9.2), T is bounded on $\dot{B}^{\alpha q}_p(W)$. □

Corollary 9.11. *Let $1 \le p < \infty$, $0 < q < \infty$, and let W satisfy any of (A1)-(A3). Suppose $0 \le \frac{\beta-n}{p} - [\frac{\beta-n}{p}] < \alpha < 1$, where β is the doubling exponent of W. Let $N = [\frac{\beta-n}{p} - \alpha]$ and $\alpha < \epsilon \le 1$. If $T \in CZO(N+1+\epsilon) \cap WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \le N+1$, then T is bounded on $\dot{B}^{\alpha q}_p(W)$.*

Proof. By Theorem 9.8, T maps any smooth $(N+1)$-atom into a smooth $(\epsilon, N+1+n+\epsilon, N+1)$-molecule, which is also a smooth $(\epsilon, N+1+n+\epsilon, N)$-molecule. This one, in its turn, is a smooth molecule for $\dot{B}_p^{\alpha q}(W)$, since (i) $\alpha < \epsilon \leq 1$, (ii) $M = N+1+n+\epsilon > J = n + \frac{\beta-n}{p} \iff [\frac{\beta-n}{p} - \alpha] + 1 > \frac{\beta-n}{p} - \epsilon$, and (iii) $N = \max([J-n-\alpha], -1) = [\frac{\beta-n}{p} - \alpha]$. By the boundedness criterion (Lemma 9.2), T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$. □

Remark 9.12. Note that the condition $T^*(y^\gamma) = 0, |\gamma| \leq N$, can be very restrictive; for example, the Hilbert transform does not satisfy this condition for $|\gamma| > 0$. On the other hand, we have considered a general class of CZOs, not necessarily of convolution type. Utilizing the convolution structure will let us drop the above condition.

Let $N \in \mathbb{N} \cup \{0\}$. Let T be a convolution operator, i.e., the kernel $K(x,y) = K(x-y)$ is defined on $\mathbb{R}^n \diagdown \{0\}$ and satisfies

(C.1) $|K(x)| \leq \dfrac{c}{|x|^n}$,

(C.2) $|D^\gamma K(x)| \leq \dfrac{c}{|x|^{n+|\gamma|}}$, for $|\gamma| \leq N+1$,

(C.3) $\displaystyle\int_{R_1<|x|<R_2} K(x)\,dx = 0$, for all $0 < R_1 < R_2 < \infty$.

Remark 9.13. We replace (II_N) and $(\mathrm{II}_{N+\epsilon})$ of the general CZO kernel with the slightly stronger smoothness condition (C.2) to make the proof below more concise. The reader can check that conditions corresponding to (II_N) and $(\mathrm{II}_{N+\epsilon})$ in the convolution case would suffice for the statements below.

Now we obtain an analog of Theorem 9.8 saying that T maps smooth atoms into smooth molecules, and then we show the boundedness of T.

Theorem 9.14. *Let $0 \leq \alpha < 1$, $0 < \epsilon \leq 1$, $N \in \mathbb{N} \cup \{0\}$. Let T be a convolution operator with a kernel K satisfying (C.1)-(C.3). Then T maps any smooth N-atom a_Q into a fixed multiple of a smooth $(\epsilon, N+1+n, N)$-molecule.*

More precisely, we will show that

(i) $\displaystyle\int x^\gamma T a_Q(x)\,dx = 0$ for $|\gamma| \leq N$, and

(ii) $\displaystyle |D^\gamma T a_Q(x)| \leq c\,|Q|^{-1/2-|\gamma|/n} \left(1 + \frac{|x-x_Q|}{l(Q)}\right)^{-(N+n+1)}$ for $|\gamma| = 0, 1$.

By the Mean Value Theorem, (ii) with $|\gamma| = 1$ implies the Lipschitz condition (M4) for $|\gamma| = 0$.

Proof. To obtain (ii) we first consider $x \notin 10\sqrt{n}\,Q$. Then

$$|T a_Q(x)|, |\bigtriangledown T a_Q(x)| = \left| \sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} D^{\gamma_0} K(x-y)\, a_Q(y)\,dy \right|$$

$$
(9.5)\qquad = \Bigg| \sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} \Bigg[D^{\gamma_0} K(x-y) - \sum_{|\gamma| \le N-|\gamma_0|} \frac{D^{\gamma} D^{\gamma_0} K(x-x_Q)}{\gamma!} (x_Q-y)^{\gamma} \Bigg] a_Q(y)\, dy \Bigg|,
$$

since a_Q is an N-atom, and thus, has N vanishing moments $\int y^{\gamma} a_Q(y)\, dy = 0$ for $|\gamma| \le N$. Then (9.5) is bounded by

$$
\sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} \sum_{|\gamma|=N+1-|\gamma_0|} \frac{|D^{\gamma+\gamma_0} K(x-x_Q+\theta(y-x_Q))|}{\gamma!} \times |x_Q-y|^{N-|\gamma_0|+1}\, |a_Q(y)|\, dy,
$$

for some $0 \le \theta \le 1$. Since $x \notin 10\sqrt{n}\, Q$ and $y \in 3Q$, $|y-x_Q| \le 2\sqrt{n}\, l(Q) \le \frac{1}{2}|x-x_Q|$. Using property (C.2) of the kernel K, we get

$$
|D^{\gamma} K(x-x_Q+\theta(y-x_Q))| \le \frac{c}{|x-x_Q+\theta(y-x_Q)|^{n+|\gamma|}} \approx \frac{c}{|x-x_Q|^{n+|\gamma|}}.
$$

So,

$$
\begin{aligned}
|Ta_Q(x)|, |\bigtriangledown Ta_Q(x)| &\le \frac{c_{n,N}}{|x-x_Q|^{n+N+1}} \int_{3Q} |x_Q-y|^{N-|\gamma_0|+1}\, |a_Q(y)|\, dy \\
&\le c\, \frac{[l(Q)]^{N-|\gamma_0|+1}}{|x-x_Q|^{n+N+1}}\, |Q|^{-1/2}\, |Q| = c\, |Q|^{-1/2-|\gamma_0|/n} \left[\frac{l(Q)}{|x-x_Q|} \right]^{n+N+1},
\end{aligned}
$$

by the properties of a_Q.

If $x \in 10\sqrt{n}\, Q$ and $y \in 3Q$, then $|x-y| \le 13nl(Q)$; so by the cancellation property (C.3) of K (using $D^{\gamma}(K * a_Q) = K * (D^{\gamma} a_Q)$), we obtain

$$
\begin{aligned}
|Ta_Q(x)|, |\bigtriangledown Ta_Q(x)| &\le \sum_{|\gamma_0|=0 \text{ or } 1} \left| \int_{3Q} K(x-y)\, D^{\gamma_0} a_Q(y)\, dy \right| \\
&= \sum_{|\gamma_0|=0 \text{ or } 1} \left| \int_{3Q} K(x-y)\, \left[D^{\gamma_0} a_Q(y) - D^{\gamma_0} a_Q(x) \right]\, dy \right| \\
&\le c \int_{|y-x| \le 13nl(Q)} \frac{1}{|x-y|^n} |Q|^{-1/2-|\gamma_0|/n-1/n}\, |x-y|\, dy \\
&\le c\, |Q|^{-1/2-|\gamma_0|/n-1/n} \int_0^{13nl(Q)} r^{-n}\, r\, r^{n-1}\, dr = c\, |Q|^{-1/2-|\gamma_0|/n}.
\end{aligned}
$$

This concludes the proof of (ii).

Property (i) comes from the fact that T is a convolution operator and a_Q has vanishing moments up to order N. Property (ii) guarantees the absolute convergence of the integral in (i). ☐

Corollary 9.15. *Convolution operators with kernels satisfying (C.1)-(C.3) are bounded on* $\dot{B}_p^{\alpha q}(W)$ *if* W *satisfies any of (A1)-(A3) and* $0 \le \alpha < \epsilon \le 1$, $0 < q < \infty$, $1 \le p < \infty$. *In particular, the Hilbert transform* $\mathbb{H}$ *(*$n = 1$*) is bounded on* $\dot{B}_p^{\alpha q}(W)$ *and the Riesz transforms* $\mathcal{R}_j$, $j = 1, ..., n$ *(*$n \ge 2$*), are bounded on* $\dot{B}_p^{\alpha q}(W)$.

Proof. This is an immediate consequence of Theorem 9.14 and Lemma 9.2: choose $N = [\frac{\beta-n}{p} - \alpha]$ in Theorem 9.14; then T maps any smooth N-atom into a smooth $(\epsilon, N+1+n, N)$-molecule, which is either a smooth $(\epsilon, N+1+n, N)$-molecule for $\dot{B}_p^{\alpha q}(W)$, if $\alpha \leq \frac{\beta-n}{p} - [\frac{\beta-n}{p}]$, or an $(\epsilon, N+1+n, N-1)$-molecule for $\dot{B}_p^{\alpha q}(W)$, if $1 > \alpha > \frac{\beta-n}{p} - [\frac{\beta-n}{p}]$. Note that both Hilbert and Riesz transforms are convolution type operators with kernels satisfying (C.1)-(C.3). □

10. Wavelets

Consider a pair (φ, ψ) from $\mathcal{A}$ with the mutual property (2.1). Then the family $\{\varphi_Q, \psi_Q\}$ behaves similarly to an orthonormal system because of the property

$$f = \sum_Q \langle f, \varphi_Q \rangle \, \psi_Q = \sum_Q s_Q \, \psi_Q \text{ for all } f \in \mathcal{S}'/\mathcal{P}.$$

However, this system does not constitute an orthonormal basis. This can be achieved by the Meyer and Lemarié construction of a wavelet basis with the generating function $\theta \in \mathcal{S}$ (see [9] and [10]):

Theorem 10.1. *There exist real-valued functions $\theta^{(i)} \in \mathcal{S}(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$, such that the collection $\{\theta_{\nu k}^{(i)}\} = \{2^{\nu n/2} \theta^{(i)}(2^\nu x - k)\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. The functions $\theta^{(i)}$ satisfy*

$$\text{supp } \hat{\theta}^{(i)} \subseteq \left\{ \left[-\frac{8}{3}\pi, \frac{8}{3}\pi \right]^n \setminus \left[-\frac{2}{3}\pi, \frac{2}{3}\pi \right]^n \right\}$$

and, hence,

$$\int_{\mathbb{R}} x^\gamma \theta(x) \, dx = 0 \textit{ for all multi-indices } \gamma.$$

Thus, we have $f = \sum_{i=1}^{2^n-1} \sum_Q \left\langle f, \theta_Q^{(i)} \right\rangle \theta_Q^{(i)}$ for all $f \in L^2(\mathbb{R}^n)$. This identity extends to all $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$.

Theorem 10.2. *Let $\alpha \in \mathbb{R}, 0 < q \leq \infty$, $1 \leq p < \infty$, and let W satisfy any of (A1)-(A3). Let $\theta^{(i)}$, $i = 1, ..., 2^n - 1$, be generating wavelet functions as in Theorem 10.1. Then*

$$\left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \left\| \left\{ \left\langle \vec{f}, \theta_Q^{(i)} \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)}.$$

Proof. Assume $i = 1, ..., 2^n - 1$. Since $\{\theta_Q^{(i)}\}_{Q,i}$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, the inequality

$$\left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W)} \equiv \left\| \sum_{Q,i} \left\langle \vec{f}, \theta_Q^{(i)} \right\rangle \theta_Q^{(i)} \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \sum_i \| \{ \left\langle \vec{f}, \theta_Q^{(i)} \right\rangle \}_Q \|_{\dot{b}_p^{\alpha q}(W)} \tag{10.1}$$

follows immediately from Theorem 5.2. Therefore, we need to focus only on the opposite direction.

Let $\varphi \in \mathcal{A}$ be such that $\sum_{\nu\in\mathbb{Z}} |\hat{\varphi}(2^\nu \xi)|^2 = 1$ for $\xi \neq 0$. Let $\vec{s}_Q = \left\langle \vec{f}, \varphi_Q \right\rangle$. Applying the boundedness of the φ-transform (Theorem 6.6), we obtain

$$\left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \leq c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W)} . \tag{10.2}$$

Now for each i and Q, define $\vec{t}_Q^{(i)} = \left\langle \vec{f}, \theta_Q^{(i)} \right\rangle$. Since $\theta_Q^{(i)} \in \mathcal{S}$, by the φ-transform decomposition with $\psi = \varphi$, we have $\theta_Q^{(i)} = \sum_P \left\langle \theta_Q^{(i)}, \varphi_P \right\rangle \varphi_P$, which gives

$$\vec{t}_Q^{(i)} = \sum_P \overline{\left\langle \theta_Q^{(i)}, \varphi_P \right\rangle} \left\langle \vec{f}, \varphi_P \right\rangle = \sum_P a_{QP}^{(i)} \, \vec{s}_P.$$

Since $\operatorname{supp} \hat{\varphi}_P \cap \operatorname{supp} \hat{\theta}_Q^{(i)} \neq \{\emptyset\}$ only if $l(Q) = 2^j l(P)$ with $j = 1, 2, 3, 4$ (recall that $\operatorname{supp} \hat{\varphi}_P \subseteq \{\xi \in \mathbb{R}^n : 2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\}$ when $l(P) = 2^{-\mu}$), we see that $a_{QP}^{(i)} = \left\langle \theta_Q^{(i)}, \varphi_P \right\rangle = 0$ unless $2 \leq \frac{l(Q)}{l(P)} \leq 16$, in which case

$$|a_{QP}| \leq c_M \left(1 + \frac{|x_Q - x_P|}{l(Q)}\right)^{-M} \quad \text{for each } M > 0,$$

as was shown in [3], p. 72. Let $M > \frac{n}{p'} + \frac{\beta}{p}$. Then $A^{(i)} := \left(a_{QP}^{(i)}\right)$ is an almost diagonal matrix for each i, and, by Theorem 1.10,

$$\|\{\vec{t}_Q^{(i)}\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \, \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \tag{10.3}$$

Combining (10.3) with (10.2), we get the opposite direction of (10.1). □

Corollary 10.3. *Let* $\{{}_N\psi^{(i)}\}$, $i = 1, ..., 2^n - 1$, *be a collection of Daubechies DN generating wavelet functions for* $L^2(\mathbb{R}^n)$ *with compact supports linearly dependent on* N *(for more details, see* [2]*). Then for any* $\vec{f}$ *with* $f_j \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, $j = 1, ..., m$,

$$\left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \left\| \left\{ \left\langle \vec{f}, \, {}_N\psi_Q^{(i)} \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \tag{10.4}$$

for sufficiently large N.

Proof. First, observe that there exists a constant c such that for all $i = 1, ..., 2^n - 1$, the functions $\frac{{}_N\psi^{(i)}}{c}$ are smooth molecules, and so $\left\{ \frac{{}_N\psi_Q^{(i)}}{c} \right\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ if we choose N sufficiently large to have the necessary smoothness and vanishing moments. Second, if $\varphi \in \mathcal{A}$, then $\left(\left\langle {}_N\psi_Q^{(i)}, \varphi_P \right\rangle_{QP} \right) \in \mathbf{ad}_p^{\alpha q}(\beta)$ by (8.2). Applying these two facts in the proof of the previous theorem, we get (10.4). □

11. Inhomogeneous Besov spaces

In this section we discuss the inhomogeneous spaces. Before we define the vector-valued inhomogeneous Besov space $B_p^{\alpha q}(W)$ with matrix weight W, we introduce a class of functions $\mathcal{A}^{(I)}$ with properties similar to those of an admissible kernel: we say $\Phi \in \mathcal{A}^{(I)}$ if $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ if $|\xi| \leq \frac{5}{3}$.

Definition 11.1 (*Inhomogeneous matrix-weighted Besov space $B_p^{\alpha q}(W)$*)**.** For $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$, W a matrix weight, $\varphi \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(I)}$, we define the Besov space $B_p^{\alpha q}(W)$ as the collection of all vector-valued distributions $\vec{f} = (f_1, ..., f_m)^{\mathrm{T}}$ with $f_i \in \mathcal{S}'(\mathbb{R}^n), 1 \le i \le m$, such that

$$\left\|\vec{f}\right\|_{B_p^{\alpha q}(W)} = \left\|\Phi * \vec{f}\right\|_{L^p(W)} + \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(W)}\right\}_{\nu \ge 0}\right\|_{l^q} < \infty,$$

where the l^q-norm is replaced by the supremum on $\nu \ge 1$ if $q = \infty$.

Note that now we consider all vector-valued distributions in $\mathcal{S}'(\mathbb{R}^n)$ (rather than $\mathcal{S}'/\mathcal{P}$ as in the homogeneous case), since $\hat{\Phi}(0) \ne 0$.

The corresponding inhomogeneous weighted sequence Besov space $b_p^{\alpha q}(W)$ is defined for the vector sequences enumerated by the dyadic cubes Q with $l(Q) \le 1$.

Definition 11.2 (*Inhomogeneous weighted sequence Besov space $b_p^{\alpha q}(W)$*)**.** For $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$, and W a matrix weight, the space $b_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{l(Q) \le 1}$ such that

$$\|\vec{s}\|_{b_p^{\alpha q}(W)} = \left\|\left\{2^{\nu\alpha}\left\|\sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}}\vec{s}_Q\chi_Q\right\|_{L^p(W)}\right\}_{\nu \ge 0}\right\|_{l^q} < \infty,$$

where the l^q-norm is again replaced by the supremum on $\nu \ge 1$ if $q = \infty$.

Following [5], given $\varphi \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(I)}$, we select $\psi \in \mathcal{A}$ and $\Psi \in \mathcal{A}^{(I)}$ such that

$$\hat{\tilde{\Phi}}(\xi) \cdot \hat{\Psi}(\xi) + \sum_{\nu \ge 1} \hat{\tilde{\varphi}}(2^{-\nu}\xi) \cdot \hat{\psi}(2^{-\nu}\xi) = 1 \text{ for all } \xi, \tag{11.1}$$

where $\tilde{\Phi}(x) = \overline{\Phi(-x)}$. Analogously to the φ-transform decomposition (2.2), we have the identity for $f \in \mathcal{S}'(\mathbb{R}^n)$:

$$f = \sum_{l(Q)=1} \langle f, \Phi_Q \rangle \Psi_Q + \sum_{\nu=1}^{\infty} \sum_{l(Q)=2^{-\nu}} \langle f, \varphi_Q \rangle \psi_Q, \tag{11.2}$$

where $\Phi_Q(x) = |Q|^{-1/2}\Phi(2^\nu x - k)$ for $Q = Q_{\nu k}$, and Ψ_Q is defined similarly.

For each $\vec{f}$ with $f_i \in \mathcal{S}'(\mathbb{R}^n)$, we define the *inhomogeneous φ-transform* $S_\varphi^{(I)}$: $B_p^{\alpha q}(W) \longrightarrow b_p^{\alpha q}(W)$ by setting $(S_\varphi^{(I)}\vec{f})_Q = \left\langle \vec{f}, \varphi_Q \right\rangle$ if $l(Q) < 1$, and $(S_\varphi^{(I)}\vec{f})_Q = \left\langle \vec{f}, \Phi_Q \right\rangle$ if $l(Q) = 1$.

The *inverse inhomogeneous φ-transform* $T_\psi^{(I)}$ is the map taking a sequence $s = \{s_Q\}_{l(Q) \le 1}$ to $T_\psi^{(I)}s = \sum_{l(Q)=1} s_Q\Psi_Q + \sum_{l(Q)<1} s_Q\psi_Q$. In the vector case, $T_\psi^{(I)}\vec{s} = \sum_{l(Q)=1} \vec{s}_Q\Psi_Q + \sum_{l(Q)<1} \vec{s}_Q\psi_Q$. By (11.2), $T_\psi^{(I)} \circ S_\varphi^{(I)}$ is the identity on $\mathcal{S}'(\mathbb{R}^n)$.

Next we show that the relation between $B_p^{\alpha q}(W)$ and $b_p^{\alpha q}(W)$ is the same as for the homogeneous spaces.

Theorem 11.3. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 < p < \infty$, and let W satisfy any of (A1)-(A3). Then*

$$\left\| \vec{f} \right\|_{B_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_{l(Q) \le 1} \right\|_{b_p^{\alpha q}(W)}. \tag{11.3}$$

Before we outline the proof, we need to adjust the notation of smooth molecules for the inhomogeneous case. Define a *family of smooth molecules* $\{m_Q\}_{l(Q)\le 1}$ *for* $B_p^{\alpha q}(W)$ as a collection of functions with the properties:

1. for dyadic Q with $l(Q) < 1$, each m_Q is a smooth (δ, M, N)-molecule with (M.i)-(M.iii) as for the homogeneous space $\dot{B}_p^{\alpha q}(W)$ (see Section 5);
2. for dyadic Q with $l(Q) = 1$, each m_Q (sometimes we denote it as M_Q to emphasize the difference) satisfies (M3), (M4) and a modification of (M2) (which makes it a particular case of (M3) when $\gamma = 0$):

$$|m_Q(x)| \le |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-M}. \tag{M2*}$$

Note that M_Q does not necessarily have vanishing moments. Now one direction of the norm equivalence (11.3) comes from the modified version of Theorem 5.2:

Theorem 11.4. *Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$ and W be a doubling matrix weight of order p. Suppose $\{m_Q\}_{l(Q)\le 1}$ is a family of smooth molecules for $B_p^{\alpha q}(W)$. Then*

$$\left\| \sum_{l(Q) \le 1} \vec{s}_Q \, m_Q \right\|_{B_p^{\alpha q}(W)} \le c \, \| \{\vec{s}_Q\}_{l(Q)\le 1} \|_{b_p^{\alpha q}(W)}. \tag{11.4}$$

Sketch of the Proof. We have

$$\left\| \sum_{l(Q) \le 1} \vec{s}_Q \, m_Q \right\|_{B_p^{\alpha q}(W)} = \left\| \sum_{l(Q) \le 1} \vec{s}_Q \left(\Phi * m_Q \right) \right\|_{L^p(W)}$$

$$+ \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q) \le 1} \vec{s}_Q \left(\varphi_\nu * m_Q \right) \right\|_{L^p(W)} \right\}_{\nu \ge 1} \right\|_{l^q} = I + II.$$

As in Theorem 5.2, which uses the convolution estimates (5.2) and (5.3), we need similar inequalities for modified molecules (the proofs are routine applications of Lemmas B.1 and B.2 from [5]):

$$|\Phi * M_Q(x)| \le c \, \left(1 + |x - x_Q| \right)^{-M} \text{ when } l(Q) = 1, \tag{11.5}$$

$$|\Phi * m_Q(x)| \le c \, |Q|^{-\frac{1}{2}} 2^{-\mu\sigma} \left(1 + |x - x_Q| \right)^{-M} \text{ for some } \sigma > J - \alpha \tag{11.6}$$

when $l(Q) = 2^{-\mu}$, $\mu \ge 1$, and

$$|\varphi_\nu * M_Q(x)| \le c \, 2^{-\nu\tau} \left(1 + |x - x_Q| \right)^{-M} \text{ for some } \tau > \alpha \tag{11.7}$$

when $\nu \ge 1$ and $l(Q) = 1$. For $\nu \ge 1$ and $l(Q) < 1$, the estimate of $|(\varphi_\nu * m_Q)(x)|$ comes from either (5.2) or (5.3).

To estimate I we use (11.5) and (11.6) (note that (11.5) is a special case of (11.6) for $\mu = 0$) and follow the steps of Theorem 1.10 by using Hölder's inequality twice to bring the p^{th} power inside of the sum, and the Squeeze and the Summation Lemmas from Section 5 (it is essential that $\sigma > J - \alpha$ for convergence purposes) to get

$$I \leq c \left\| \{\vec{s}_Q\}_{l(Q)\leq 1} \right\|_{b_p^{\alpha q}(W)}.$$

The second term II is also estimated by $\|\{\vec{s}_Q\}_{l(Q)\leq 1}\|_{b_p^{\alpha q}(W)}$, which is obtained by exact repetition of the proof of Theorem 5.2, only restricting the sum over $\mu \in \mathbb{Z}$ to the sum over $\mu \geq 0$. Also note that (11.7) is a particular case of (5.3) when $\mu = 0$ and, thus, $l(Q) = 1$. Therefore, (11.4) is proved. □

In particular, since Φ and Ψ generate families of smooth molecules for $B_p^{\alpha q}(W)$, we get

$$\left\| \vec{f} \right\|_{B_p^{\alpha q}(W)} \leq c \left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_{l(Q)\leq 1} \right\|_{b_p^{\alpha q}(W)},$$

which gives one direction of the norm equivalence (11.3). To show the other direction, i.e., that the (inhomogeneous) φ-transform is bounded, we simply observe that $\tilde{\Phi} * \vec{f} \in E_0$, which is true since $\left(\tilde{\Phi} * \vec{f}\right)_i^{\widehat{}} \in \mathcal{S}'$ and supp $\hat{\tilde{\Phi}} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$. Hence, Lemmas 6.3 and 6.5 apply to $\vec{g} = \Phi * \vec{f}$ as stated. We have

$$\left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_{l(Q)\leq 1} \right\|_{b_p^{\alpha q}(W)} \approx \left\| \sum_{k\in\mathbb{Z}^n} \left(\tilde{\Phi} * \vec{f} \right)(k)\, \chi_{Q_{0k}} \right\|_{L^p(W)}$$

$$+ \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \chi_Q \left\langle \vec{f}, \varphi_Q \right\rangle \right\|_{L^p(W)} \right\}_{\nu\geq 1} \right\|_{l^q} = I + II.$$

Using $\tilde{\Phi} * \vec{f} \in E_0$ and repeating the proof of Theorem 6.6 for both terms (in the second term we take the l^q norm only over $\nu \in \mathbb{N}$), we get the desired estimate:

$$\left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_{l(Q)\leq 1} \right\|_{b_p^{\alpha q}(W)} \leq c \left\| \vec{f} \right\|_{B_p^{\alpha q}(W)}.$$

Note that as a consequence we also get independence of $B_p^{\alpha q}(W)$ from the choices of Φ and φ.

Now we will briefly discuss operators on the inhomogeneous spaces. An *almost diagonal matrix* on $b_p^{\alpha q}(W)$ is the matrix $A = (a_{QP})_{l(Q),l(P)\leq 1}$ whose entries satisfy (8.1), i.e., $|a_{QP}|$ is bounded by (8.1) only for dyadic Q, P with $l(Q), l(P) \leq 1$. Such a matrix A is a bounded operator on $b_p^{\alpha q}(W)$ for the following reasons: let $\vec{s} \in b_p^{\alpha q}(W)$ and then define $\dot{\vec{s}} = \{\dot{\vec{s}}_Q\}_{Q\,\text{dyadic}}$ by setting $\dot{\vec{s}}_Q = \vec{s}_Q$ if $l(Q) \leq 1$ and $\dot{\vec{s}}_Q = 0$ if $l(Q) > 1$. Note that $\vec{s}$ is a restriction of $\dot{\vec{s}}$ on $b_p^{\alpha q}(W)$. Also, set $\dot{A} = (\dot{a}_{QP})_{Q,P\,\text{dyadic}}$, putting $\dot{a}_{QP} = a_{QP}$ if $l(Q), l(P) \leq 1$ and $\dot{a}_{QP} = 0$ otherwise.

Then

$$\|A\,\vec{s}\|_{b_p^{\alpha q}(W)} = \left\| \left\{ \sum_{l(P)\le 1} a_{QP}\,\vec{s}_P \right\}_{l(Q)\le 1} \right\|_{b_p^{\alpha q}(W)}$$

$$= \left\| \left\{ \sum_{P\ dyadic} \dot{a}_{QP}\,\vec{\dot{s}}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \le c\, \left\|\vec{\dot{s}}\right\|_{\dot{b}_p^{\alpha q}(W)},$$

by Theorem 1.10. By the construction, $\|\vec{\dot{s}}\|_{\dot{b}_p^{\alpha q}(W)} = \|\vec{s}\|_{b_p^{\alpha q}(W)}$, and so we get boundedness of A on $b_p^{\alpha q}(W)$.

It is easy to see that the class of almost diagonal matrices on $b_p^{\alpha q}(W)$ is closed under composition. The same statements (boundedness and being closed under composition) are true for the corresponding almost diagonal operators on $B_p^{\alpha q}(W)$ by combining the norm equivalence (11.3) and the above results about almost diagonal matrices on $b_p^{\alpha q}(W)$. For Calderón-Zygmund operators on inhomogeneous matrix-weighted Besov spaces, some minor notational changes should be made. The collection of smooth N-atoms $\{a_Q\}_{Q\,\mathrm{dyadic}}$ in the homogeneous case ought to be replaced by the set of atoms $\{a_Q\}_{l(Q)<1} \cup \{A_Q\}_{l(Q)=1}$, where the a_Q's have the same properties as before and the A_Q's are such that supp $A_Q \subseteq 3Q$ and $|D^\gamma A_Q(x)| \le 1$ for $\gamma \in \mathbb{Z}_+^n$. This leads to a slight change of the smooth atomic decomposition (see [5], p. 132):

$$f = \sum_{l(Q)<1} s_Q\, a_Q + \sum_{l(Q)=1} s_Q\, A_Q.$$

With these adjustments, all corresponding statements about CZOs hold with essentially the same formulations for the inhomogeneous spaces. Thus, all results obtained for the matrix-weighted homogeneous Besov spaces are essentially the same for the inhomogeneous case.

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References

1. M.Christ and M.Goldberg, *Vector A_2 weights and a Hardy-Littlewood maximal function*, Trans. Amer. Math. Soc., **353**, no. 5, (2001), 1995-2002.
2. I.Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909-996. MR **90m:**42039
3. M.Frazier, B.Jawerth and G.Weiss, *Littlewood-Paley Theory and Study of Function Spaces*, CBMS Regional Conference Series in Mathematics **79**, Amer. Math. Soc., Providence, RI (1991). MR **92m:**42021
4. M.Frazier and B.Jawerth, *Decomposition of Besov Spaces*, Indiana Univ. Math. J. **34** (1985), 777-799. MR **87b:**46083
5. M.Frazier and B.Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34-170. MR **92a:**46042

6. M.Frazier, R.Torres and G.Weiss, *The boundedness of Calderón-Zygmund operators on the spaces $\dot{F}_p^{\alpha q}(W)$*, Rev. Mat. Iberoamericana **4**, no.1, (1988), 41-72. MR **90k:**42029
7. D.Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), 27-42. MR **80h:**46052
8. R.Hunt, B.Muckenhoupt and R.Wheeden, *Weighted norm inequalities for conjugate function and Hilbert transform*, Trans. Amer. Math. Soc., **176** (1973), 227-251. MR **47:**701
9. P.G.Lemarié and Y.Meyer, *Ondelettes et bases hilbertiennes,* Rev. Mat. Iberoamericana **2** (1986), 1-18.
10. Y.Meyer, *Principe d'incertitude, bases Hilbertiennes et algèbres d'opérateurs,* Séminaire Bourbaki 1985/86, Exposé 662, Astérisque, no. 145-146, Soc. Math. France, Paris, 1987, pp. 209-223. MR **88g:**42012
11. Y.Meyer, *Wavelets and Operators,* Cambridge University Press, 1992. MR **92f:**42001
12. F.Nazarov and S.Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz **8** (1996), no. 5, 32-162; English transl., St. Petersburg Math. J. **8** (1997), 721-824. MR **99d:**42026
13. S.Roudenko, *The theory of function spaces with matrix weights*, Ph.D. thesis, Michigan State Univ., 2002.
14. E.Stein, *Harmonic Analysis,* Princeton University Press, 1993. MR **95c:**42002
15. H.Triebel, *Theory of function spaces*, Monographs in Math., vol. 78, Birkhäuser-Verlag, Basel, 1983. MR **86j:**46026
16. S.Treil and A.Volberg, *Wavelets and the angle between past and future*, J. Funct. Anal. **143** (1997), no.2, 269-308. MR **99k:**42073
17. A.Volberg, *Matrix A_p weights via S-functions*, J. Amer. Math. Soc. **10** (1997), no.2, 445-466. MR **98a:**42013

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