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LINKING NUMBERS IN RATIONAL HOMOLOGY 3-SPHERES, CYCLIC BRANCHED COVERS AND INFINITE CYCLIC COVERS

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Abstract. We study the linking numbers in a rational homology 3-sphere and in the infinite cyclic cover of the complement of a knot. They take values in $\mathbb{Q}$ and in $Q(\mathbb{Z}[t,t^{-1}])$, respectively, where $Q(\mathbb{Z}[t,t^{-1}])$ denotes the quotient field of $\mathbb{Z}[t,t^{-1}]$. It is known that the modulo-$\mathbb{Z}$ linking number in the rational homology 3-sphere is determined by the linking matrix of the framed link and that the modulo-$\mathbb{Z}[t,t^{-1}]$ linking number in the infinite cyclic cover of the complement of a knot is determined by the Seifert matrix of the knot. We eliminate 'modulo $\mathbb{Z}$' and 'modulo $\mathbb{Z}[t,t^{-1}]$'. When the finite cyclic cover of the 3-sphere branched over a knot is a rational homology 3-sphere, the linking number of a pair in the preimage of a link in the 3-sphere is determined by the Goeritz/Seifert matrix of the knot.

Introduction

Let $K \cup K_1 \cup \cdots \cup K_m$ ($m \geq 1$) be an oriented $(m+1)$-component link in the three sphere S^3. If the linking number $\mathrm{lk}(K, K_i)$ is even for all $i(=1,...,m)$, then there is an unoriented, possibly nonorientable surface F bounded by K disjoint from $K_1 \cup \cdots \cup K_m$. Let G_α be the *Goeritz matrix* [5], [6] with respect to a basis $\alpha = (a_1,...,a_n)$ of $H_1(F)$, i.e., the (i,j)-entry of G_α is equal to $\mathrm{lk}(a_i, \tau a_j)$, where τa_j is a 1-cycle in $S^3 - F$ obtained by pushing off $2a_j$ in both normal directions.* Let $V_\alpha(K_i) = (\mathrm{lk}(K_i,a_1),...,\mathrm{lk}(K_i,a_n))$. For i,j ($1 \leq i,j \leq m$, possibly $i=j$) we define

$$\lambda_F(K_i,K_j) = V_\alpha(K_i) G_\alpha^{-1} V_\alpha(K_j)^T,$$

and $\lambda_F(K_i,K_j)=0$ for a 2-disk F. Note that $\lambda_F(K_i,K_j) = \lambda_F(K_j,K_i)$. The number $\lambda_F(K_i,K_j)$ is independent of the choice of a basis and S^*-equivalence class of F in $S^3 - (K_i \cup K_j)$ (Proposition 2.1), and if $i=j$ it is an invariant of links (Corollary 2.4).

If $\mathrm{lk}(K,K_i)=0$ for all $i(=1,...,m)$, then there is a Seifert surface F of K with $F \cap (K_1 \cup \cdots \cup K_m) = \emptyset$. Let M_α be the *Seifert matrix* with respect to a basis $\alpha = (a_1,...,a_n)$, i.e., $m_{ij} = \mathrm{lk}(a_i^+, a_j) (= \mathrm{lk}(a_i, a_j^-))$, where $a_i^\pm$ means a curve that is obtained by pushing off in the $\pm$-direction. Let $G_{\alpha,\omega}$ be the Hermitean matrix

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*$2a_j$ can be thought as the double cover of a_j lying in the boundary of the regular neighborhood of F.

$(1-\overline{\omega})M_\alpha + (1-\omega)M_\alpha^T$, where $\omega(\neq 1)$ is a root of unity different from a root of the Alexander polynomial of K. Since $G_{\alpha,\omega} = (\omega - 1)(\overline{\omega}M_\alpha - M_\alpha^T)$, $G_{\alpha,\omega}$ is nonsingular. For i, j ($1 \leq i, j \leq m$, possibly $i = j$) we define

$$\lambda_F(K_i, K_j; \omega) = V_\alpha(K_i) G_{\alpha,\omega}^{-1} V_\alpha(K_j)^T,$$

and $\lambda_F(K_i, K_j; \omega) = 0$ for a 2-disk F. Let $G_\alpha(t) = tM_\alpha - M_\alpha^T$. Then we define

$$\lambda_F(K_i, K_j)(t) = V_\alpha(K_i) G_\alpha(t)^{-1} V_\alpha(K_j)^T,$$

and $\lambda_F(K_i, K_j)(t) = 0$ for a 2-disk F. Let $M_{p,\alpha}$ be a $(p-1)n \times (p-1)n$ matrix defined by

$$M_{p,\alpha} = \begin{pmatrix} M_\alpha + M_\alpha^T & -M_\alpha^T & O & \cdots & O \\ -M_\alpha & M_\alpha + M_\alpha^T & \ddots & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & M_\alpha + M_\alpha^T & -M_\alpha^T \\ O & \cdots & O & -M_\alpha & M_\alpha + M_\alpha^T \end{pmatrix},$$

where O is the $n \times n$ zero matrix. Note that $M_{p,\alpha}$ is a presentation matrix of the first homology group of the p-fold cyclic cover of S^3 branched over K [10]. Let $V_{p,\alpha}^k(K_i) = (\mathbf{0}_{(k-1)n}, V_\alpha(K_i), \mathbf{0}_{(p-k-1)n})$, where $\mathbf{0}_l$ is the $1 \times l$ zero vector. When $M_{p,\alpha}$ is nonsingular, i.e., the p-fold cyclic cover of S^3 branched over K is a rational homology 3-sphere, we define

$$\lambda_F^{(k,l)}(K_i, K_j) = V_{p,\alpha}^k(K_i) M_{p,\alpha}^{-1} V_{p,\alpha}^l(K_j)^T,$$

and $\lambda_F^{(k,l)}(K_i, K_j) = 0$ for a 2-disk F. Note that $\lambda_F(K_i, K_j; \omega) = \lambda_F(K_j, K_i; \omega)$, $\lambda_F^{(k,l)}(K_i, K_j) = \lambda_F^{(k,l)}(K_j, K_i)$, and $\lambda_F(K_i, K_j)(t)$ is equal to $\lambda_F(K_j, K_i)(t^{-1})$ up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$. We shall show that $\lambda_F(K_i, K_j; \omega)$, $\lambda_F(K_i, K_j)(t)$ and $\lambda_F^{(k,l)}(K_i, K_j)$ are independent of the choice of basis and S-equivalence class of F in $S^3 - (K_i \cup K_j)$ (Proposition 3.1), and if $i = j$, then $\lambda_F(K_i, K_i; \omega)$ and $\lambda_F(K_i, K_i)(t)$ are invariants for links (Corollary 4.2). The definitions of $\lambda_F(K_i, K_j)$, etc., were given by Y.W. Lee [14], [15]. But his definitions require some additional conditions. We make his definitions more general.

Let M be a rational homology 3-sphere and $K_1 \cup K_2$ a 2-component oriented link in M. Then there is a 2-chain F in M such that F bounds cK_1, where cK_1 is a disjoint union of c copies of K_1 in a small neighborhood of K_1. We define

$$\mathrm{lk}_M(K_1, K_2) = \frac{F \cdot K_2}{c} \in \mathbb{Q},$$

where $F \cdot K_2$ is the intersection number of F and K_2 [21]. It is known that this linking number is well defined and $\mathrm{lk}_M(K_1, K_2) = \mathrm{lk}_M(K_2, K_1)$. Note that lk_{S^3} is the same as the linking number lk in the usual sense.

Let S be a 3-manifold whose boundary is composed of some tori. Let S_μ and S_δ be 3-manifolds obtained from S by Dehn fillings with respect to systems of curves μ and δ on ∂S, respectively. Suppose that both S_μ and S_δ are rational homology 3-spheres. In Section 1, we show that the difference of the linking number $\mathrm{lk}_{S_\delta} - \mathrm{lk}_{S_\mu}$ is determined by a matrix obtained from μ and δ (Theorem 1.1). It generalizes a result of J. Hoste [7] proved for *integral* homology 3-spheres. As a corollary, for a rational homology 3-sphere M obtained by Dehn surgery along a rational framed link in S^3, we obtain that the linking number lk_M is determined by the linking matrix of the framed link (Corollary 1.2). This was shown by J.C. Cha and K.H. Ko [3] for *integral* framed links. It is known that the linking number modulo $\mathbb{Z}$ is obtained via the matrix; see [6] for example. Our results do not require 'modulo $\mathbb{Z}$'.

In Sections 2 and 3 we show that, for a 3-component link $K \cup K_1 \cup K_2$ with $\mathrm{lk}(K, K_i)$ even (resp. $= 0$), $\mathrm{lk}_{X_2}(K_{ik}, K_{jl})$ (resp. $\mathrm{lk}_{X_p}(K_{ik}, K_{jl})$) is determined by $\lambda_F(K_i, K_j)$ (resp. $\lambda_F^{(k,l)}(K_i, K_j)$) (Theorems 2.3 and 3.2), where X_p is the p-fold cyclic cover of S^3 branched over K, and K_{ik} ($\subset X_p$) is a component of the preimage of K_i.

Let X_∞ be the infinite cyclic cover of the complement of a knot and τ a covering translation that shifts X_∞ along the positive direction with respect to the knot. Let $K_1 \cup K_2$ be a 2-component oriented link in X_∞ with $\tau^i K_1 \cap K_2 = \emptyset$ for all $i \in \mathbb{Z}$. Note that there is a 2-chain F in X_∞ such that

$$\partial F = \bigcup_{k \in \mathbb{Z}} c_k \tau^k K_1,$$

where the c_k's are integers. Then we define

$$\tilde{\mathrm{lk}}_{X_\infty}(K_1, K_2) = \frac{\sum_{h \in \mathbb{Z}} t^h (F \cdot \tau^h K_2)}{\sum_{k \in \mathbb{Z}} c_k t^k} \in Q(\mathbb{Z}[t, t^{-1}]).$$

Since $H_2(X_\infty; \mathbb{Z}) \cong 0$ [2], this is well defined. We do not need to treat this linking pairing modulo $\mathbb{Z}[t, t^{-1}]$. Note that $\tilde{\mathrm{lk}}_{X_\infty}(\tau K_i, K_j) = \tilde{\mathrm{lk}}_{X_\infty}(K_i, \tau^{-1} K_j) = t\tilde{\mathrm{lk}}_{X_\infty}(K_i, K_j)$.

In Section 4 we show that, for a 3-component link $K \cup K_1 \cup K_2$ in S^3 with $\mathrm{lk}(K, K_i) = 0$, $\tilde{\mathrm{lk}}_{X_\infty}(K_{ik}, K_{jl})$ is determined by $\lambda_F(K_i, K_j)(t)$ (Theorem 4.1), where K_{ik} ($\subset X_\infty$) is a component of the preimage of K_i. For a parallel copy K_i' of K_i with $\mathrm{lk}(K_i, K_i') = 0$, the linking pairing $\tilde{\mathrm{lk}}_{X_\infty}(K_{ik}, K_{ik}')$ is called *Kojima-Yamazaki's η-function* $\eta(K, K_i; t)$ [12]. This means the linking pairing is obtained via the Seifert matrix of K. It is known that the linking pairing modulo $\mathbb{Z}[t, t^{-1}]$ is determined by the matrix [11], [16], [25]. Our result does not require 'modulo $\mathbb{Z}[t, t^{-1}]$'. As a corollary we have that $(1-t)\lambda_F(K_i, K_i)(t)$ is equal to Kojima-Yamasaki's η-function $\eta(K, K_i; t)$ and that $(1-t)\lambda_F(K_1, K_2)(t) + \mathrm{lk}(K_1, K_2)$ is a topological concordance invariant of $K \cup K_1 \cup K_2$ up to multiplication by $t^{\pm n}$.

1. Rational homology 3-sphere

Let S be a 3-manifold with a boundary composed of n tori, $T_1^2, T_2^2, ..., T_n^2$. Suppose that $\mu = (\mu_1, \mu_2, ..., \mu_n)^T$ and $\delta = (\delta_1, \delta_2, ..., \delta_n)^T$ are two systems of curves (written as columns), $\mu_i, \delta_i \subset T_i^2$, such that the intersection number $\mu_i \cdot \delta_i = q_i \neq 0$. Furthermore we suppose that *μ and δ represent two bases of* $H_1(S; \mathbb{Q})$. This condition can be restated as: Dehn fillings of S with respect to μ and δ give rational homology spheres S_μ and S_δ, respectively. Since μ and δ represent two bases $[\mu] = ([\mu_1], [\mu_2], ..., [\mu_n])^T$ and $[\delta] = ([\delta_1], [\delta_2], ..., [\delta_n])^T$ of $H_1(S; \mathbb{Q})$, there is an $n \times n$-matrix $B = (b_{ij})$ changing the basis, which is an invertible matrix with rational coefficients such that $\delta_i = \sum_{j=1}^n b_{ij} \mu_j$ or for short $[\delta] = B[\mu]$ (and $[\mu] = B^{-1}[\delta]$). Let J_i (resp. $\widehat{J}_i$) be the core of a solid torus attached to T_i^2 in S_μ (resp. S_δ). Let $G = (g_{ij})$ be an $n \times n$-matrix with $g_{ij} = \mathrm{lk}_{S_\mu}(J_i, J_j)$ for $i \neq j$ and $g_{i,i} = b_{ii}/q_i$. Note that b_{ii}/q_i is a Dehn surgery coefficient used to change S_μ to S_δ. In particular $[\delta_i - b_{ii}\mu_i]$ is zero in $H_1(S_\mu - J_i; \mathbb{Q})$. We call $G = (g_{ij})$ a *surgery-linking matrix* from S_μ to S_δ. We can consider the surgery-linking matrix $H = (h_{ij})$ from S_δ to

S_μ in an analogous manner, i.e., $h_{ij} = \mathrm{lk}_{S_\delta}(\widehat{J}_i, \widehat{J}_j)$ for $i \neq j$ and $h_{ii} = \overline{b_{ii}}/(-q_i)$, where $\overline{b_{ij}}$ is the (i,j)-entry of B^{-1}. Note that $q_i = \mu_i \cdot \delta_i = -\delta_i \cdot \mu_i$. Let Q be a diagonal matrix with $q_{ii} = q_i$. Then we have the following theorem.

Theorem 1.1. (1) *$B = QG$ and $B^{-1} = -QH$.*

(2) *For a two component oriented link $K_1 \cup K_2$ in S,*

$$\begin{aligned}&\mathrm{lk}_{S_\delta}(K_1, K_2) - \mathrm{lk}_{S_\mu}(K_1, K_2)\\ &\quad= -(\mathrm{lk}_{S_\mu}(K_1, J_1), ..., \mathrm{lk}_{S_\mu}(K_1, J_n))G^{-1}(\mathrm{lk}_{S_\mu}(K_2, J_1), ..., \mathrm{lk}_{S_\mu}(K_2, J_n))^T.\end{aligned}$$

In Theorem 1.1(2), the case that both S_μ and S_δ are *integral* homology 3-spheres was shown by J. Hoste [7].

Before proving Theorem 1.1, we formulate a useful corollary. Let $J_1 \cup \cdots \cup J_n$ be an n-component oriented link in S^3. We say that $J_1 \cup \cdots \cup J_n$ is a (*rational*) *framed link* if every component J_i is equipped with a rational number p_i/q_i with $p_i, q_i \in \mathbb{Z}$. Let N_i be a small neighborhood of J_i in S^3 such that $N_i \cap N_j = \emptyset$ for $i \neq j$. Let m_i be a meridian of N_i with $\mathrm{lk}(m_i, J_i) = 1$ and l_i a longitude that is null-homologous in $S^3 - J_i$. Then we obtain a new 3-manifold M in the following way: Remove the interiors of the solid tori $N_1, ..., N_n$ from S^3, attach 2-handles $D_1^2 \times [0,1], ..., D_n^2 \times [0,1]$ so that $[\partial D_i] = p_i[m_i] + q_i[l_i] \in H_1(\partial N_i)$ $(i = 1, ..., n)$, and cap it off with 3-balls. We say that M is *obtained by Dehn surgery along the* (*rational*) *framed link* $J_1 \cup \cdots \cup J_n$. Let $G = (g_{ij})$ be the *linking matrix* of the framed link, i.e., $g_{ij} = \mathrm{lk}_{S^3}(J_i, J_j)$ if $i \neq j$ and $g_{ii} = p_i/q_i$. Since G is a surgery-linking matrix from S^3 to M, by Theorem 1.1(2), we have the following corollary, which was shown in [3] for integral framed links.

Corollary 1.2. *Let M be a rational homology 3-sphere obtained by Dehn surgery along a rational framed, oriented link $J_1 \cup \cdots \cup J_n$ in S^3. Let G be the linking matrix of the framed link. Then for a 2-component oriented link $K_1 \cup K_2$ in the complement of the framed link,*

$$\begin{aligned}&\mathrm{lk}_M(K_1, K_2) - \mathrm{lk}_{S^3}(K_1, K_2)\\ &\quad= -(\mathrm{lk}_{S^3}(K_1, J_1), ..., \mathrm{lk}_{S^3}(K_1, J_n))G^{-1}(\mathrm{lk}_{S^3}(K_2, J_1), ..., \mathrm{lk}_{S^3}(K_2, J_n))^T. \ \Box\end{aligned}$$

Proof of Theorem 1.1. (1) By the definitions of G and H, we have $b_{ii} = q_i g_{ii}$ and $\overline{b_{ii}} = -q_i h_{ii}$. We may assume $i \neq j$. Since $[\delta] = B[\mu]$, each $d\delta_i$ is homologous to $d\sum_{k=1}^n b_{ik}\mu_k$ in S for some integer d. This implies $\mathrm{lk}_{S_\mu}(\delta_i, J_j) = \mathrm{lk}_{S_\mu}(\sum_{k=1}^n b_{ik}\mu_k, J_j) = b_{ij}$. Meanwhile δ_i is homologous to $q_i J_i$ in the solid torus attached to T_i^2 since $\mu_i \cdot \delta_i = q_i$. Therefore $\mathrm{lk}_{S_\mu}(\delta_i, J_j) = \mathrm{lk}_{S_\mu}(q_i J_i, J_j) = q_i g_{ij}$. Notice that $\delta_i \cdot \mu_i = -\mu_i \cdot \delta_i = -q_i$. By the same arguments as above, we have $\overline{b_{ij}} = \mathrm{lk}_{S_\delta}(\sum_{k=1}^n \overline{b_{ik}}\delta_k, \widehat{J}_j) = \mathrm{lk}_{S_\delta}(\mu_i, \widehat{J}_j) = \mathrm{lk}_{S_\delta}(-q_i\widehat{J}_i, \widehat{J}_j) = -q_i h_{ij}$.

(2) Since dK_k is homologous to $d\sum_{i=1}^n \mathrm{lk}_{S_\mu}(K_k, J_i)\mu_i$ in S for some integer d, there is a 2-chain F_k in S that realizes the homologous above. This implies that

$$\begin{aligned}\mathrm{lk}_{S_\delta}(K_1, K_2) - \frac{F_1 \cdot K_2}{d} &= \mathrm{lk}_{S_\delta}\left(\sum_{i=1}^n \mathrm{lk}_{S_\mu}(K_1, J_i)\mu_i, K_2\right)\\ &= \mathrm{lk}_{S_\delta}\left(\sum_{i=1}^n \mathrm{lk}_{S_\mu}(K_1, J_i)\mu_i, \sum_{j=1}^n \mathrm{lk}_{S_\mu}(K_2, J_j)\mu_j\right)\\ &= \sum_{i=1}^n\sum_{j=1}^n \mathrm{lk}_{S_\mu}(K_1, J_i)\mathrm{lk}_{S_\mu}(K_2, J_j)\mathrm{lk}_{S_\delta}(\mu_i, \mu_j)\end{aligned}$$

and

$$\begin{aligned}\mathrm{lk}_{S_\mu}(K_1,K_2)-\frac{F_1\cdot K_2}{d} &= \mathrm{lk}_{S_\mu}\left(\sum_{i=1}^{n}\mathrm{lk}_{S_\mu}(K_1,J_i)\mu_i, K_2\right)\\ &= \mathrm{lk}_{S_\mu}\left(\sum_{i=1}^{n}\mathrm{lk}_{S_\mu}(K_1,J_i)\mu_i, \sum_{j=1}^{n}\mathrm{lk}_{S_\mu}(K_2,J_j)\mu_j\right)\\ &= 0,\end{aligned}$$

where $\mathrm{lk}_{S_\delta}(\mu_i,\mu_i)$ and $\mathrm{lk}_{S_\mu}(\mu_i,\mu_i)$ mean the linking numbers of μ_i and a parallel copy of μ_i in T_i. Hence we have

$$\mathrm{lk}_{S_\delta}(K_1,K_2)-\mathrm{lk}_{S_\mu}(K_1,K_2)=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{lk}_{S_\mu}(K_1,J_i)\mathrm{lk}_{S_\mu}(K_2,J_j)\mathrm{lk}_{S_\delta}(\mu_i,\mu_j).$$

Since μ_i is homologous to $-q_i\widehat{J}_i$ in the solid torus attached to T_i^2 and since $d\mu_i$ is homologous to $d\sum_{k=1}^{n}\overline{b_{ik}}\delta_k$ in S for some integer d, $\mathrm{lk}_{S_\delta}(\mu_i,\mu_j)=\mathrm{lk}_{S_\delta}(-q_i\widehat{J}_i,-q_j\widehat{J}_j)$ $=q_iq_jh_{ij}$ for $i\neq j$ and $\mathrm{lk}_{S_\delta}(\mu_i,\mu_i)=\mathrm{lk}_{S_\delta}(-q_i\widehat{J}_i,\sum_{k=1}^{n}\overline{b_{ik}}\delta_k)=-q_i\overline{b_{ii}}=q_iq_ih_{ii}$. So we have $\mathrm{lk}_{S_\delta}(\mu_i,\mu_j)=q_iq_jh_{ij}$ for any i,j. Theorem 1.1(1) completes the proof. □

Remark 1.3. In Theorem 1.1(1), the assumption that both $[\mu]$ and $[\delta]$ are bases of $H_1(S;\mathbb{Q})$ is not necessarily needed. We can obtain the same result if $[\mu]$ and $[\delta]$ are bases of the same subspace of $H_1(S;\mathbb{Q})$.

Remark 1.4. (a) Let $K_1\cup K_2$ be a 2-component oriented link in an oriented manifold M each of whose components represents an element in $\mathrm{Tor}H_1(M)$. For a 2-chain F in M with $\partial F=cK_1$, we define

$$\mathrm{lk}_M(K_1,K_2)=\frac{F\cdot K_2}{c}\in\mathbb{Q}.$$

Since $[K_2]$ is in $\mathrm{Tor}H_1(M)$, $(c'F\cup(-cF'))\cdot K_2=0$ for any 2-chain F' with $\partial F'=c'K_1$. This implies that lk_M is well defined.

(b) Let M be an oriented 3-manifold. We define a function mul: $H_1(M)\to\mathbb{Z}$ as follows: For an element $a\in H_1(M)$, let mul(a) be the greatest common divisor of the integers in $\{a\cdot F|F$ is a 2-cycle in $M\}$. We put mul(a)=0 if $a\cdot F=0$ for any F. Set $T(H_1(M))=\{a\in H_1(M)|\mathrm{mul}(a)=0\}$. Note that $\mathrm{Tor}(H_1(M))\subset T(H_1(M))$ for any M and that $\mathrm{Tor}(H_1(M))\neq T(H_1(M))$ for some M, e.g. $M=S^1\times S^1\times[0,1]$. Moreover, we note that, for a compact 3-manifold M, $T(H_1(M))=H_1(M)$ if and only if M can be embedded in a rational homology 3-sphere. Let $K\cup K_1$ be a 2-component oriented link in M such that K_1 represents an element in $\mathrm{Tor}(H_1(M))$, and let $c=|\mathrm{Tor}(H_1(M))|$. For a 2-chain F in M with $\partial F=cK_1$, we define

$$\mathrm{L}_M(K_1;K)\equiv F\cdot K_2 \pmod{\mathrm{mul}([K])}.$$

Since $(F\cup(-F'))\cdot K$ is divisible by mul($[K]$) for any 2-chain F' with $\partial F'=cK_1$, $\mathrm{L}_M(\ ;K)$ is well defined. In the case that $[K]\in T(H_1(M))$, that is mul($[K]$) $=0$, we may delete 'modulo mul($[K]$)' from the definition above. If $[K]\in\mathrm{Tor}(H_1(M))$, then $\mathrm{L}_M(K_1;K)/c=\mathrm{lk}_M(K_1,K)$. □

Remark 1.5. R.H. Kyle [13] showed that for any symmetric integral matrix M, there is an integral unimodular matrix P such that PMP^T is a block sum of a nonsingular matrix and a zero matrix. This guarantees that any closed oriented 3-manifold M

is obtained by Dehn surgery along a framed link $J_1 \cup \cdots \cup J_n \cup J'_1 \cup \cdots \cup J'_m$ in S^3 of which the linking matrix is a block sum of a nonsingular matrix B and a zero matrix O, where B (resp. O) is the linking matrix of $J_1 \cup \cdots \cup J_n$ (resp. $J'_1 \cup \cdots \cup J'_m$). By arguments similar to those in the proof of Theorem 1.1(2), we have the following: For a 2-component oriented link $K_1 \cup K_2$ in $S^3 - J_1 \cup \cdots \cup J_n \cup J'_1 \cup \cdots \cup J'_m$ each of which component represents an element in $\mathrm{Tor}H_1(M)$,

$$\begin{aligned}&\mathrm{lk}_M(K_1, K_2) - \mathrm{lk}_{S^3}(K_1, K_2)\\ &\quad = -(\mathrm{lk}_{S^3}(K_1, J_1), ..., \mathrm{lk}_{S^3}(K_1, J_n))B^{-1}(\mathrm{lk}_{S^3}(K_2, J_1), ..., \mathrm{lk}_{S^3}(K_2, J_n))^T. \quad \square\end{aligned}$$

2. Double branched cover of S^3

Let $K \cup K_1 \cup \cdots \cup K_m$ be an $(m+1)$-component oriented link in S^3 and F and F' unoriented surfaces bounded by K without intersecting $K_1 \cup \cdots \cup K_m$. These two surfaces are *S^*-equivalent rel.* $K_1 \cup \cdots \cup K_m$ if they can be transposed into each other by the following operations: (1) attaching a half twisted band locally, (2) attaching a hollow 1-handle (1-surgery), and (3) deleting a hollow 1-handle (0-surgery), where these operations can be done in the complement of $K_1 \cup \cdots \cup K_m$.

By the argument similar to that in the proof of Theorem 1 in [14], we have the following.

Proposition 2.1. *Let $K \cup K_1 \cup \cdots \cup K_m$ $(m \geq 1)$ be an oriented $(m+1)$-component link with the linking number $\mathrm{lk}(K, K_i)$ even for any $i(= 1, ..., m)$. Let F and F' be unoriented, possibly nonorientable surfaces bounded by K that does not intersect $K_1 \cup \cdots \cup K_m$. If F and F' are S^*-equivalent rel. $K_i \cup K_j$, then $\lambda_F(K_i, K_j) = \lambda_{F'}(K_i, K_j)$.*

This theorem implies that $\lambda_F(K_i, K_j)$ is independent of the choice of a basis of $H_1(F)$.

Remarks 2.2. (1) Let $K \cup K_1 \cup K_2$ be a split sum of a trivial knot K and the Hopf link $K_1 \cup K_2$. Let F be the Seifert surface of K illustrated in Figure 1 and D a disk bounded by K with $D \cap (K_1 \cup K_2) = \emptyset$. Then $\lambda_F(K_1, K_2) \neq \lambda_D(K_1, K_2)$. It follows from Proposition 2.1 that F and D are not S^*-equivalent rel. $K_1 \cup K_2$. On the other hand, M. Saito [20] showed that, for an oriented 2-component link $K \cup K_1$ with $\mathrm{lk}(K, K_1)$ even, any two unoriented surfaces bounded by K without intersecting K_1 are S^*-equivalent rel. K_1.

(2) In the next section, we will define S-equivalence, which is an orientable version of S^*-equivalence, rel. $K_1 \cup K_2$ for Seifert surfaces for K in $S^3 \setminus (K_1 \cup K_2)$.

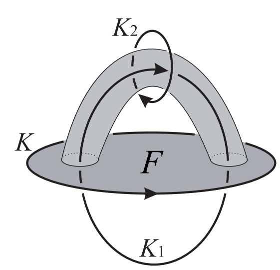


Figure 1.

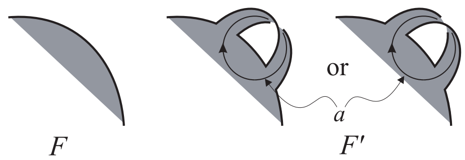


FIGURE 2.

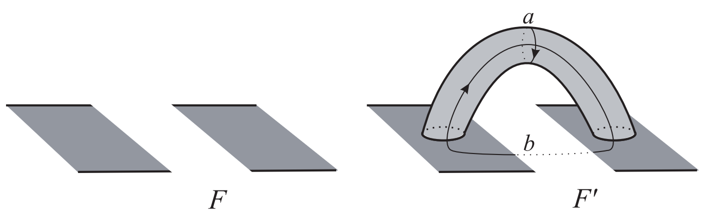


FIGURE 3.

As a special case of [17, Lemma 4] or [9, 4.1.5 Proposition], we have that two Seifert surfaces for K in $S^3 \setminus (K_1 \cup K_2)$ are S-equivalent rel. $K_1 \cup K_2$ if and only if they are homologous in $H_2(S^3 \setminus (K_1 \cup K_2), \partial N(K); \mathbb{Z})$, where $N(K)$ is a regular neighborhood of K in $S^3 \setminus (K_1 \cup K_2)$. □

Proof of Proposition 2.1. Let β be another basis of $H_1(F)$. Then there is a unimodular matrix P such that $\beta = \alpha P$, $G_\beta = P^T G_\alpha P$, $V_\beta(K_i) = V_\alpha(K_i)P$ and $V_\beta(K_j)^T = P^T V_\alpha(K_j)^T$. Thus we have

$$V_\beta(K_i)G_\beta^{-1}V_\beta(K_j)^T = V_\alpha(K_i)PP^{-1}G_\alpha^{-1}(P^T)^{-1}P^T V_\alpha(K_j)^T = V_\alpha(K_i)G_\alpha^{-1}V_\alpha(K_j)^T.$$

We may assume that F' is obtained from F by attaching a half twisted band or by attaching a hollow 1-handle.

In the case that F' is obtained from F by attaching a half twisted band, let a be a cycle as illustrated in Figure 2. Let α be a basis of $H_1(F)$ and $\beta = (a, \alpha)$ a basis of $H_1(F')$. Then we have

$$G_\beta = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & G_\alpha & \\ 0 & & & \end{pmatrix},$$

$V_\beta(K_i) = (0, V_\alpha(K_i))$ and $V_\beta(K_j) = (0, V_\alpha(K_j))$. Thus we have

$$V_\beta(K_i)G_\beta^{-1}V_\beta(K_j)^T = V_\alpha(K_i)G_\alpha^{-1}V_\alpha(K_j)^T.$$

Suppose that F' is obtained from F by attaching a hollow 1-handle. Let a and b be cycles as illustrated in Figure 3. Let α be a basis of $H_1(F)$ and $\beta = (a, b, \alpha)$ a basis $H_1(F')$. Then we have

$$G_\beta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & x & x_1 & \cdots & x_n \\ 0 & x_1 & & & \\ \vdots & \vdots & & G_\alpha & \\ 0 & x_n & & & \end{pmatrix},$$

$V_\beta(K_i) = (0, \mathrm{lk}(K_i, b), V_\alpha(K_i))$ and $V_\beta(K_j) = (0, \mathrm{lk}(K_j, b), V_\alpha(K_j))$. Then it is not hard to see that there are unimodular matrices P and Q such that

$$PG_\beta Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & G_\alpha & \\ 0 & 0 & & & \end{pmatrix},$$

$V_\beta(K_i)Q = V_\beta(K_i)$ and $PV_\beta(K_j)^T = V_\beta(K_j)^T$. Thus we have

$$V_\beta(K_i)G_\beta^{-1}V_\beta(K_j)^T = V_\beta(K_i)(PG_\beta Q)^{-1}V_\beta(K_j)^T = V_\alpha(K_i)G_\alpha^{-1}V_\alpha(K_j)^T.$$

This completes the proof. □

Let $K \cup K_1 \cup K_2$ be an oriented 3-component link with $\mathrm{lk}(K, K_i)$ even ($i = 1, 2$). Let F be an unoriented surface bounded by K that does not intersect $K_1 \cup K_2$. According to the construction by S. Akbulut and R. Kirby [1], we may assume that the double cover X_2 of S^3 branched over K is obtained from two copies M_1 and M_2 of S^3-cut-along-F by gluing each to the other along their boundaries suitably. Let K_{1k} and K_{2k} ($k = 1, 2$) be the preimages in M_k of K_1 and K_2, respectively.

Theorem 2.3. *For any* i, j, k, l $((i, k) \neq (j, l))$,

$$\mathrm{lk}_{X_2}(K_{ik}, K_{jl}) - (1 - \delta_{ij})\delta_{kl}\mathrm{lk}(K_i, K_j) = (-1)^{\delta_{kl}}\lambda_F(K_i, K_j),$$

where $\mathrm{lk}(K_i, K_i) = 0$.

We note that $\mathrm{lk}_{X_2}(K_{i1}, K_{i2}) = \lambda_F(K_i, K_i)$ for each i and $|\mathrm{lk}_{X_2}(K_{11}, K_{21}) - \mathrm{lk}_{X_2}(K_{11}, K_{22})| = |\mathrm{lk}_{X_2}(K_{12}, K_{21}) - \mathrm{lk}_{X_2}(K_{12}, K_{22})| = |2\lambda_F(K_1, K_2) - \mathrm{lk}(K_1, K_2)|$. Since K_{ik}'s are the preimage of K_i, we have the following corollary.

Corollary 2.4. *Both* $\lambda_F(K_i, K_i)$ $(i = 1, 2)$ *and* $|2\lambda_F(K_1, K_2) - \mathrm{lk}(K_1, K_2)|$ *are invariants of* $K \cup K_1 \cup K_2$. □

Now we denote $\lambda_F(K_i, K_i)$ by $\lambda_K(K_i)$.

Remark 2.5. Let $K \cup K_1$ be an oriented link, $K_1(2, 1)$ the $(2, 1)$-cable knot of K_1. Since $\mathrm{lk}(K, K_1(2, 1))$ is even, we can define

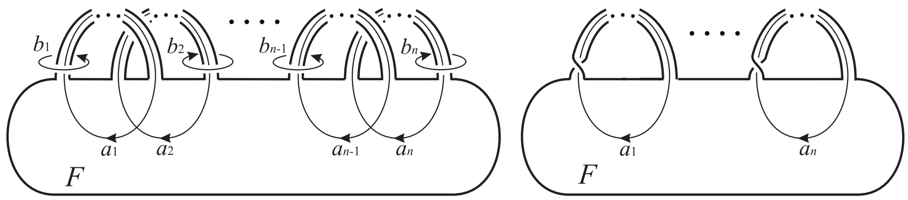
$$\overline{\lambda}_K(K_1) = \frac{1}{4}\lambda_K(K_1(2, 1)).$$

Note that $\overline{\lambda}_K(K_1) = \lambda_K(K_1)$ if $\mathrm{lk}(K, K_1)$ is even. Let $K \cup K_1 \cup \cdots \cup K_m$ be an $(m + 1)$-component oriented link. Then we define

$$\overline{\lambda}_K(K_1 \cup \cdots \cup K_m) = \sum_{i=1}^{m} \overline{\lambda}_K(K_i).$$

Thus we have an invariant for oriented links. □

Proof of Theorem 2.3. Let F be an unoriented surface bounded by K with $F \cap (K_1 \cup K_2) = \emptyset$. Then we may assume that F is a surface as illustrated in Figure 4(a) or (b). Let a_i be a curve in F as in Figure 4(a) or (b) ($i = 1, ..., n$). Then $(a_1, ..., a_n)$ is a basis α of $H_1(F)$. By [1], we have that the double branched cover X_2 is obtained from S^3 by Dehn surgery along an framed oriented link $J_1 \cup \cdots \cup J_n$ with $\mathrm{lk}(J_i, J_j) = \mathrm{lk}(a_i, \tau a_j)$ for any $i \neq j$ and with the framing of J_i equal to



(a) (b)

FIGURE 4.

$\text{lk}(a_i, \tau a_i)$ for any i. By the construction, we note that $K_{11} \cup K_{21} \cup K_{12} \cup K_{22}$ is in the complement of the framed link in S^3, $\text{lk}_{S^3}(K_{ik}, K_{jl}) = \delta_{kl}\text{lk}_{S^3}(K_i, K_j)$ and

$$(\text{lk}_{S^3}(K_{ik}, J_1), ..., \text{lk}_{S^3}(K_{ik}, J_n)) = \begin{cases} V_\alpha(K_i) & \text{if } k = 1, \\ -V_\alpha(K_i) & \text{if } k = 2. \end{cases}$$

By Corollary 1.2, we have the conclusion. □

3. CYCLIC BRANCHED COVER OF S^3

Let $K \cup K_1 \cup \cdots \cup K_m$ be an $(m+1)$-component oriented link and F and F' Seifert surfaces of K that do not intersect $K_1 \cup \cdots \cup K_m$. These two surfaces are *S-equivalent rel.* $K_1 \cup \cdots \cup K_m$ if they can be transposed into each other by the following operations: (1) attaching a hollow orientable 1-handle (1-surgery), and (2) deleting a hollow 1-handle (0-surgery), where these operations can be done in the complement of $K_1 \cup \cdots \cup K_m$.

By the argument similar to that in the proof of Proposition 2.1, we have the following theorem.

Proposition 3.1. *Let $K \cup K_1 \cup \cdots \cup K_m$ $(m \geq 1)$ be an oriented $(m+1)$-component link with $\text{lk}(K, K_i) = 0$ for any $i (= 1, ..., m)$. Let F and F' be Seifert surfaces of K that do not intersect $K_1 \cup \cdots \cup K_m$. If F and F' are S-equivalent rel. $K_i \cup K_j$, then $\lambda_F(K_i, K_j; \omega) = \lambda_{F'}(K_i, K_j; \omega)$, $\lambda_F(K_i, K_j)(t) = \lambda_{F'}(K_i, K_j)(t)$ and $\lambda_F^{(k,l)}(K_i, K_j) = \lambda_{F'}^{(k,l)}(K_i, K_j)$.* □

This theorem implies that $\lambda_F(K_i, K_j; \omega)$, $\lambda_F(K_i, K_j)(t)$ and $\lambda_F^{(k,l)}(K_i, K_j)$ are independent of the choice of a basis of $H_1(F)$.

Let $K \cup K_1 \cup K_2$ be an oriented 3-component link with $\text{lk}(K, K_i) = 0$ $(i = 1, 2)$. Let F be a Seifert surface of K with $F \cap (K_1 \cup K_2) = \emptyset$. By [1], we may assume that the p-fold cyclic cover X_p of S^3 branched over K is obtained from p copies $M_1, ..., M_p$ of S^3 by identifying $F \times [0, 1] \subset M_i$ and $F \times [-1, 0] \subset M_{i+1}$ suitably $(i = 1, ..., p-1)$, where $F \times \{0\} = F$. Let K_{1k} and K_{2k} be the preimages in M_k $(k = 1, ..., p)$ of K_1 and K_2, respectively.

Theorem 3.2. *For any i, j, k, l $((i,k) \neq (j,l))$,*

$$\begin{aligned}
&\mathrm{lk}_{X_p}(K_{ik}, K_{jl}) - (1-\delta_{ij})\delta_{kl}\mathrm{lk}(K_i, K_j) \\
&= \begin{cases}
-\lambda_F^{(k-1,l-1)}(K_i,K_j) + \lambda_F^{(k-1,l)}(K_i,K_j) & \\
\quad +\lambda_F^{(k,l-1)}(K_i,K_j) - \lambda_F^{(k,l)}(K_i,K_j) & \text{if } 2 \leq k \leq l \leq p-1, \\
\lambda_F^{(1,l-1)}(K_i,K_j) - \lambda_F^{(1,l)}(K_i,K_j) & \text{if } k=1,\ 2 \leq l \leq p-1, \\
-\lambda_F^{(k-1,p-1)}(K_i,K_j) + \lambda_F^{(k,p-1)}(K_i,K_j) & \text{if } 2 \leq k \leq p-1,\ l=p, \\
-\lambda_F^{(1,1)}(K_i,K_j) & \text{if } k=l=1, \\
-\lambda_F^{(p-1,p-1)}(K_i,K_j) & \text{if } k=l=p, \\
\lambda_F^{(1,p-1)}(K_i,K_j) & \text{if } k=1,\ l=p.
\end{cases}
\end{aligned}$$

Proof. Let F be a Seifert surface of K with $F \cap (K_1 \cup K_2) = \emptyset$. Then we may assume that F is a surface as illustrated in Figure 4(a). Let a_i be a curve in F as in Figure 4(a) $(i = 1, ..., n)$. Then we may regard that $(a_1, ..., a_n)$ is a basis α of $H_1(F)$. Let M_α be a Seifert matrix with respect to α. By [1], we have that the p-fold cyclic branched cover X_p is obtained from S^3 by Dehn surgery along an framed oriented link $J_{11} \cup \cdots \cup J_{n1} \cup \cdots \cup J_{1(p-1)} \cup \cdots \cup J_{n(p-1)}$ with the linking matrix is equal to $M_{p,\alpha}$. By the construction, we note that $K_{11} \cup K_{21} \cup \cdots \cup K_{ip} \cup K_{2p}$ is in the complement of the framed link in S^3, $\mathrm{lk}_{S^3}(K_{ik}, K_{jl}) = \delta_{kl}\mathrm{lk}_{S^3}(K_i, K_j)$ and

$$\begin{aligned}
&(\mathrm{lk}_{S^3}(K_{ik}, J_{11}), ..., \mathrm{lk}_{S^3}(K_{ik}, J_{n1}), ..., \mathrm{lk}_{S^3}(K_{ik}, J_{1(p-1)}), ..., \mathrm{lk}_{S^3}(K_{ik}, J_{n(p-1)})) \\
&\qquad = \begin{cases}
V^1_{p,\alpha}(K_i) & \text{if } k=1, \\
-V^{p-1}_{p,\alpha}(K_i) & \text{if } k=p, \\
-V^{k-1}_{p,\alpha}(K_i) + V^k_{p,\alpha}(K_i) & \text{if } 2 \leq k \leq p-1.
\end{cases}
\end{aligned}$$

By Corollary 1.2, we have the conclusion. □

4. Infinite cyclic cover of the complement of a knot

Theorem 4.1. *Let $K \cup K_1 \cup K_2$ be an oriented link with $\mathrm{lk}(K, K_1) = \mathrm{lk}(K, K_2) = 0$ and F a Seifert surface of K with $F \cap (K_1 \cup K_2) = \emptyset$. Let (X_∞, p) be the infinite cyclic cover of the complement of K, F_0 a component of $p^{-1}(F)$, and K_{ik} a component of $p^{-1}(K_i)$ contained in a subspace bounded by $\tau^k F_0$ and $\tau^{k+1}F_0$ $(i = 1, 2,\ k \in \mathbb{Z})$. Then*

$$\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}) - \mathrm{lk}(K_1, K_2) = (1-t)\lambda_F(K_1, K_2)(t).$$

Here τ is a covering translation that shifts X_∞ along the positive direction of $p^{-1}(F)$.

Take a parallel copy K_i' of K_i in S^3 with $\mathrm{lk}(K_i, K_i') = 0$. Then we have $\tilde{\mathrm{lk}}_{X_\infty}(K_{ik}, K_{ik}') = (1-t)\lambda_F(K_i, K_i')(t) = (1-t)\lambda_F(K_i, K_i)(t)$. Meanwhile we note that

$$\begin{aligned}
\tilde{\mathrm{lk}}_{X_\infty}(\tau^m K_{1k}, \tau^n K_{2k}) &= t^{m-n}\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}) \\
&= t^{m-n}((1-t)\lambda_F(K_1, K_2)(t) + \mathrm{lk}(K_1, K_2)).
\end{aligned}$$

Hence we have the following corollary.

Corollary 4.2. (1) *$\lambda_F(K_i, K_i)(t)$ is an invariant of $K \cup K_i$ and so is $\lambda_F(K_i, K_i; \omega)$.*

(2) $(1-t)\lambda_F(K_1, K_2)(t) + \mathrm{lk}(K_1, K_2)$ *is an invariant of* $K \cup K_1 \cup K_2$ *up to multiplication by* $t^{\pm n}$. □

Remarks 4.3. (1) As we mentioned in the Introduction, for a parallel copy K_i' of K_i with $\mathrm{lk}(K_i, K_i') = 0$, the linking pairing $\tilde{\mathrm{lk}}_{X_\infty}(K_{ik}, K_{ik}')$ is called Kojima-Yamazaki's η-function $\eta(K, K_i; t)$. Thus $\eta(K, K_i; t) = (1-t)\lambda_F(K_i, K_i)(t)$, and hence $\lambda_F(K_i, K_i)(t)$ is a topological concordance invariant. A different way to calculate the value of Kojima-Yamazaki's η-function was given in [8].

(2) By the argument similar to that in [12, Proof of Theorem 2], we see that $\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k})(= (1-t)\lambda_F(K_1, K_2)(t) + \mathrm{lk}(K_1, K_2))$ is a topological concordance invariant of $K \cup K_1 \cup K_2$ up to multiplication by $t^{\pm n}$.* □

Let $\lambda_K(K_i)(t)$ and $\lambda_K(K_i; \omega)$ denote $\lambda_F(K_i, K_i)(t)$ and $\lambda_F(K_i, K_i; \omega)$, respectively. Note that $\lambda_K(K_i)(\overline{\omega}) = (\omega - 1)\lambda_K(K_i; \omega)$.

Remark 4.4. For an oriented 2-component link $K \cup K_1$ and for the untwisted double $K_1(2)$ of K_1, we define

$$\overline{\lambda}_K(K_1)(t) = \begin{cases} \lambda_K(K_1)(t) & \text{if } \mathrm{lk}(K, K_1) = 0, \\ \lambda_K(K_1(2))(t) & \text{otherwise.} \end{cases}$$

For an $(m+1)$-component oriented link $K \cup K_1 \cup \cdots \cup K_m$, we define

$$\overline{\lambda}_K(K_1 \cup \cdots \cup K_m)(t) = \sum_{i=1}^{m} \overline{\lambda}_K(K_i)(t).$$

Hence we have an invariant for oriented links. □

By the definition of the linking pairing, we have the following lemma.

Lemma 4.5. *Let* X_∞ *be the infinite cyclic cover of the complement of a knot and* $K \cup K_1 \cup \cdots \cup K_m$ *(resp.* $K \cup K_1' \cup \cdots \cup K_n'$*) an oriented* $(m+1)$*-component (resp.* $(n+1)$*-component) link in* X_∞*. If there is a 2-chain* F *such that* $\partial F = K_1 \cup \cdots \cup K_m \cup (-K_1') \cup \cdots \cup (-K_n')$*, then*

$$\tilde{\mathrm{lk}}_{X_\infty}(K, K_1 \cup \cdots \cup K_m) = \tilde{\mathrm{lk}}_{X_\infty}(K, K_1' \cup \cdots \cup K_n') + K \cdot F$$

and

$$\tilde{\mathrm{lk}}_{X_\infty}(K_1 \cup \cdots \cup K_m, K) = \tilde{\mathrm{lk}}_{X_\infty}(K_1' \cup \cdots \cup K_n', K) + K \cdot F.$$

Here $\tilde{\mathrm{lk}}_{X_\infty}(K, K_1 \cup \cdots \cup K_m) = \sum_{i=1}^{m} \tilde{\mathrm{lk}}_{X_\infty}(K, K_i)$ *and* $\tilde{\mathrm{lk}}_{X_\infty}(K_1 \cup \cdots \cup K_m, K) = \sum_{i=1}^{m} \tilde{\mathrm{lk}}_{X_\infty}(K_i, K)$. □

Let K be a knot and F a Seifert surface of K. We may assume that F is a surface as illustrated in Figure 4(a). Let $a_1, ..., a_n$ be curves as in Figure 4(a) and $M = (m_{ij})$ the Seifert matrix of F with respect to a basis $[a_1], ..., [a_n]$. Take curves $b_1, ..., b_n$ so that $\mathrm{lk}(a_i, b_j) = \delta_{ij}$ for any i, j as illustrated in Figure 4(a). Then we have the following lemma.

Lemma 4.6. *Let* (X_∞, p) *be the infinite cyclic cover of the complement of* K*,* F_0 *a component of* $p^{-1}(F)$*, and* b_{ik} *a component of* $p^{-1}(b_i)$ *contained in a subspace bounded by* $\tau^k F_0$ *and* $\tau^{k+1} F_0$ $(i = 1, ..., n,\ k \in \mathbb{Z})$*. Then* $\mathrm{lk}_{X_\infty}(b_{ik}, b_{jk})$ *is equal to the* (i, j)*-entry of* $(1-t)(tM - M^T)^{-1}$.

*U. Kaiser pointed out that the invariant $\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k})$ was given by U. Dahlmeier [4].

Proof. We denote by t both a covering translation and a unit of $\mathbb{Z}[t,t^{-1}]$ since it is well known that there is natural correspondence between them. Take curves $a_1^{\pm},...,a_n^{\pm}$ so that $\mathrm{lk}(a_i^+,b_j)=\mathrm{lk}(a_i^-,b_j)=0$ for any i,j. Then a_i^+ is homologous to $\mathrm{lk}(a_i^+,a_1)b_1+\cdots+\mathrm{lk}(a_i^+,a_n)b_n$ and a_i^- is homologous to $\mathrm{lk}(a_i^-,a_1)b_1+\cdots+\mathrm{lk}(a_i^-,a_n)b_n$. Moreover there are surfaces E_i^+ and E_i^- that realize these homologous such that $E_i^+\cap F=E_i^-\cap F=\emptyset$ and E_i^+ (resp. E_i^-) is bounded by $-a_i^+$ (resp. a_i^-) and some copies of b_j's $(j=1,...,n)$. Then we have

$$\begin{pmatrix}[a_1^+]\\ \vdots\\ [a_n^+]\end{pmatrix}=M\begin{pmatrix}[b_1]\\ \vdots\\ [b_n]\end{pmatrix},\ \begin{pmatrix}[a_1^-]\\ \vdots\\ [a_n^-]\end{pmatrix}=M^T\begin{pmatrix}[b_1]\\ \vdots\\ [b_n]\end{pmatrix}.$$

Let $A_i=a_i\times[-1,1]$ be an annulus in S^3 with $\partial A_i=\pm a_i\times\{\pm1\}=\pm a_i^{\pm}$ and A_{ik} a component of $p^{-1}(A_i)$ with $A_{ik}\cap t^{k+1}F_0\neq\emptyset$. Then we have $\partial A_{ik}=ta_{ik}^+-a_{ik}^-$, where $a_{ik}^{\pm}$ is a component of $p^{-1}(a_i^{\pm})$ contained in a subspace between t^kF_0 and $t^{k+1}F_0$. Let E_{ik}^+ (resp. E_{ik}^-) be a component of $p^{-1}(E_i^+)$ (resp. $p^{-1}(E_i^-)$) contained in a subspace between t^kF_0 and $t^{k+1}F_0$. Let $B_{ik}=E_{ik}^-\cup A_{ik}\cup tE_{ik}^+$. Then

$$\begin{pmatrix}[\partial B_{1k}]\\ \vdots\\ [\partial B_{nk}]\end{pmatrix}=(tM-M^T)\begin{pmatrix}[b_{1k}]\\ \vdots\\ [b_{nk}]\end{pmatrix}.$$

Set $G(t)=tM-M^T$. Since $G(t)$ is nonsingular, we have

$$\det(G(t))G(t)^{-1}\begin{pmatrix}[\partial B_{1k}]\\ \vdots\\ [\partial B_{nk}]\end{pmatrix}=\det(G(t))\begin{pmatrix}[b_{1k}]\\ \vdots\\ [b_{nk}]\end{pmatrix}.$$

Set $\det(G(t))G(t)^{-1}=(l_{ij}(t))$. Since the boundary of each B_{ik} is a disjoint union of some copies of b_{jk}'s and tb_{jk}'s $(j=1,...,n)$, $l_{i1}(t)B_{1k}\cup\cdots\cup l_{in}(t)B_{nk}$ is a 2-chain of which boundary is a disjoint union of t^sb_{jk}'s $(s\in\mathbb{Z},\ j=1,...,n)$. Hence we have

$$\partial(l_{i1}(t)B_{1k}\cup\cdots\cup l_{in}(t)B_{nk})=(\det(G(t))b_{ik})\cup\bigcup_{1\le j\le n,s\in\mathbb{Z}}c_{ijs}(t^sb_{jk}\cup(-t^sb_{jk})).$$

Note that $\bigcup_{1\le j\le n,s\in\mathbb{Z}}c_{ijs}(t^sb_{jk}\cup(-t^sb_{jk}))$ bounds a disjoint union A of embedded annuli in $X_\infty-p^{-1}(F)$. Since $B_{ik}\cdot b_{jk}=A_{ik}\cdot b_{jk}=\delta_{ij}$, $B_{ik}\cdot tb_{jk}=A_{ik}\cdot tb_{jk}=-\delta_{ij}$ and $B_{ik}\cdot t^sb_{jk}=0$ for any i,j and $s(\neq 0,1)$, we have

$$\tilde{\mathrm{lk}}_{X_\infty}(b_{ik},b_{jk})=\sum_{s\in\mathbb{Z}}\frac{t^s((A\cup l_{i1}(t)B_{1k}\cup\cdots\cup l_{in}(t)B_{nk})\cdot t^sb_{jk})}{\det(G(t))}=\frac{(1-t)l_{ij}(t)}{\det(G(t))}.$$

This completes the proof. □

Proof of Theorem 4.1. It is not hard to see that there is a 2-component link $K_1\cup K_2'$ in S^3-F such that K_2 and $-K_2'$ cobound a surface E_0 in S^3-F and $\mathrm{lk}(K_1\cup K_2')=0$. Let E_{0k} (resp. K_{2k}') be a component of $p^{-1}(E_0)$ (resp. $p^{-1}(K_2')$) contained in a subspace between τ^kF_0 and $\tau^{k+1}F_0$. Then, by Lemma 4.5, we have

$$\begin{aligned}\tilde{\mathrm{lk}}_{X_\infty}(K_{1k},K_{2k})&=\tilde{\mathrm{lk}}_{X_\infty}(K_{1k},K_{2k}')+K_{1k}\cdot E_{0k}\\&=\tilde{\mathrm{lk}}_{X_\infty}(K_{1k},K_{2k}')+K_1\cdot E_0\\&=\tilde{\mathrm{lk}}_{X_\infty}(K_{1k},K_{2k}')+\mathrm{lk}(K_1,K_2).\end{aligned}$$

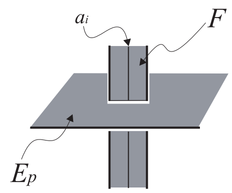


FIGURE 5.

Let E_1 (resp. E_2) be a Seifert surface of K_1 (resp. K_2') in S^3 such that $E_1 \cap K_2' = K_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2$ intersects F as illustrated in Figure 5. Let $N(F)$ be a small neighborhood of F and $E_p^o = E_p - \mathrm{int}N(F)$ $(p = 1, 2)$. Let E_{pk}^o be a component of $p^{-1}(E_p^o)$ contained in a subspace between $\tau^k F_0$ and $\tau^{k+1} F_0$. We note that

$$\partial E_{1k}^o = K_{1k} \cup \bigcup_{i=1}^{n} (-\mathrm{lk}(K_1, a_i) b_{ik} \cup c_i b_{ik} \cup (-c_i b_{ik}))$$

and

$$\partial E_{2k}^o = K_{2k}' \cup \bigcup_{i=1}^{n} (-\mathrm{lk}(K_2', a_i) b_{ik} \cup d_i b_{ik} \cup (-d_i b_{ik})).$$

By Lemma 4.5, we have

$$\begin{aligned}
\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}') &= \tilde{\mathrm{lk}}_{X_\infty}\left(\bigcup_{i=1}^{n} (\mathrm{lk}(K_1, a_i) b_{ik} \cup (-c_i b_{ik}) \cup c_i b_{ik}), K_{2k}' \right) \\
&= \sum_{i=1}^{n} \mathrm{lk}(K_1, a_i) \tilde{\mathrm{lk}}_{X_\infty}(b_{ik}, K_{2k}') \\
&= \sum_{i=1}^{n} \mathrm{lk}(K_1, a_i) \tilde{\mathrm{lk}}_{X_\infty}\left(b_{ik}, \bigcup_{j=1}^{n} (\mathrm{lk}(K_2', a_j) b_{jk} \cup (-d_i b_{jk}) \cup d_i b_{jk}) \right) \\
&= \sum_{i=1}^{n} \mathrm{lk}(K_1, a_i) \sum_{j=1}^{n} \mathrm{lk}(K_2', a_j) \tilde{\mathrm{lk}}_{X_\infty}(b_{ik}, b_{jk}).
\end{aligned}$$

Combining this and Lemma 4.6, we have

$$\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}') = (1 - t)\lambda_F(K_1, K_2')(t).$$

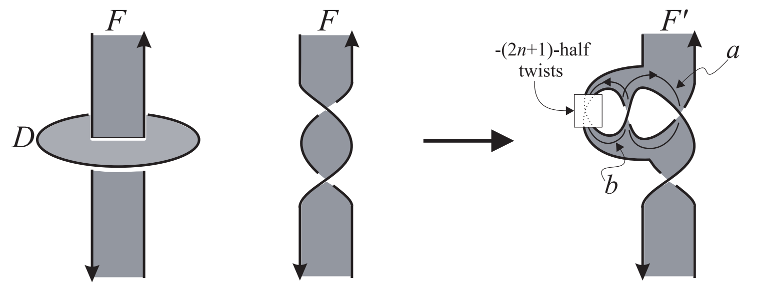
Since $\lambda_F(K_1, K_2)(t) = \lambda_F(K_1, K_2')(t)$ and $\tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}) = \tilde{\mathrm{lk}}_{X_\infty}(K_{1k}, K_{2k}') + \mathrm{lk}(K_1, K_2)$, we have the required result. □

5. CONNECTIONS BETWEEN λ_K AND SIGNATURES

Let K be a knot and D a disk intersecting K transversely in its interior with $|K \cap D| = 2$. Performing $1/n$-Dehn surgery along ∂D, we obtain a new knot K_n. Note that if $\mathrm{lk}(\partial D, K) = 0$ (resp. $\neq 0$), $K_{\pm 1}$ (resp. $K_{\mp 1}$) is obtained from K by changing a $\mp$-crossing into a $\pm$-crossing. Then we have the following two theorems. These results were partially shown by Lee [14], [15]. We modify his proofs.

Theorem 5.1. *If* $\mathrm{lk}(\partial D, K) = 0$*, then the following hold:*

(1) $n(1 - \omega)(1 - \overline{\omega})\lambda_K(\partial D; \omega) \neq 1$ *and* $\lambda_K(\partial D, \omega)$ *is a real number.*



(a)

FIGURE 6.

(2) *$\sigma_\omega(K_n) = \sigma_\omega(K) - 2n/|n|$ (resp. $= \sigma_\omega(K)$) if and only if $n(1-\omega)(1-\overline{\omega})\lambda_K(\partial D;\omega) > 1$ (resp. < 1).*
(3) *$n(1-\omega)(1-\overline{\omega})\lambda_K(\partial D;\omega) = -\nabla_{K_n}(\sqrt{-1}|1-\omega|)/\nabla_K(\sqrt{-1}|1-\omega|) + 1$.*
(4) *$n(t-1)\lambda_K(\partial D)(t) \neq 1$.*
(5) *$\Delta_{K_n}(t)$ is equal to $(1 - n(t-1)\lambda_K(\partial D)(t))\Delta_K(t)$ up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$.*

Here σ_ω is the Tristram-Levine signature [23]*, $\nabla_K(z)$ is the Conway polynomial, and $\Delta_K(t)$ is the Alexander polynomial.*

By combining (2) and (3) in the theorem above, we have the following corollary.

Corollary 5.2. *Suppose $\mathrm{lk}(\partial D, K) = 0$. Then $\sigma_\omega(K_n) = \sigma_\omega(K) - 2n/|n|$ (resp. $= \sigma_\omega(K)$) if and only if $\nabla_{K_n}(\sqrt{-1}|1-\omega|)/\nabla_K(\sqrt{-1}|1-\omega|) < 0$ (resp. > 0).* □

Remark 5.3. Note that a crossing change of a knot K is realized by ± 1-surgery along the boundary of disk D with $|K \cap D| = 2$ and $\mathrm{lk}(\partial D, K) = 0$. By the corollary above and the induction on the unknotting number of a knot, we have that $\nabla_K(\sqrt{-1}|1-\omega|)/|\nabla_K(\sqrt{-1}|1-\omega|)| = \sqrt{-1}^{\sigma_\omega(K)}$ for any knot K. This implies that $\sigma_\omega(K) \neq 0$ if $\nabla_K(\sqrt{-1}|1-\omega|) < 0$. □

Theorem 5.4. (1) *$2n\lambda_K(\partial D) \neq 1$.*
(2) *$\sigma(K_n) = \sigma(K) - 2n/|n| + n|\mathrm{lk}(\partial D, K)|$ (resp. $= \sigma(K) + n|\mathrm{lk}(\partial D, K)|$) if and only if $2n\lambda_K(\partial D) > 1$ (resp. < 1).*
Here $\sigma (= \sigma_{-1})$ is the signature of a knot in the usual sense [24], [18]

Proof of Theorem 5.1. We note that there ia a Seifert surface F of K with $F \cap D$ is an arc as illustrated in Figure 6(a). We construct a Seifert surface F' of K_n from F as illsutrated in Figure 6. Let α be a basis of $H_1(F)$. Let a and b be cycles as illustrated in Figure 6. We may assume that $\beta = (a, b, \alpha)$ is a basis of $H_1(F')$.

Then we have

$$G_{\beta,\omega} = \begin{pmatrix} 0 & 1-\overline{\omega} & \varepsilon(1-\overline{\omega})V_\alpha(\partial D) \\ 1-\omega & -n(1-\overline{\omega})(1-\omega) & 0 \quad \cdots \quad 0 \\ & 0 & \\ \varepsilon(1-\omega)V_\alpha(\partial D)^T & \vdots & G_{\alpha,\omega} \\ & 0 & \end{pmatrix},$$

where $\varepsilon = 1$ or $= -1$. This matrix is congruent to

$$G'_{\beta,\omega} = \begin{pmatrix} 1/n & 0 & & \varepsilon(1-\overline{\omega})V_\alpha(\partial D) & \\ 0 & -n(1-\overline{\omega})(1-\omega) & 0 & \cdots & 0 \\ & 0 & & & \\ \varepsilon(1-\omega)V_\alpha(\partial D)^T & \vdots & & G_{\alpha,\omega} & \\ & 0 & & & \end{pmatrix}.$$

Let

$$U = \begin{pmatrix} 1 & 0 & & -\varepsilon(1-\overline{\omega})V_\alpha(\partial D)G^{-1}_{\alpha,\omega} & \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I & \\ 0 & 0 & & & \end{pmatrix}.$$

Then we have

$$UG^{-1}_{\beta,\omega}\overline{U}^T = \begin{pmatrix} 1/n-(1-\overline{\omega})(1-\omega)\lambda_K(\partial D;\omega) & 0 & 0 & \cdots & 0 \\ 0 & -n(1-\overline{\omega})(1-\omega) & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & G_{\alpha,\omega} & \\ 0 & 0 & & & \end{pmatrix}.$$

Thus $\lambda_K(\partial D, \omega)$ is a real number. Since this matrix is nonsingular, $\lambda_K(\partial D;\omega) \neq 1/n(1-\overline{\omega})(1-\omega)$. Moreover

$$\sigma_\omega(K_n) = \operatorname{sign}\begin{pmatrix} 1/n-(1-\overline{\omega})(1-\omega)\lambda_K(\partial D;\omega) & 0 \\ 0 & -n(1-\overline{\omega})(1-\omega) \end{pmatrix} + \sigma_\omega(K).$$

This implies (1) and (2).

Since (4) follows directly from (5), we shall prove (5). By the argument similar to that in the above, we have

$$\begin{aligned} |G_\beta(t)| &= \begin{vmatrix} 0 & t & & \varepsilon t V_\alpha(\partial D) & \\ -1 & -n(t-1) & 0 & \cdots & 0 \\ & 0 & & & \\ -\varepsilon V_\alpha(\partial D)^T & \vdots & & G_\alpha(t) & \\ & 0 & & & \end{vmatrix} \\ &= \begin{vmatrix} -t/n(t-1)+t\lambda_K(\partial D)(t) & 0 & 0 & \cdots & 0 \\ 0 & -n(t-1) & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & G_\alpha(t) & \\ 0 & 0 & & & \end{vmatrix} \\ &= t(1-n(t-1)\lambda_K(\partial D)(t))|G_\alpha(t)|. \end{aligned}$$

Thus we have (5).

In the proof of (5), replace $G_\alpha(t)$ and $G_\beta(t)$ with $t^{-1}M_\alpha - tM_\alpha^T$ and $t^{-1}M_\beta - tM_\beta^T$, respectively. By the argument similar to that in the proof of (5), we have

$$\Omega_{K_n}(t) = (1-n(t^{-1}-t)V_\alpha(\partial D)(t^{-1}M_\alpha - tM_\alpha^T)^{-1}V_\alpha(\partial D)^T)\Omega_K(t),$$

where $\Omega_K(t) = |t^{-1}M_\alpha - tM_\alpha^T|$. Put $t = \sqrt{-1}(1-\omega)/|1-\omega|$. Then we have

$$\Omega_{K_n}\left(\frac{\sqrt{-1}(1-\omega)}{|1-\omega|}\right) = (1 - n(1-\omega)(1-\overline{\omega})\lambda_K(\partial D;\omega))\Omega_K\left(\frac{\sqrt{-1}(1-\omega)}{|1-\omega|}\right).$$

Since $\Omega_K(t) = \nabla_K(t - t^{-1})$, we have

$$\nabla_{K_n}(\sqrt{-1}|1-\omega|) = (1 - n(1-\omega)(1-\overline{\omega})\lambda_K(\partial D;\omega))\nabla_K(\sqrt{-1}|1-\omega|).$$

The fact that $G_{\alpha,\omega}(= (1-\overline{\omega})M_\alpha + (1-\omega)M_\alpha^T)$ is nonsingular implies

$$\nabla_K(\sqrt{-1}|1-\omega|) = \Omega_K(\sqrt{-1}(1-\omega)/|1-\omega|) \neq 0.$$

Hence we have

$$n(1-\omega)(1-\overline{\omega})\lambda_K(\partial D;\omega) = -\frac{\nabla_{K_n}(\sqrt{-1}|1-\omega|)}{\nabla_K(\sqrt{-1}|1-\omega|)} + 1.$$

This completes the proof. □

Proof of Theorem 5.4. Let F, F', α and β be the same as in the proof of Theorem 5.1. The only difference is that the surfaces are not necessarily orientable. Then we have

$$G_{\beta,\omega} = \begin{pmatrix} 0 & 1 & & \varepsilon V_\alpha(\partial D) & \\ 1 & -2n & 0 & \cdots & 0 \\ & 0 & & & \\ \varepsilon V_\alpha(\partial D)^T & \vdots & & G_\alpha & \\ & 0 & & & \end{pmatrix}.$$

By the argument similar to that in the proof of Theorem 5.1, this matrix is congruent to

$$\begin{pmatrix} 1/2n - \lambda_K(\partial D) & 0 & 0 & \cdots & 0 \\ 0 & -2n & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & G_\alpha & \\ 0 & 0 & & & \end{pmatrix}.$$

Since this matrix is nonsingular, $\lambda_K(\partial D) \neq 1/2n$. Moreover

$$\sigma(K_n) = \operatorname{sign}\begin{pmatrix} 1/2n - \lambda_K(\partial D) & 0 \\ 0 & -2n \end{pmatrix} + \operatorname{sign}(G_\alpha) + \frac{1}{2}e(F'),$$

where $e(F')$ is the *normal Euler number* of F' [6]. Since $e(F') = e(F) + 2n|\mathrm{lk}(\partial D, K)|$,

$$\operatorname{sign}(G_\alpha) + \frac{1}{2}e(F') = \sigma(K) + n|\mathrm{lk}(\partial D, K)|.$$

This completes the proof. □

References

1. S. Akbulut and R. Kirby, Branched covers of surfaces in 4-manifolds, *Math. Ann.* **252** (1980), 111-131. MR **82j:**57001
2. G. Burde and H. Zieschang, *Knots*, De Gruyter Studies in Mathematics, **5**, Walter de Gruyter, Berlin, New York, 1985. MR **87b:**57004
3. J.C. Cha and K.H. Ko, Signatures of links in rational homology spheres, *Topology* **41** (2002), 1161-1182. MR **2003g:**57036
4. U. Dahlmeier, Verkettungshomotopien in Mannigfaltigkeiten, Doktorarbeit Siegen 1994.
5. L. Goeritz, Knoten und quadratische Formen, *Math. Z.* **36** (1933), 647-654.

6. C.McA. Gordon and R.A. Litherland, On the signature of a link, *Invent. Math.* **47** (1978), 53-69. MR **58:**18407
7. J. Hoste, A formula for Casson's invariant, *Trans. Amer. Math. Soc.* **297** (1986), 547-562. MR **88b:**57009
8. G.T. Jin, On Kojima's η-function of links, *Differential topology*, pp. 14-30, Lecture Notes in Math. **1350**, Springer-Verlag, 1988. MR **90c:**57003
9. U. Kaiser, *Link theory in manifolds*, Lecture Notes in Math. **1669**, Springer-Verlag, 1997. MR **98j:**57010
10. L.H. Kauffman, Branched covering, open books and knot periodicity, *Topology* **13** (1974), 143-160. MR **51:**11532
11. C. Kearton, Blanchfield duality and simple knots, *Trans. Amer. Math. Soc.* **202** (1975), 141-160. MR **50:**11255
12. S. Kojima and M. Yamasaki, Some new invariants of links, *Invent. Math.* **54** (1979), 213-228. MR **81b:**57004
13. R.H. Kyle, Branched covering spaces and the quadratic forms of links, *Ann. of Math.* **59** (1954), 539-548. MR **15:**979a
14. Y.W. Lee, A rational invariant for knot crossings, *Proc. Amer. Math. Soc.* **126** (1998), 3385-3392. MR **99c:**57026
15. Y.W. Lee, Alexander polynomial for link crossings, *Bull. Korean Math. Soc.* **35** (1998), 235-258. MR **99k:**57021
16. J. Levine, Knot modules I, *Trans. Amer. Math. Soc.* **229** (1977), 1-50. MR **57:**1503
17. R. Mandelbaum and B. Moishezon, Numeric invariants in 3-manifolds, *Low dimensional topology*, pp. 285-304, Contemporary Math. **20** ed. S.J. Lomonaco Jr., American Math. Soc., 1983. MR **85c:**57006
18. K. Murasugi, On a certain numerical invariant of link type, *Trans. Amer. Math. Soc.* **117** (1965), 387-422. MR **30:**1506
19. D. Rolfsen, *Knots and links*, Publish or Perish, Inc., Berkeley, 1976. MR **58:**24236
20. M. Saito, On the unoriented Sato-Levine invariant, *J. Knot Theory Ramifications* **2** (1993), 335-358. MR **94h:**57017
21. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig 1934.
22. M.V. Sokolov, Quantum invariants, skein modules, and periodicity of 3-manifolds, Ph.D. Thesis, The George Washington University, Washington, D.C., 2000.
23. A. G. Tristram, Some cobordism invariants for links, *Proc. Cambridge Philos. Soc.* **66** (1969), 251-264. MR **40:**2104
24. H.F. Trotter, Homology of group systems with applications to knot theory, *Ann. of Math.* **76** (1962), 464-498 MR **26:**761
25. H.F. Trotter, On S-equivalence of Seifert matrices, *Invent. Math.* **20** (1973), 173-207. MR **58:**31100

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