[5]K. Maki, Physics 1, 21 (1964).
[6]C. Caroli, M. Cyrot, and P. G. de Gennes, Solid State Commun. 4, 17 (1966).
[7]M. F. Merriam, M. A. Jensen, and B. R. Coles. Phys. Rev. 130, 1719 (1963).
[8]In the case where 1 is negligible compared to $2K_2^2$, for $T = T_c$, the slope becomes $A(1) = \pi^4/28\beta\xi(3) = 2.49$.

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Fermi-Liquid Effects on Surface Impedance in the Anomalous-Skin-Effect Regime

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The reflection properties of a semi-infinite plasma in the presence of a magnetic field normal to its surface are calculated on the assumption that the response of the electrons is governed by Fermi-liquid theory and that the surface of the plasma randomizes the motion of the electrons incident upon it. A resonancelike magnetic field dependence of the surface impedance is found near that field for which propagation of the correlation-produced magnetoplasma mode first becomes possible. The size of the structure is calculated to be much enhanced over that of the corresponding structure calculated using specular scattering as a boundary condition for the electrons. Numerical and analytic results are presented and discussed for a range of parameters relevant to possible experiments in the alkali metals.

I. INTRODUCTION

The electromagnetic excitations propagated by conduction electrons of a metal immersed in a strong magnetic field[1] are becoming an increasingly useful tool for obtaining information about the many-body correlations between these electrons.[2-5] Of special interest in this regard is the wave which propagates parallel to the magnetic field at a frequency close to the cyclotron frequency. In the absence of correlations, i.e., on the assumption that each carrier is an independent particle whose motion is governed by the self-consistent field arising from the motion of all the other carriers, no such wave can exist. However, by using the Landau-Silin[6-9] form of Fermi-liquid theory (a phenomenological scheme for introducing the effects of correlation into the self-consistent-field description above), Cheng, Clarke, and Mermin[10] (CCM) predicted that a wave *can* propagate near the cyclotron frequency. Because this wave owes its very existence to the correlations, its properties should serve as a sensitive probe for measuring them. The wave should, by altering the shielding currents and electromagnetic fields at the surface, cause a change in the reflection properties of the metal. Considerations addressed to this question are presented in this paper; we calculate the change in surface admittance caused by Fermi-liquid correlation effects under the conditions ($\omega_c \approx \omega$ and anomalous-skin-effect regime) necessary to the propagation of this wave.

At first thought, such a calculation might seem to be a pointless one: Any information obtained by surface experiments could also, in principle, be obtained from a direct study of the wave in bulk. In particular, the dispersion relation for the wave in the infinite medium is available in a relatively simple form and, by fitting the observed wavelength to this dispersion relation, values of the Fermi-liquid parameters might be obtained. Reference 3 provides a beautiful example of this technique as applied to the high-frequency waves (HFW) which propagate *across* the magnetic field, and rightly points out the simplicity of an infinite-medium calculation versus the difficulty of even the simplest boundary-value problem. Why pose a difficult calculation when the same results might be obtained from a simple one?

The answer to this question lies in the observability of the phenomena. Unlike the HFW waves, the magnetoplasma mode (which we shall denote as the CCM mode) turns out *not* to be the least damped excitation which the system will support (Sec. V). The Gantmakher-Kaner (GK) oscillations,[11] single-particle excitations rather strongly coupled to the electromagnetic field, are of longer spatial range. If the sample is made thick enough to justify an analysis based on the infinite-medium situation, there is a strong chance that the GK oscillations will overwrite the CCM mode. On the other hand, if the sample is made thin, say, less than a wavelength, there is reason for serious doubt that an infinite-medium mode will even exist.

In this situation, it may well turn out that with the magnetic field normal to the surface, a reflection experiment is the only way to observe the mode. Since the dynamics of the HFW waves propagating across the field is so different from those of the CCM mode propagating along the field, it will be of considerable interest to see if the same estimates of the correlation effects emerge from the two classes of observations.

The model used for these calculations is a simple one: The metal is treated as a zero-temperature gas of electrons which interact with each other via the self-consistent field and the short-range forces embodied in the Landau-Silin theory of the Fermi liquid. This gas is immersed in a uniform magnetic field B_0, normal to its surface. A transverse electromagnetic wave falls on that surface at normal incidence. In the bulk of the metal, the interaction between the electrons and the background lattice is characterized by a scattering time τ. Electrons impinging upon the surface of the metal are reflected so that they are prevented from leaving the sample.

Were we content to assume that electrons striking the surface suffer specular reflection, our calculation would reduce to the evaluation of an integral already given by Platzman and Jacobs[12] as an extension of the Reuter-Sondheimer[13] calculation of the surface impedance. However, use of this integral to estimate the change in surface impedance caused by correlations has led to remarkably small values[14] – well below the limits of observability. The only hope of seeing the mode is to abandon the specular boundary condition as a description of the electron-surface interaction. Hence, the major part of this paper deals with the situation where electrons striking the surface are redirected with equal probability at all angles back into the metal – the so-called diffuse boundary condition. We find that in the diffuse case, the effect of correlations is enhanced over their effect in the specular case by a factor of order λ/δ, where λ is the wavelength of the mode and δ is the anomalous skin depth.

The intensity of the effects calculated here, even though enormously enhanced over the specular calculation, is still rather small when compared to the size of the smooth changes in surface admittance (or impedance) caused by reducing the magnetic field from its cyclotron resonance value to zero. Nonetheless, the correlations are predicted to produce a characteristic resonancelike structure in the surface admittance as the magnetic field is swept through the cyclotron-resonance region, structure which is totally absent in any calculation ignoring correlations.

Because the change from specular to diffuse scattering has such a large effect on the predicted amplitude of the resonance structure, we can speculate that if the surface of the sample were rough[15] instead of being diffuse, in a way which hindered the establishment of shielding currents, then the fields in the bulk of the material would be stronger, and their interaction with the mode enhanced thereby. Indeed, the form of the expressions which appear in the calculation supports this point of view and we shall return to a brief consideration of the effect of surface roughness at the end of the paper. This is a matter which is peripheral to the theme of the main investigation but which, unfortunately, must be faced when theory and experiment are to be compared.

Leaving these matters aside, we return to the problem of central interest – calculating the change in surface admittance of a simple metal caused by Fermi-liquid effects under diffuse boundary conditions. The calculation is a rather long one and it may be useful to outline how it was done and to provide, at the same time, a guide to the contents of this paper.

Section II is merely preliminary to the start of the calculation. We write the Landau-Silin transport equation, establish notation, linearize, and then restrict the linearized equation to the geometry of interest. The calculation starts in Sec. III, where we convert the transport equation to a set of coupled integral equations which embody the diffuse scattering boundary condition.

In Sec. IV, we relate these integral equations to a variational principle. The stationary value of the functional to be varied gives the surface admittance. We introduce suitable trial functions, and in Appendix A, we evaluate the quantity whose variation must vanish. This yields the equations which fix the variational parameters.

These equations simplify so much if one stays near cyclotron resonance that an analytic solution is possible. We show this in Appendix B and point out a relationship which exists between the stationary quantity and the dispersion relation for the CCM mode. The analytic solution is carried far enough to exhibit the main features of the numerical solution presented in Sec. V.

In Sec. V, we obtain numerical solutions to the CCM dispersion relation in the presence of collisions and compare the propagation and damping constant for the mode with those of the GK oscillations into which the mode merges at its Doppler-shifted cyclotron resonance. We also obtain numerical solutions to the simplified equations of Appendix B and evaluate, thereby, the change in surface admittance as a function of magnetic field and of the bulk scattering time τ.

Finally, since the contact with experiment is

via the surface impedance, and, in the alkalis at least, is likely to involve a rough surface rather than a diffuse one, we conclude by converting the admittance variations to impedance variations and discussing briefly what changes might be engendered by a rough surface.

II. FORMULATION OF PROBLEM

A complete and readily available account of the Landau theory of Fermi liquids and Silin's extension of it to charged fermions is to be found in the text by Pines and Nozières.[16] An equally lucid exposition of exactly those parts we shall need here is provided in Ref. 3.

Our starting point is the spin-independent part of the transport equation which governs the time evolution of $n(p, r, t)$, the quasiparticle distribution function:

$$\frac{\partial n}{\partial t}+\left(\frac{\partial n}{\partial \vec{r}}\cdot\frac{\partial \epsilon}{\partial \vec{p}}-\frac{\partial \epsilon}{\partial \vec{r}}\cdot\frac{\partial n}{\partial \vec{p}}\right)+q(\vec{E}+\vec{V}\times\vec{B})\cdot\frac{\partial n}{\partial \vec{p}}=\left(\frac{\partial n}{\partial t}\right)_{\text{coll}} . \tag{2.1}$$

Here, $\epsilon(\vec{p}, \vec{r}, t)$ is the energy which a quasiparticle of momentum $\vec{p}$ has when it is at position $\vec{r}$ at time t. This energy is composed of two terms,

$$\epsilon = E^0(\vec{p})+\delta\epsilon(\vec{p}, \vec{r}, t) \quad ,$$

the second of which,

$$\delta\epsilon(\vec{p}, \vec{r}, t)\equiv\int\frac{d^3p'}{(2\pi\hbar)^3}f(\vec{p}, \vec{p}')[n(\vec{p}, \vec{r}, t)-n_0], \tag{2.2}$$

represents the amount by which the energy of the quasiparticle is changed by the short-range interactions with other quasiparticles in its immediate vicinity. Its long-range interaction with distant quasiparticles is electromagnetic and appears in (2.1) in the $q(\vec{E}+\vec{V}\times\vec{B})$ term where the fields $\vec{E}$ and $\vec{B}$ are the total fields, external plus self-consistent, acting at r and t. The velocity which appears in this term is the Hamiltonian one, $\vec{V}=\vec{\nabla}_p\epsilon$. The last term in (2.1) represents the effect of collisions between the quasiparticles and the lattice.

The transport equation (2.1) is linearized by introducing a quantity $g(\vec{p}, \vec{r}, t)$ which measures the deviation of n from its equilibrium value n_0:

$$n = n_0+\frac{\partial n_0}{\partial E^0}g \quad .$$

The linearized form is

$$\frac{\partial g}{\partial t}+\left(\frac{1}{\tau}+\vec{V}\cdot\vec{\nabla}+q(\vec{V}\times\vec{B}_0)\cdot\vec{\nabla}_p\right)(g-\delta\epsilon) = -q\vec{V}\cdot\vec{E} . \tag{2.3}$$

In (2.3), and from here on, the symbol $\vec{V}$ will be used only for the interaction-independent part of the velocity:

$$\vec{V}\equiv\vec{\nabla}_p E^0(\vec{p}) \quad . \tag{2.4}$$

The term $1/\tau$ arises from the collision integral (see Pines and Nozières[16]) under the assumption that the quasiparticles are scattered isotropically by the lattice and that there are no density fluctuations. Choosing the Z axis to lie along the direction of the dc magnetic field, and introducing spherical coordinates in $\vec{p}$ space, $p_x = p\sin\theta\cos\varphi$, etc., one has

$$q(\vec{V}\times\vec{B}_0)\cdot\vec{\nabla}_p = -\omega_c\frac{\partial}{\partial\varphi} , \tag{2.5}$$

where $$\omega_c = qB_0/m^* \tag{2.6a}$$

and $$m^* = p/V \quad . \tag{2.6b}$$

In all of the equations used here, the momentum $\vec{p}$ is understood to be confined to the Fermi surface, so that $|p| = p_F$. Thus, the dependence of g and $\delta\epsilon$ on $\vec{p}$ is really a dependence on $\vec{\Omega}$, the direction of the momentum. We restrict attention here to the situation where distribution functions and fields vary only in the Z direction, and where the field is transverse and circularly polarized $[E_y/i = E_x \equiv e(z), E_z = 0]$. It is useful to represent g, $\delta\epsilon$, $\bar{g}\equiv g-\delta\epsilon$, and $\vec{E}\cdot\vec{V}$ as

$$g = \sum_{lm}\delta N_{lm}(Z)\,Y_{lm}(\vec{\Omega}) \quad , \tag{2.7}$$

$$\delta\epsilon = \sum_{lm}\delta\epsilon_{lm}(Z)\,Y_{lm}(\vec{\Omega}) \quad , \tag{2.8}$$

$$\bar{g}\equiv g-\delta\epsilon = \sum_{lm}\delta\bar{N}_{lm}(Z)\,Y_{lm}(\vec{\Omega}) \quad , \tag{2.9}$$

$$\vec{E}\cdot\vec{V} = (8\pi/3)^{1/2}V_F e(Z)\,Y_{11}(\vec{\Omega}) \quad , \tag{2.10}$$

where the $Y_{lm}(\vec{\Omega})$ are the normalized spherical harmonics.

The quasiparticle scattering amplitude $f(\vec{p}, \vec{p}')$ is expanded in Legendre polynomials of the angle between $\vec{p}$ and $\vec{p}'$:

$$f(\vec{p}, \vec{p}') = \sum_n f_n P_n(\vec{\Omega}\cdot\vec{\Omega}') \quad . \tag{2.11}$$

Making use of (2.7)–(2.10) in (2.2) gives

$$\delta\epsilon_{lm} = -A_l\delta N_{lm} = -[A_l/(1+A_l)]\,\delta\bar{N}_{lm} \quad , \tag{2.12}$$

where $$A_l = (m^*P_F/\pi^2\hbar^3)f_l/(2l+1) \quad . \tag{2.13}$$

Within the framework of the Landau theory, the set of A_l and the parameter m^* fully characterize the spin-independent properties of the quasiparticles.

We assume a time dependence $e^{-i\omega t}$ and, using (2.5), (2.8), (2.10), and (2.12) in the transport equation (2.5), we see that there is nothing coupling the various m values. Hence, we can have a solution where only $m = 1$ appears, i.e., where

$$\delta\bar{N}_{lm}(Z)=\delta\bar{N}_l(Z), \quad \text{if } m=1$$
$$=0, \quad \text{otherwise} \tag{2.14}$$

and the transport equation becomes

$$\left(\frac{1}{\tau}+V_F\cos\theta\frac{\partial}{\partial Z}-i(\omega+\omega_c)\right)\bar{g}$$
$$+i\omega\sum_{l=1}^{\infty}\left(\frac{A_l}{1+A_l}\right)\delta\bar{N}_l(Z)\,Y_{l1}(\vec{\Omega})$$
$$=-q\,V_F\left(\tfrac{8}{3}\pi\right)^{1/2}e(Z)\,Y_{11}(\vec{\Omega}). \tag{2.15}$$

Finally, consider the current,

$$\vec{j}(\mathrm{r},\,t)=q\int\frac{d^3p}{(2\pi\hbar)^3}\,V(g-\delta\epsilon)\frac{\partial n_0}{\partial E^0}\quad.$$

Using the definitions (2.9) and (2.14) in the form

$$\delta\bar{N}_l(Z)=\int d\Omega\,Y^*_{l1}(\vec{\Omega})\bar{g} \tag{2.16}$$

allows us to express the circularly polarized current as

$$j(Z)\equiv j_x(Z)-ij_y(Z)$$
$$=-\frac{2q}{(2\pi\hbar)^3}\,P_F^2\left(\frac{8\pi}{3}\right)^{1/2}\delta\bar{N}_1(Z)\quad. \tag{2.17}$$

This current is related to the field e through Maxwell's equations, which can be combined to give the wave equation

$$\left(\frac{d^2}{dZ^2}+\frac{\omega^2}{c^2}\right)e(Z)=-i\omega\mu_0 j(Z)\quad. \tag{2.18}$$

This set of equations, (2.15)–(2.18), augmented by spatial boundary conditions as given in Sec. III, constitutes the full statement of the problem if we had assumed a plane-wave solution in which all spatial dependences are as e^{ikZ}, it would be easy to solve for $\bar{g}$, take moments, and study the resulting set of coupled algebraic equations for $\delta\bar{N}_l$. Here, the same sort of approach is useful; solve for $\bar{g}$, take moments, and study the resulting set of coupled equations. These are integral equations rather than algebraic.

III. INTEGRAL EQUATIONS

After division by $V_F\cos\theta$, Eq. (2.15) becomes a first-order differential equation for $\bar{g}$. This is to be integrated subject to the usual diffuse scattering boundary conditions, namely, $\bar{g}(Z=0,\vec{\Omega})$ vanishes for $\cos\theta>0$, i.e., in the direction which would carry a particle on the surface into the bulk, while for $\cos\theta<0$, i.e., particles moving towards the surface from the bulk, we use the fact that $\bar{g}(Z=\infty,\vec{\Omega})$ vanishes. Having solved for $\bar{g}$, we take angular moments and obtain the following set of coupled integral equations:

$$\delta\bar{N}_l(Z)+\frac{i\omega}{V_F}\sum_{m=1}\frac{A_m}{1+A_m}\int_0^\infty K_{lm}(Z-Z')\,\delta\bar{N}_m(Z')\,dZ'$$
$$=-(8\pi/3)^{1/2}\,q\int_0^\infty K_{l1}(Z-Z')\,e(Z')\,dZ', \tag{3.1}$$

where

$$K_{mn}(Z-Z')\equiv\int_{\cos\theta\geq 0}\frac{d\Omega}{\cos\theta}\,Y^*_{m1}(\vec{\Omega})\,Y_{n1}(\vec{\Omega})$$
$$\times\exp[-\alpha(Z-Z')/\cos\theta],\quad Z>Z' \tag{3.2a}$$
$$\equiv\int_{\cos\theta\leq 0}\frac{d\Omega}{\cos\theta}\,Y^*_{m1}(\vec{\Omega})\,Y_{n1}(\vec{\Omega})$$
$$\times\exp[-\alpha(Z-Z')/\cos\theta],\quad Z'>Z. \tag{3.2b}$$

The kernels defined in (3.2) are central to the development. $K_{11}(Z-Z')$ is, to within dimensional factors, equal to the nonlocal conductivity of the noninteracting electron gas. Its Fourier transform determines the properties of all excitations which propagate parallel to the magnetic field in the infinite medium. The Fourier transforms of the other K_{lm} kernels are, to within a factor i/k (k = transform variable), equal to the algebraic quantities K_{lm} used in the CCM paper in the derivation of the infinite-medium dispersion relation for the mode.

The reader can easily derive a corresponding set of coupled integral equations assuming specular boundary conditions and obtain, thereby, the justification for using the Reuter-Sondheimer expression for the surface impedance,

$$Z\approx\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2-k_0^2-i\omega\mu_0\sigma(k,\,\omega)}\quad.$$

Here $\sigma(k,\omega)$ is the Fourier transform of the nonlocal conductivity, calculated for the infinite medium within the framework of Fermi-liquid theory.

Unfortunately, the Wiener-Hopf technique used by Reuter and Sondheimer for the *diffuse* situation does not generalize to a system of equations such as (3.1), and no closed-form expression analogous to their diffuse result for the surface admittance appears to exist. The use of a variational principle utilizing the nonlocal conductivity[17] is also precluded because (3.1) cannot be solved in closed form. The approach adopted in Sec. IV rests on a variational principle using (3.1) directly, bypassing the need for having the nonlocal conductivity, but paying for that gain by raising the number of fields which must be varied.

IV. VARIATIONAL PRINCIPLE

Let us specialize (3.1) immediately by retaining only A_1 and A_2 but assuming that A_3 and all higher parameters are zero. The approximation here will not be severe if, as is commonly believed,[3,18] the Landau parameters converge rapidly. Retention of A_2, though it may be small, is essential to

the existence of the CCM mode.

It is convenient to use the mean free path $l = V_F\tau$ as the unit of length,

$$Z = xl \quad , \tag{4.1}$$

and to rescale the dependent variables $\delta\bar{N}_1$ and $\delta\bar{N}_2$ so as to collect all the dimensional constants together. Accordingly, we define

$$\delta\bar{N}_1(Z) \equiv ql(8\pi/3)^{1/2}\Psi_1(x) , \tag{4.2a}$$

$$\delta\bar{N}_2(Z) \equiv ql(8\pi/3)^{1/2}\Psi_2(x) \quad , \tag{4.2b}$$

$$e(Z) \equiv \Psi_3(x) \quad . \tag{4.2c}$$

The set of Eqs. (3.1) and (2.17) and (2.18) becomes

$$\Psi_m(x) + \sum_{n=1}^{2}\int_0^\infty K_{mn}(x-x')h_n\Psi_n(x')\,dx' + \int_0^\infty K_{m1}(x-x')\Psi_3(x')\,dx' = 0 \quad , \tag{4.3a}$$

$$-ib\Psi_1(x) + \left(\frac{d^2}{dx^2} + k_0^2 l^2\right)\Psi_3(x) = 0 \quad , \tag{4.3b}$$

where $$h_j = i\omega\tau A_j/(1+A_j) \quad , \tag{4.4a}$$

$$b = (\omega\omega_p^2 l^3/V_F C^2) \quad , \tag{4.4b}$$

and $$K_{nm}(x-x') = \int_0^{2\pi} d\varphi \int_0^1 \frac{du}{u}\, Y_{n1}^*(u,\varphi) \times Y_{m1}^*(u,\varphi)\exp[-a(x-x')/u], \quad x > x' \tag{4.5}$$

$$= (-1)^{n+m}K_{nm}(x'-x) , \quad x' > x \tag{4.6}$$

and $$a = 1 - i(\omega + \omega_c)\tau \quad . \tag{4.7}$$

Equations (4.3) are to be solved subject to the boundary conditions

$$\Psi_3(x=0) = 1 \quad , \tag{4.8a}$$

$$\Psi_3(x=\infty) = 0 \quad . \tag{4.8b}$$

For the purposes of devising a variational principle, we consider the equations adjoint to (4.3). Denoting their solutions as $\varphi_i(x)$, the adjoint equations are

$$\varphi_1(x) + h_1\int_0^\infty K_{11}(x-x')\varphi_1(x')\,dx' - h_1\int_0^\infty K_{21}(x-x')\varphi_2(x') - ib\varphi_3(x) = 0 \, , \tag{4.9a}$$

$$-h_2\int_0^\infty K_{12}(x-x')\varphi_1(x')\,dx' + \varphi_2(x) + h_2\int_0^\infty K_{22}(x-x')\varphi_2(x')\,dx' = 0 \, , \tag{4.9b}$$

$$\int_0^\infty K_{11}(x-x')\varphi_1(x')\,dx' - \int_0^\infty K_{21}(x-x')\varphi_2(x')\,dx' + \left(\frac{d^2}{dx^2} + k_0^2 l^2\right)\varphi_3(x) = 0, \tag{4.9c}$$

and the boundary conditions we impose are

$$\varphi_3^0(x=0) = 1 \quad , \tag{4.10a}$$

$$\varphi_3^0(x=\infty) = 0 \quad . \tag{4.10b}$$

The principle is most easily described in terms of the following matrix operator:

$$M_{ij}(x-x') = \begin{pmatrix} \delta(x-x') + h_1K_{11}(x-x') & h_2K_{12}(x-x') & K_{11}(x-x') \\ h_1K_{21}(x-x') & \delta(x-x') + h_2K_{22}(x-x') & K_{21}(x-x') \\ -ib\delta(x-x') & 0 & k_0^2 l^2\delta(x-x') \end{pmatrix} \quad . \tag{4.11}$$

Consider the following functional:

$$M[\varphi,\Psi] \equiv -\int_0^\infty dx \left(\frac{d\varphi_3}{dx}\right)\left(\frac{d\Psi_3}{dx}\right) + \sum_{ij}\int_0^\infty\int_0^\infty dx\,dx'\,\varphi_i(x)M_{ij}(x-x')\Psi_j(x') \, . \tag{4.12}$$

It is readily shown that the condition that this functional be stationary with respect to variations of the fields φ which maintain the boundary conditions (4.10), i.e.,

$$\delta M/\delta\varphi_i(x) = 0 \quad , \tag{4.13a}$$

$$\delta\varphi_3(0) = 0 \quad , \tag{4.13b}$$

is equation set (4.3). Stationarity with respect to the fields Ψ,

$$\delta M/\delta\psi_i(x) = 0 \quad , \tag{4.14a}$$

$$\delta\psi_3(0) = 0 \quad , \tag{4.14b}$$

implies equation set (4.9). Moreover, the stationary value of this functional is readily shown to be

$$M[\varphi,\psi] = \left(\frac{d\psi_3}{dx}\right)_{x=0} = \left(\frac{d\varphi_3}{dx}\right)_{x=0} \quad . \tag{4.15}$$

Hence, the surface admittance

$$Y = \frac{c}{i\omega}\left(\frac{de/dZ}{e(Z)}\right)_{Z=0} \tag{4.16}$$

is given by

$$Y = M[\varphi, \psi]/ik_0 l \quad . \tag{4.17}$$

The program, of course, is to propose trial functions for the fields φ and ψ, vary the parameters on which they depend until M is stationary, and use that stationary value in (4.17) to obtain the admittance. If the best trial fields deviate from the true fields by a quantity which is first order small, the error in Y computed in this way will be second order small. However, we are interested in the *changes* in Y caused by the Fermi-liquid theory, and we should therefore make some effort to get an exact evaluation for the part of Y which is independent of correlations. We choose the following functions as trial fields:

$$\varphi_i(x) = \varphi_i^0(x) + a_i e^{ipx} + b_i e^{iqx} = \varphi^0 + \varphi^c \quad , \tag{4.18a}$$

$$\psi_i(x) = \psi_i^0(x) + c_i e^{irx} + d_i e^{isx} = \psi^0 + \psi^c \quad . \tag{4.18b}$$

The φ_i^0 and ψ_i^0 are the *exact* fields when there are no correlations, i.e., that satisfy (4.3) and (4.9) when the A_j are zero, and which satisfy the normalization (4.8) and (4.10). In order that the trial fields (4.18) satisfy this same normalization, it is necessary that p, q, r, s all have positive imaginary parts, and that

$$a_3 + b_3 = 0 \quad , \tag{4.19a}$$

$$c_3 + d_3 = 0 \quad . \tag{4.19b}$$

Aside from this constraint which is to be used to eliminate b_3 and d_3, the other 14 parameters in (4.18) are to be considered as independent, with values to be fixed by solving (4.13) and (4.14).

This program is carried through in the Appendixes. In Appendix A, we show that to lowest order in the parameter $(ab^{-1/3})^{1/2}$ (essentially, the square root of the ratio of the anomalous skin depth to the wavelength of the GK oscillations), one is left with two equations to solve for p and r appearing in (4.18). In Appendix B, an analytic solution to these equations is obtained which is valid near threshold, i.e., where the CCM dispersion relation would predict $k = 0$ in the infinite $\omega\tau$ limit. The resulting expression for the surface admittance is

$$Y = Y_0 + \Delta Y^{(1)} + \Delta Y^{(2)} \quad , \tag{4.20}$$

where $\Delta Y^{(1)} = (\Phi^0 | \Delta M | \psi^0)/ik_0 l$.

Y_0 is just the surface admittance in the absence of correlations. $Y^{(1)}$ is computed using Φ^0 and ψ^0, the fields which are calculated in the absence of correlations, and ΔM, using (4.11), is

$$\Delta M_{ij}(x - x') = K_{ij}(x - x')\, h_j \quad , \quad i, j = 1, 2$$
$$= 0 \quad , \qquad i = 3 \text{ or } j = 3 \quad .$$

Neither the fields Φ^0, ψ^0, nor the operator ΔM exhibit any structure (i.e., unusual dependence on ω_c/ω) in the neighborhood of threshold and so the term $\Delta Y^{(1)}$ contributes no structure to the admittance in that region. (Estimates of $\Delta Y^{(1)}$, based on material in the Appendixes, also reveal almost no magnetic field dependence at $\omega_c/\omega = 1$,[19] where both the fields Φ^0, ψ^0 and the kernels do have structure. The kernels have their longest range at cyclotron resonance, and the fields, because of shielding currents, have their least amplitude. The combinations which enter into $\Delta Y^{(1)}$ are just such that these changes at cyclotron resonance cancel each other.) We can ignore $\Delta Y^{(1)}$ because of its small size compared to Y_0 and its lack of magnetic field dependence. Using the expression for $\Delta Y^{(2)}$ derived in Appendix B,

$$Y = Y_0 + \left(\frac{C}{V_F}\right)\left(\frac{A_2}{1 + A_2}\right)^2 \left(\frac{a_1 + a_2(1 - \eta)^{1/2}}{(\omega_c/\omega - 1) - i/\omega\tau}\right) \quad , \tag{4.21}$$

where a_1 and a_2 are real numbers of order unity, and where

$$\eta = \frac{-C_2}{(\omega_c/\omega - 1) - i/\omega\tau}, \quad C_2 = \frac{A_2}{1 + A_2} \quad . \tag{4.22}$$

This expression has several features which reinforce its identification with the CCM mode. First, recall that near threshold, the propagation constant for the mode is, to within a real constant,

$$k \approx (1 - \eta)^{1/2} \quad ,$$

so that the admittance contains a part which is conductive or susceptive depending on whether the mode is propagating or evanescent. Second, note that, like the dispersion relation for the mode, the expression does not depend on A_1. Finally, as we point out in Appendix B, there is an intimate connection between $\Delta Y^{(2)}$, regarded as a variational principle for p and r, and the dispersion relation for the CCM mode.

V. NUMERICAL RESULTS

The discussion given in Sec. IV was restricted to the immediate neighborhood of threshold and to the $1/\omega\tau \to 0$ limit. However, the numerical work to explore finite values of $1/\omega\tau$ and larger neighborhoods of threshold, including cyclotron resonance, is modest. For the surface admittance, we have, from (4.17), (A18), and (B1),

$$Y = Y_0 + \Delta Y^{(2)} = Y_0 + MA^{-1}J/ik_0 l \quad , \tag{5.1}$$

where $MA^{-1}J$ is given by (B4) and where the parameters p and r are to be fixed by evaluating

$$\frac{d}{dp}(MA^{-1}J)=0 \quad , \qquad \frac{d}{dr}(MA^{-1}J)=0 \quad . \tag{5.2}$$

The term $\Delta Y^{(1)}$ has been omitted from (5.1) because, as explained in Sec. IV, its contribution is small compared to Y_0 and is without structure.

It turns out that by writing

$$\Delta Y^{(2)}=(-C_2 C/V_F)\Delta\hat{Y} \quad , \tag{5.3}$$

the quantity $\Delta\hat{Y}$ is a function of two variables

$$x\equiv(\omega_c/\omega-1)/(-C_2) \tag{5.4a}$$

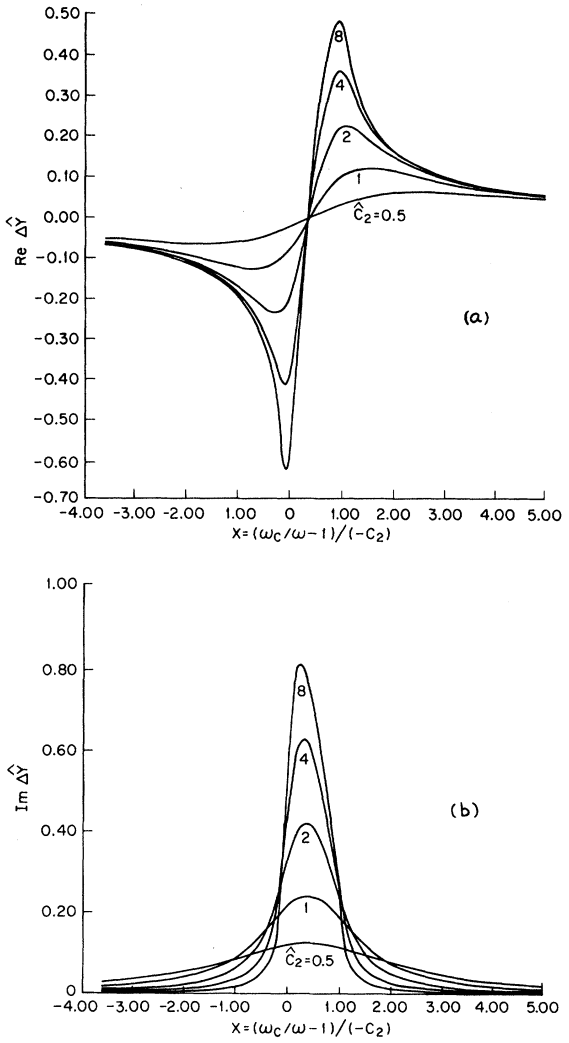

FIG. 1. (a) Real part and (b) imaginary part of the dimensionless change in the surface admittance versus dimensionless magnetic field. Cyclotron resonance occurs at $x=0$, mode threshold at $x=1$. The parameter $\hat{C}_2=-\omega\tau C_2$ measures the importance of correlation effects relative to damping effects.

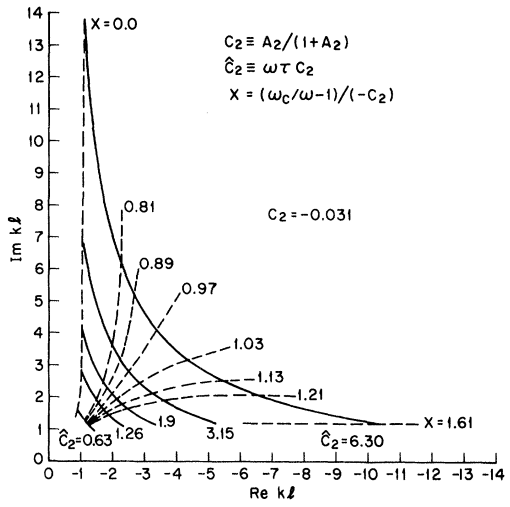

FIG. 2. Dispersion relation for the CCM mode showing real and imaginary parts of the propagation constant k in a system of units where the mean free path is the unit of length. Values of x run from zero (cyclotron resonance) to about $x=\frac{5}{3}$ where, in the $\tau\to\infty$ limit, the mode would suffer Doppler-shifted cyclotron damping.

and $$\hat{C}_2\equiv-\omega\tau C_2=ih_2 \quad . \tag{5.4b}$$

The variable x is proportional to the magnetic field, and has been arranged so that cyclotron resonance occurs at $x=0$ and mode threshold at $x=1$. The variable $\hat{C}_2$ is a measure of the importance of correlations relative to collisions.

In Figs. 1(a) and 1(b), we have plotted the real and imaginary parts of $\Delta\hat{Y}$ obtained from the numerical solution of (5.1) and (5.2), for representative positive values of $\hat{C}_2$. Values of $\Delta\hat{Y}$ for the corresponding negative values of $\hat{C}_2$ are the complex conjugate of those given here, i.e., the real part is the same but the imaginary part is reversed. It is clear from (4.3) that the quantities $\hat{C}_j=ih_j$ are the dimensionless measure of the importance of the correlations. Consequently, the possibility of having the CCM mode exist is also tied to the size of $\hat{C}_2$. To illustrate this point, we have solved the CCM dispersion relation for k, the propagation constant of the mode (again, the mean free path is the unit of length), and have exhibited the results in Figs. 2 and 3, both of which are calculated for $A_2=-0.03$. In Fig. 2, we have given the locus of k in the complex plane as x varies between zero and $\frac{5}{3}$, where Doppler-shifted cyclotron damping destroys the mode. In Fig. 3, the same information appears plotted as the real part of k [Fig. 3(a)] and as the imaginary part of k [Fig. 3(b)] versus x, for a single value of h_2. The dashed line on the same graph indicates the real and imaginary parts of the propagation constant for the GK oscil-

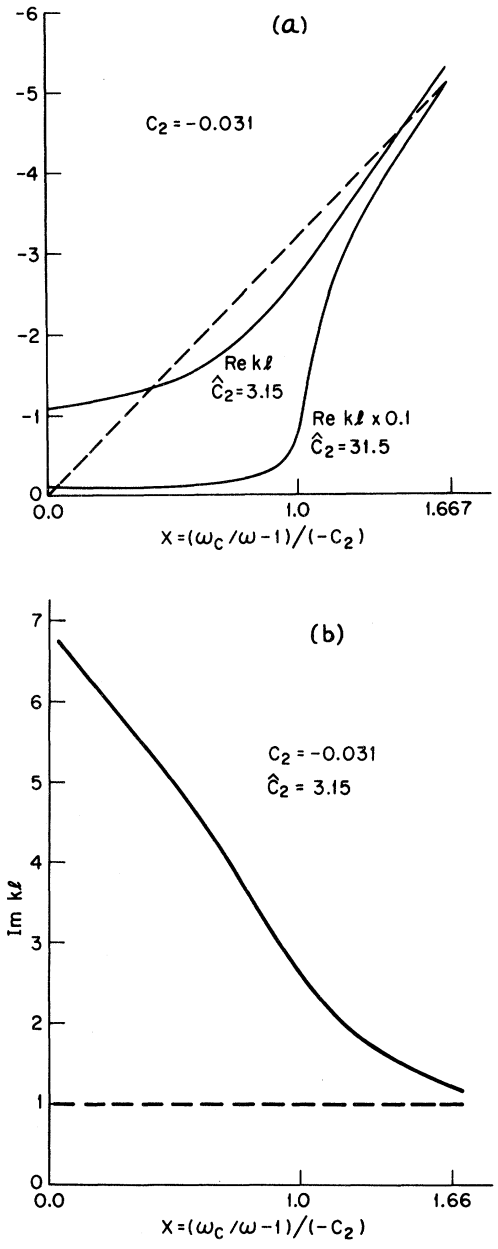

FIG. 3. (a) Real part and (b) imaginary part of the propagation constant for the CCM mode as a function of the dimensionless magnetic field variable $x = (\omega_c/\omega - 1)/(-C_2)$. The values of the parameters chosen for display would correspond to $A_2 = -0.03$ and $l = 0.05$ mm, a value somewhat better than found in good potassium. Also shown in Fig. 3(a) is the dispersion relation for $l = 0.5$ mm, an unrealistically long mean free path, in order to display how the long mean-free-path results approach the infinite mean-free-path curve shown in Ref. 10. The dashed lines give the propagation constant for the Gantmakher-Kaner oscillation. Doppler-shifted cyclotron damping occurs when $|(\mathrm{Re}kl)_{\mathrm{mode}}| > |(\mathrm{Re}kl)_{\mathrm{GK}}|$. Note that the mode is always more heavily damped than the GK oscillations.

lations. It is included to show that the CCM mode is always more strongly damped than the GK oscillations, and, at mode cutoff, has the same wavelength.

The information contained in Fig. 1 is all that one needs to predict the effects of Fermi-liquid correlations on the admittance of a diffuse surface. The quantity usually studied experimentally is the surface impedance, the reciprocal of the admittance. There, the effects of interest are transformed somewhat by the background impedance, i.e., the impedance contributed by the electrons in the metal in the absence of correlation. What we mean is, that although the correlation effects *added* a small term to the surface admittance as in (5.1), the term which augments the surface impedance,

$$Z \equiv Y^{-1} = (Y_0 + \Delta Y)^{-1} \approx Z_0 - Z_0^2 \Delta Y = Z_0 + \Delta Z \ , \tag{5.5}$$

depends explicitly on the value of the surface impedance Z_0 in the absence of correlations. To the order we have been working, Z_0 should, in the second term of (5.5), be considered as being constant, and equal to its extreme anomalous limit value of

$$Z_0 = (\tfrac{1}{2}\sqrt{3})\, e^{-i\pi/3} \left(\frac{3\pi}{4} \frac{\omega_p^2 C}{\omega^2 V_F} \right)^{-1/3} \ . \tag{5.6}$$

The factor $e^{-i\pi/3}$ appearing in (5.6) yields a ΔZ in which the real and imaginary parts are linear combinations of the real and imaginary parts of $\Delta \hat{Y}$, and which scales the result using the ω_p, V_F, and ω appropriate to the particular metal and the particular frequency being studied. In Fig. 4, we have multiplied the $\Delta \hat{Y}$ presented in Fig. 1 by $-e^{-2\pi i/3}$. The resulting curves are proportional to the change in the real and imaginary parts of the surface impedance caused by correlations, the proportionality factor being

$$\left(\frac{-3C\,C_2}{4\,V_F} \right) \left(\frac{3\pi}{4} \frac{\omega_p^2 C}{\omega^2 V_F} \right)^{-2/3} \ . \tag{5.7}$$

The change ΔZ caused by correlations has to be compared with the change in Z_0 which is caused by sweeping the magnetic field (or frequency) through the cyclotron resonance region. The change in Z_0 is conveniently given by using the second term of Dingle's[20] expansion for Z_0 in the anomalous-skin-effect regime. [We take complex conjugates because Dingle used $e^{+i\omega t}$ time dependence, and multiply by $k_0 l = \omega l/C$, in conformity with Eq. (4.17).]

$$\Delta Z(\mathrm{Dingle}) = \frac{\omega l e^{-i\pi/6}}{aC} \left(\frac{3\pi b}{4} \right)^{-2/3}$$

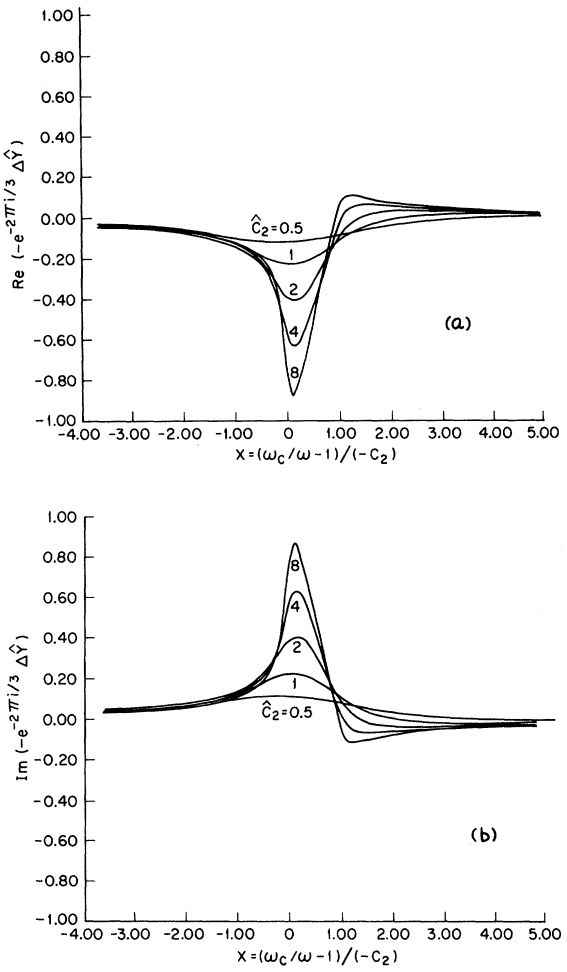

FIG. 4. Real and imaginary parts of $-e^{-2\pi i/3}(\Delta\hat{Y})$ as a function of x and $\hat{C}_2$. The phase shift chosen corresponds to that which would be applied to ΔY, the change in surface admittance, to obtain ΔZ, the change in surface impedance, in the case of a diffuse surface.

$$\times\left[0.5330+0.1351\ln\left(\frac{3\pi b}{4ia^3}\right)^{1/3}\right], \qquad (5.8)$$

where, as in (4.4) and (4.7),

$$b=(\omega\omega_p^2 l^3/V_F C^2) \quad , \qquad (5.9a)$$

$$a=1-i(\omega-\omega_c)\tau \quad . \qquad (5.9b)$$

As the mean free path l becomes longer and longer, ΔZ (Dingle), the variation in surface impedance of a free-electron gas, goes as $1/l$ while ΔZ, (5.5), the change caused by correlations, saturates to a size independent of l. However, at the values of l which are currently available, the correlation-produced changes are typically 10^{-2} of those produced by sweeping the magnetic field through the region where correlation effects are important.

VI. SURFACE ROUGHNESS, EXPERIMENT, AND CONCLUSIONS

The calculations presented to this point have focused on the change in the surface admittance of a diffuse surface when the magnetic field, normal to the surface, is swept through the cyclotron resonance region. The changes arise from two separate effects. First, the properties of the conduction-electron gas bounded by the surface, treated as a gas of noninteracting particles, are influenced by the steady magnetic field. This effect has long been known to produce a smooth monotonic increase in the size of the real part of the surface impedance as the magnetic field increases from zero in the direction of cyclotron resonance. The second, and by far the more interesting effect, arises because correlations between the electrons give rise to the possibility that a relatively undamped wave can propagate normal to the surface when the magnetic field is close to, but not exactly at, the value for cyclotron resonance. Our analytic result describing the size of the latter change, Eq. (4.21), exhibits the following features:

(1) The over-all change is greatest at cyclotron resonance, and decreases like $(\omega_c/\omega-1-i/\omega\tau)^{-1}$ as the field or frequency departs from resonance.

(2) There is a part of the admittance change which is conductive or susceptive, depending on whether the mode is propagating or evanescent, the crossover occurring at threshold.

(3) To lowest order in (anomalous skin depth/wavelength of the mode)$^{1/2}$, the structure in the admittance, like the dispersion relation for the mode, does not depend on the A_1 correlation parameter.

Our numerical results, Fig. 1 and Eqs. (5.5)–(5.7), indicate that the change in surface impedance caused by correlations will be enhanced by about three orders of magnitude over what they are calculated to be for a specular surface. This is surprising; it raises the question of why this occurs, since, as is well known, the surface impedance itself is calculated to be rather independent of whether the surface is assumed diffuse or specular. We shall return to this question almost immediately below.

There are at present no experimental results with which the predictions of this calculation can be directly compared. The single measurement of surface impedance as a function of magnetic field thus far reported for an alkali metal near cyclotron resonance[14] has an over-all shape which is far better understood on the assumption that the surface of the metal was rough, rather than that it was diffuse.[15] There was structure reported in

this measurement, a sharp peak in the real part of the surface resistance slightly above cyclotron resonance and a slight dip at cyclotron resonance. The height of the peak was roughly comparable to the size of the change in the background surface impedance caused by reducing the field from its value at cyclotron resonance to zero. This result, obtained from measurements on what appeared to be a rough surface, is $\sim 10^2$–10^3 times larger than what we would have expected from these calculations, had the surface been diffuse. The shape of the observed structure does not correspond at all with what is shown in Fig. 4(a) as the real part of the impedance of a diffuse surface.

Since the size of the correlation effect on the impedance is so drastically affected by whether the surface is assumed specular or diffuse, we must ask whether the sensitivity to surface conditions would reconcile calculation and observation if the surface were rough. In other words, how would the assumption of a rough surface, rather than a diffuse one, change the size and shape of the structure we calculate? We cannot answer this question exactly, but we do wish to make the following observations which may suggest physically what is going on:

(1) The expression (5.1), for the size of the structure contributed by the mode, contains factors M and J, each of which is more or less the Fourier transform of the field calculated at the small k values characteristic of the mode; i.e., it is as though M and J were overlap integrals between the *unperturbed* field and the field of the mode. This overlap integral appears twice as a factor of ΔY.

(2) The enhancement factor by which the structure in the diffuse calculation, given here, exceeds that in the specular calculation, estimated in Ref. 14, is essentially the square of the factor by which the long-range (i.e., low-k) part of the unperturbed diffuse field exceeds the long-range part of the unperturbed specular field.

These two observations suggest that, if the effect of a rough surface is to decrease the shielding currents and enhance the deep field penetration, the structure contributed by the correlations will be multiplied by the square of the enhancement factor. A real enhancement factor would increase the size of the structure. A complex enhancement factor would, besides increasing the size of the structure, cause a mixing between the real and imaginary parts of ΔZ.

Our preliminary calculations of enhancement factors using the surface roughness model[15] indicate that there can be a large enhancement factor. The enhancement factors which emerge are complex, with the phase not too strongly dependent on the applied magnetic field. One thus expects to observe a surface resistance which is a linear combination of the real and imaginary parts of the calculated ΔZ. The amount of the phase shift and the size of the enhancement depend on the details of the parameters describing the roughness. For our purposes here, it suffices to characterize the effect of the roughened surface simply as a field-independent phase shift of the long-range tail of the unperturbed fields. A phase shift of $e^{-i\pi/3}$ applied to the field gives the surface resistance curve exhibited in Fig. 5, which exhibits structure quite similar to that reported in Ref. 14. The various curves given in Fig. 5 correspond to different mean free paths and depict, therefore, how the impedance change of the rough surface might change with temperature. We must emphasize, however, that the details of the coupling between the fields outside a physically imperfect surface and the electron motion inside the bulk is still a difficult and unsolved problem. Its ultimate solution is likely to involve details of the surface which are not of any deep significance to the bulk problems usually considered.

Although the phase shift chosen for the purpose of display in Fig. 5 was chosen to optimize the fit between the calculation and experiment, the underlying calculation of $\Delta\hat{Y}$ shows that there are only two field values, cyclotron resonance and threshold (rather than mode cutoff by Doppler-shifted cyclotron damping), where structure occurs. It may well turn out that when the surface impedance of a rough surface in the presence of correlations is calculated properly, this feature will remain. In that case, the sharp peak reported in the observa-

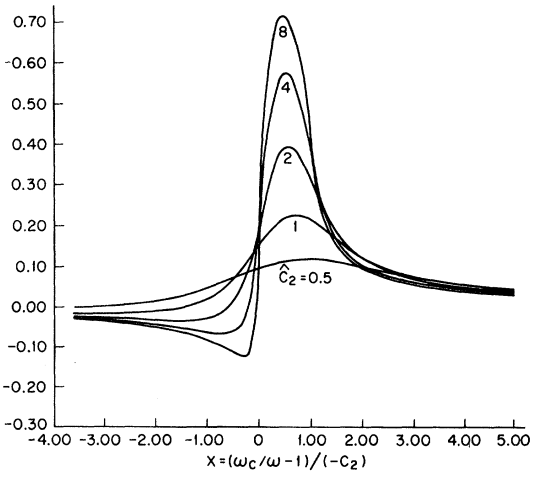

FIG. 5. Real part of the change in surface impedance as calculated assuming that the fields in the interior of the metal suffer a phase shift of $e^{-i\pi/3}$ caused by surface roughness.

tions in Ref. 14 is properly identified as threshold. Such identification leads to a value of the Landau parameter A_2 equal to about -0.03. This is in substantial agreement with that obtained by observing the HFW which propagate across the magnetic field, and whose dynamics are totally different from those of the CCM mode which propagates along the field. On the other, if the field enhancement idea underlying Fig. 5 turns out to be substantially correct, then the peak in the real part of the surface resistance occurs at $x=0.5$–0.7 rather than at $x=1.0$. This would yield a value of A_2 some 1.5 to 2.0 times the value implied by having the peak at $x=1.0$. This discrepancy is probably within the range of uncertainty occasioned by imprecision in the field value which should be assigned to cyclotron resonance.

Note added in proof. The field enhancement calculations have been completed, and will appear shortly. They yield an enhancement factor which does depend on magnetic field and which, in the neighborhood of the threshold, is very close to the $e^{-i\pi/3}$ chosen here for display purposes. Using these calculations, the peak appears at $x=1.0$, indicating that $A_2 \approx -0.03$, and confirming the value deduced from HFW. The uncertainty in size of A_2 could be substantially reduced by measuring both the real and imaginary parts of ΔZ, taking the linear combination of the two which resulted in the most dispersive line shape, and measuring the separation between the positive and the negative peaks. According to Fig. 1(a), the separation will tend toward $\Delta x=1$, i.e., to $\Delta\omega_c/\omega = C_2$, as the mean free path increases. For finite mean free path, the separation is greater than $\Delta x=1$, and this will establish an upper bound on the size of C_2.

A description of the way the surface roughness model may be used to calculate the deep field enhancement factor will appear shortly.

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATION OF ADMITTANCE IN ANOMALOUS-SKIN-EFFECT REGIME

Our first step is to evaluate M: We write (4.12) in a symbolic form as

$$M[\Phi,\psi] \equiv (\Phi|M|\psi) = (\Phi^0|M|\psi^0) + (\Phi^0|M|\psi^c) + (\Phi^c|M|\psi^0) + (\Phi^c|M|\psi^c) \ , \tag{A1}$$

and we write the matrix operator as

$$M = M^0 + \Delta M \ , \tag{A2}$$

where M^0 are the parts of M which survive when there are no correlations and ΔM is the rest – parts which have h_1 or h_2 as a factor. Because Φ^0 and ψ^0 are exact solutions, they render M_0 stationary and, as a result,

$$(\Phi^0|M^0|\psi^0+\psi^c) \equiv (\Phi^0+\Phi^c|M^0|\psi^0) = (\Phi^0|M^0|\psi^0) \equiv M_0 \ . \tag{A3}$$

Hence,

$$M[\Phi,\psi] = M_0 + (\Phi^0|\Delta M|\psi^0) + (\Phi^c|M|\psi^c) + (\Phi^c|\Delta M|\psi^0) + (\Phi^0|\Delta M|\psi^c) \ . \tag{A4}$$

Using (4.18), we now have

$$(\Phi^c|M|\psi^c) = \sum_{i,j}^{3,3} [a_i f_{ij}(p,r)\, c_j + a_i f_{ij}(p,s)\, d_j + b_i f_{ij}(q,r)\, c_j + b_i f_{ij}(q,s)\, d_j] \ , \tag{A5}$$

where

$$f_{ij}(p,r) = \begin{pmatrix} i/(p+r) + H_{11}(p,r)\,h_1 & H_{12}(p,r)\,h_2 & H_{11}(p,r) \\ H_{21}(p,r)\,h_1 & i/(p+r) + H_{22}(p,r)\,h_2 & H_{21}(p,r) \\ b/(p+r) & 0 & ipr/(p+r) + ik_0^2 l^2/(p+r) \end{pmatrix} \ , \tag{A6}$$

and the integrals $H_{ij}(p,r)$ are defined as

$$H_{ij}(p,r) = \int_0^\infty \int_0^\infty dx\,dx'\, e^{ipx} K_{ij}(x-x') e^{irx'} \ . \tag{A7}$$

We also have

$$(\Phi^c|\Delta M|\psi^0) = \sum_{i,j}^{2} [a_i S_{ij}(p) + b_i S_{ij}(q)] \ , \tag{A8}$$

$$(\Phi^0|\Delta M|\psi^c) = \sum_{ij} [c_j T_{ij}(r) + d_j T_{ij}(s)] \ , \tag{A9}$$

where

$$S_{ij}(p) = \int_0^\infty \int_0^\infty dx\,dx'\, e^{ipx} K_{ij}(x-x') h_j \psi_j^0(x') \ , \tag{A10}$$

$$T_{ij}(r) = \int_0^\infty \int_0^\infty dx\,dx'\,\Phi_i^0(x) K_{ij}(x-x') h_j e^{irx'} \quad . \tag{A11}$$

We use (4.19) to eliminate b_3 and d_3, and introduce the following notation for the independent parameters:

$$(U_1, U_2, U_3, U_4, U_5) \equiv (c_1, c_2, c_3, d_1, d_2) \quad , \tag{A12a}$$

$$(U_1^\dagger, U_2^\dagger, U_3^\dagger, U_4^\dagger, U_5^\dagger) \equiv (a_1, a_2, a_3, b_1, b_2) \, , \tag{A12b}$$

$$(k_1, k_2, k_3, k_4) \equiv (k) = (p, q, r, s) \quad . \tag{A12c}$$

This casts (A4) into the form

$$M = M_0 + (\Phi^0 | \Delta M | \psi^0) + \underline{U}^\dagger \cdot \underline{\underline{g}}(k) \cdot \underline{U} + \underline{U}^\dagger \cdot \underline{s}(k) + \underline{t}(k) \cdot \underline{U} \quad . \tag{A13}$$

The $g_{ij}(k)$ are linear combinations of the f_{ij} – linear combinations which arise by eliminating b_3 and d_3 from (A5). The $s(k)$ and $t(k)$ are also linear combinations of the S_{ij} and T_{ij}, respectively.

From $\delta M/\delta U^\dagger = 0$, we obtain

$$\underline{\underline{g}}(k) \cdot \underline{U} + \underline{s}(k) = 0$$

$$\text{or} \quad \underline{U} = -\underline{\underline{g}}^{-1}(k) \cdot \underline{s}(k) \quad . \tag{A14}$$

Similarly from $\delta M/\delta U = 0$, we obtain

$$\underline{U}^\dagger = -\underline{t}(k) \cdot \underline{\underline{g}}^{-1}(k) \quad . \tag{A15}$$

Finally, the variation of (A13) with respect to k_i is set equal to zero.

$$U^\dagger \left(\frac{\partial}{\partial k_i} \underline{\underline{g}}(k) \right) \cdot \underline{U} + \underline{U}^\dagger \cdot \left(\frac{\partial}{\partial k_1} \underline{s}(k) \right) + \left(\frac{\partial}{\partial k_i} \underline{t}(k) \right) \cdot \underline{U} = 0 \quad . \tag{A16}$$

We substitute (A14) and (A15) into (A16) and, using the operator identity

$$\frac{\partial}{\partial k_i} \underline{\underline{g}}^{-1}(k) = -\underline{\underline{g}}^{-1} \left(\frac{\partial}{\partial k_i} \underline{\underline{g}} \right) \underline{\underline{g}}^{-1} \, ,$$

Eq. (A16) becomes

$$-\frac{\partial}{\partial k_i} [\underline{t}(k) \cdot \underline{\underline{g}}^{-1}(k) \cdot \underline{s}(k)] = 0 \quad . \tag{A17}$$

We may also substitute (A15) and (A14) into (A13), with the result

$$M = M_0 + (\Phi^0 | \Delta M | \psi^0) - \underline{t}(k) \cdot \underline{\underline{g}}^{-1}(k) \cdot \underline{s}(k) \quad . \tag{A18}$$

In (A17), we have four equations which fix the four unknowns k_i. Having fixed them, we may evaluate (A18). It is interesting to note that if we had regarded (A18) as a trial form for M (never mind where it came from) then (A17) would be the associated stationarity condition.

These equations, involving derivatives of the inverse of a 5×5 matrix of known but rather complicated functions, simplify remarkably if we take advantage of the fact that they are to be used in the anomalous-skin-effect regime. Here, the number b will be huge – typically of order 10^6 to 10^9 depending on mean free path, and, as we shall now show, the number $b^{-1/3}$ (essentially, the anomalous skin depth in units of the mean free path), provides a useful expansion parameter.

Consider the trial fields proposed in (4.18b): One effect of correlation is to allow the system to support a new mode whose fields are of relatively long range and long wavelength compared to ψ^0, the fields in the absence of correlation. Hence, one of the propagation constants in (4.18b), say, r, should be of the order of the k of the mode. (In our units, k will be of order ia.) Another effect of correlations is to change the conductivity slightly, affecting the shielding currents and altering the fields within the anomalous skin depth. This change can be represented in (4.18b) only if the other propagation constant, say, s, is of the order of an inverse anomalous skin depth, namely, $b^{1/3}$. The close relation between the adjoint fields Φ and the physical fields ψ leads us to surmise that the propagation constants p and q appearing in (4.18a) will be roughly the same as r and s. On this basis, we may expect p and r to be of order ia while q and s will be of order $b^{1/3}$. This is a large disparity in size which we can exploit in evaluating the matrix $\underline{\underline{g}}$ and taking its inverse.

The specific form for $\underline{\underline{g}}$ appears when we compare (A5) with (A13), using (A12) and (4.19) to effect the comparison. We obtain

$$\underline{\underline{g}}(k) = \begin{pmatrix} f_{11}^1 & f_{12}^1 & f_{13}^1 - f_{13}^2 & f_{11}^2 & f_{12}^2 \\ f_{21}^1 & f_{22}^1 & f_{23}^1 - f_{23}^2 & f_{21}^2 & f_{22}^2 \\ f_{31}^1 - f_{31}^3 & f_{32}^1 - f_{32}^3 & f_{33}^1 - f_{33}^2 - f_{33}^3 + f_{33}^4 & f_{31}^2 - f_{31}^4 & f_{32}^2 - f_{32}^4 \\ f_{11}^3 & f_{12}^3 & f_{13}^3 - f_{13}^4 & f_{11}^4 & f_{12}^4 \\ f_{21}^3 & f_{22}^3 & f_{23}^3 - f_{23}^4 & f_{21}^4 & f_{22}^4 \end{pmatrix} \, , \tag{A19}$$

where we have used a superscript to denote the arguments of the f_{ij} as follows:

$$f^1_{ij} \equiv f_{ij}(p, r), \quad f^2_{ij} \equiv f_{ij}(p, s), \quad f^3_{ij} \equiv f_{ij}(q, r), \quad f^4_{ij} \equiv f_{ij}(q, s) . \tag{A20}$$

If we examine the form of f given by (A6) and (A7), it is clear that, with the exception of the (3.3) element, all other elements of $f_{ij}(p, r)$ have either a $(p+r)^{-1}$ or a $(pr)^{-1}$ behavior. On this basis, we can state that f^1 is of the order of $\zeta^{-1} \equiv b^{1/3}/ia$ larger than f^2, f^3, or f^4.

We may now write $\underline{\underline{g}}$ and $\underline{\underline{g}}^{-1}$ in partitioned form as

$$\underline{\underline{g}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \underline{\underline{g}}^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} , \tag{A21}$$

where A and E are 3×3 matrices while D and H are 2×2 matrices. Using our estimates of the sizes of f^1, f^2, f^3, and f^4, in (A19), we find that the matrices A, B, C, D stand roughly in the size

$$A{:}B{:}C{:}D = \zeta^{-1}{:}1{:}1{:}1 . \tag{A22}$$

The components of $\underline{\underline{g}}^{-1}$ are given by

$$E = (A - BD^{-1}C)^{-1} ,$$

$$F = -A^{-1}B(D - CA^{-1}B)^{-1} ,$$

$$G = -D^{-1}C(A - BD^{-1}C)^{-1} ,$$

$$H = (D - CA^{-1}B)^{-1} .$$

By partitioning $\underline{s}$ and $\underline{t}$ in the same way,

$$\underline{s} = (J/L) , \quad \underline{t} = (M|N) , \tag{A23}$$

we obtain

$$\underline{t}\cdot\underline{\underline{g}}^{-1}\cdot\underline{s} = (M - ND^{-1}C)(A - BD^{-1}C)^{-1}J + (N - MA^{-1}B)(D - CA^{-1}B)^{-1}L .$$

However, to lowest order in ζ, the estimate (4.41) tells us that

$$(A - BD^{-1}C)^{-1} = A^{-1} ,$$

$$(D - CA^{-1}B)^{-1} = D^{-1} .$$

Furthermore, to lowest order in ζ, the matrix D is a multiple of the unit matrix: $D = i/(q+s)\times1 = d\times1$.

Hence,

$$\underline{t}\cdot\underline{\underline{g}}^{-1}\cdot\underline{s} = (M - NC/d)A^{-1}J + (N - MA^{-1}B)L/d . \tag{A24}$$

The disparity in sizes of p and r versus q and s can also be expected to be reflected in the relative sizes of $J{:}L$ and $M{:}N$. By comparing (A8) and (A9) with (A13), using (A12) to effect the comparison, we have

$$\begin{bmatrix} S_{11}(p) + S_{12}(p) \\ S_{21}(p) + S_{22}(p) \\ 0 \\ S_{11}(q) + S_{12}(q) \\ S_{21}(q) + S_{22}(q) \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ L_1 \\ L_2 \end{bmatrix} , \tag{A25}$$

$$\begin{bmatrix} T_{11}(r) + T_{21}(r) \\ T_{12}(r) + T_{22}(r) \\ 0 \\ T_{11}(S) + T_{21}(S) \\ T_{12}(S) + T_{22}(S) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ N_1 \\ N_2 \end{bmatrix} \tag{A26}$$

One might expect, from considering the forms (A10) and (A11), that each term $S_{ij}(k)$ or $T_{ij}(k)$ has a $1/k$ dependence at large k, and on that basis, one would have $J{:}L \approx M{:}N \approx 1{:}\zeta$. The S_{ij} integrals do behave that way, and the estimate $J{:}L \approx 1{:}\zeta$ is correct. The T_{ij} integrals do not take on this limiting behavior until $k \approx b^{1/3}$, and it turns out that $M{:}N \approx 1{:}\zeta^{1/2}$. The only convincing way to obtain this result is to carry out the evaluation of S and T, which we describe briefly below, and carry out in more detail in later Appendixes. Using these estimates, however, (A24) reduces to

$$\underline{t}\cdot\underline{\underline{g}}^{-1}\cdot\underline{s} = MA^{-1}J + N(L - CA^{-1}J)/d , \tag{A27}$$

and the second term is of order $\zeta^{1/2}$ smaller than the first.

Even the labor of evaluating (A27) for use in (A17) and (A18) is substantially reduced by the circumstance that A has one element, (3, 1), so much larger than the others that it dominates the inversion. Deleting from A^{-1} all terms of order ζ, we have

$$A^{-1} = (f^1_{12}f^1_{23} - f^1_{13}f^1_{22})^{-1} \begin{pmatrix} 0 & 0 & 0 \\ f^1_{23} & -f^1_{13} & 0 \\ -f^1_{22} & f^1_{12} & 0 \end{pmatrix} . \tag{A28}$$

Finally, the evaluation of the integrals which make up s and t must be considered briefly: By setting $h_j = 0$ in (4.3) we obtain

$$\psi^0_j(x) = -\int_0^\infty K_{j1}(x - x')\psi^0_3(x')\,dx', \quad j = 1, 2 \tag{A29}$$

while ψ^0_3, the electric field in the absence of correlations, satisfies the familiar integrodifferential equation[13]

$$\left(\frac{d^2}{dx^2} + k_0^2 l^2\right)\psi^0_3(x) + ib\int_0^\infty K_{11}(x - x')\psi^0_3(x')\,dx' = 0,$$

$$\psi^0_3(0) = 1 . \tag{A30}$$

By setting $h_j = 0$ in (4.6), we obtain

$$\Phi_1^0(x) = ib\Phi_3^0(x) \quad , \qquad \Phi_2^0(x) = 0 \quad , \tag{A31}$$

while Φ_3^0 satisfies the same equation and boundary conditions as does ψ_3^0 and hence is identical to it. The integrals in (A9) and (A10) thus become

$$S_{ij}(p) = -h_j \int_0^\infty \int_0^\infty \int_0^\infty dx\,dx'\,dx'' \times e^{ipx} K_{ij}(x-x') K_{j1}(x'-x'') \psi_3^0(x'') \ , \tag{A32a}$$

$$T_{1j}(r) = ibh_j \int_0^\infty \int_0^\infty dx\,dx'\,\psi_3^0(x) K_{1j}(x-x') e^{irx'} . \tag{A32b}$$

If the kernel which operates on $\psi_3^0(x)$ is K_{11}, we can use (A30) to effect the integration. If the kernel is some K_{ij} other than K_{11}, then we express the kernel as a spatial integral of K_{11} (Appendix C) and again use (A30). In this way, we arrive eventually at expressions from which the kernels have been eliminated, i.e., essentially the Fourier transform of ψ_3^0. This transform is available from the Wiener-Hopf solution of (A30) (Appendix D), and we use it to evaluate the integrals (A32) in Appendix E.

APPENDIX B: ANALYTIC SOLUTION IN IMMEDIATE VICINITY OF ANOMALY

The derivation of (A27) required that

$$\zeta \approx \frac{(\omega - \omega_c)/V_F}{(\omega\omega_p^2/V_F C^2)^{1/3}}$$

be small in comparison with unity.[19] Typical values of this parameter are so small that, in the neighborhood of cyclotron resonance over which the mode propagation occurs, $\sqrt{\zeta}$ can also be used as a small (but not terribly small) parameter. By neglecting the second term in (A27) which is of order $\sqrt{\zeta}$ times the first, we can obtain a solution of (A17) and an evaluation of (A18) which are of interest in their own right. Should the labor of working with the full (A27) be deemed rewarding, this solution can be used as a starting point for tackling the more general problem.

The first term in (A27),

$$\underline{t} \cdot \underline{\underline{g}}^{-1} \cdot \underline{s} \approx MA^{-1}J \quad , \tag{B1}$$

depends only on p and r, not on s and q. Hence, to order ζ, the two variational equations

$$\frac{\partial}{\partial q}(\underline{t} \cdot \underline{\underline{g}}^{-1} \cdot \underline{s}) = 0$$

and

$$\frac{\partial}{\partial s}(\underline{t} \cdot \underline{\underline{g}}^{-1} \cdot \underline{s}) = 0$$

are satisfied automatically. The remaining two variational equations,

$$\frac{\partial}{\partial p}(\underline{t} \cdot \underline{\underline{g}}^{-1} \cdot \underline{s}) = 0 \quad , \tag{B2a}$$

$$\frac{\partial}{\partial r}(\underline{t} \cdot \underline{\underline{g}}^{-1} \cdot \underline{s}) = 0 \quad , \tag{B2b}$$

are now to be studied.

In evaluating (B1), it turns out to be useful to remove a factor of $i/(p+r)$ from the $H_{mn}(p, r)$ integrals defined by (A7), so that we have functions $D_{mn}(p, r)$ defined by

$$H_{mn}(p, r) = i D_{mn}(p, r)/(p+r) \quad . \tag{B3}$$

Using (A28) in (B1), and referring to (A6), we obtain

$$MA^{-1}J = \frac{i(p+r)M_2(r)[D_{21}(p, r)J_1(p) - D_{11}(p, r)J_2(p)]}{D_{11}(p, r) + h_2[D_{11}(p, r)D_{22}(p, r) - D_{12}(p, r)D_{21}(p, r)]} \quad . \tag{B4}$$

There is already, at this stage, a hint of the CCM mode lurking in (B4), for if one sets $p + r = 0$ so that the numerator vanishes, $MA^{-1}J$ is zero *unless* the denominator also vanishes, *which it does if either p or r satisfies the dispersion relation given by* CCM (Appendix C). The k of the CCM mode vanishes when $\omega_c/\omega = (1 + A_2)^{-1}$. This suggests that near $\omega_c/\omega = (1 + A_2)^{-1}$, we can also expect p and r to be small and suggests the utility of a power-series expansion. Accordingly, we introduce new independent variables, R and ρ, defined by

$$\rho \equiv \tfrac{1}{2}(r + p), \quad R \equiv \tfrac{1}{2}(r - p) \ , \tag{B5}$$

in terms of which (B2) becomes

$$\frac{\partial}{\partial \rho}(MA^{-1}J) = 0 \quad , \tag{B6a}$$

$$\frac{\partial}{\partial R}(MA^{-1}J) = 0 \quad . \tag{B6b}$$

We rewrite (B4) in terms of these variables as

$$MA^{-1}J = \rho N(R, p)/D(R, \rho) \quad , \tag{B7}$$

where

$$N(R, \rho) = 2iM_2(r)[D_{21}(p, r)J_1(p) - D_{11}(p, r)J_2(p)] \quad , \tag{B8a}$$

$$D(R, \rho) = D_{11}(p, r) + h_2[D_{11}(p, r)D_{22}(p, r) - D_{12}(p, r)D_{21}(p, r)] \quad . \tag{B8b}$$

The numerator and denominator of (B7) are to be ex-

panded about $R=0$, $\rho=0$, and terms up to second order are to be retained. Thus, we need terms up to first order in N,

$$N(R,\rho)=N_{00}+N_{10}R+N_{01}\rho \quad , \tag{B9}$$

and terms up to second order in D. However, the symmetry (4.6) results in D being even with respect to interchange of p and r, so no odd powers of R can appear in the expansion. To second order, then,

$$D(R,\rho)=d_{00}+d_{01}\rho+d_{02}\rho^2+d_{20}R^2 \quad . \tag{B10}$$

We now insert (B9), (B8), and (B7) into (B6), and retain terms up to second order in the coefficient of $D(R,\rho)^{-2}$. This gives us

$$d_{00}N_{00}+RN_{10}d_{00}+2\rho N_{01}d_{00}+R^2N_{00}d_{20}+\rho^2(N_{01}d_{01}-N_{00}d_{02})=0 \quad , \tag{B11a}$$

$$\rho(N_{10}d_{00}+\rho N_{10}d_{01}-2RN_{00}d_{20})=0 \quad . \tag{B11b}$$

It is clear that there will be small R and ρ solutions only if d_{00} is small, and, since d_{00} is smallest when $\omega_c/\omega=(1+A_2)^{-1}$, this is consistent with what we expected. If we regard d_{00} as being first order small, i.e., like R or ρ, then all terms in (B11a) are second order small *except* for $d_{00}N_{00}$ and no solution is possible. Hence, d_{00} must be *second* order small, i.e., like R^2 or $R\rho$ or ρ^2. If we delete quantities third order small from (B11a) and (B11b), there remains

$$d_{00}N_{00}+R^2N_{00}d_{20}+\rho^2(N_{01}d_{01}-N_{00}d_{02})=0 \ ,$$

$$\rho N_{10}d_{01}-2RN_{00}d_{20}=0 \ ,$$

which gives the solution

$$R=\rho N_{10}d_{01}/2N_{00}d_{20} \quad , \tag{B12a}$$

$$\rho^2=\frac{-d_{00}/d_{01}}{(N_{10}/2N_{00})^2\,d_{01}/d_{20}+N_{01}/N_{00}-d_{02}/d_{01}} \ . \tag{B12b}$$

Using (B9) and (B10) in (B7),

$$MA^{-1}J=-\left(\frac{N_{00}+(N_{10}R+N_{01}\rho)}{d_{01}+(d_{00}+d_{02}\rho^2+d_{20}R^2)/\rho}\right), \tag{B13}$$

which is of the form

$$\frac{N_{00}}{d_{01}}\left(\frac{1+A(d_{00}/d_{01})^{1/2}}{1+B(d_{00}/d_{01})^{1/2}}\right) \quad , \tag{B14}$$

where A and B can be determined by comparing (B12), (B13), and (B14).

It is possible to evaluate the coefficients appearing in (B9) and (B10), making use of the material in Appendixes C, D, and E. Using this material, we have

$$d_{00}/d_{01}=(1-\eta)/\beta(\eta-\tfrac{3}{8}) \quad , \tag{B15a}$$

where $\beta=ia=(\omega-\omega_c)\tau+i$, (B15b)

$$\eta=-ih_2/\beta=\frac{-A_2(1+A_2)^{-1}}{(\omega_c/\omega-1)-i/\omega\tau} \quad . \tag{B15c}$$

In the large $\omega\tau$ limit, d_{00} vanishes at $\eta=1$, which we designate as threshold. Evaluating (B14) in the limit $\eta\to 1$, we obtain

$$MA^{-1}J=\frac{-h_2^2}{i\beta}\left(\frac{a+a_2(1-\eta)^{1/2}}{a_3+a_4(1-\eta)^{1/2}}\right) \quad , \tag{B16}$$

where a_1, a_2, a_3, and a_4 are *real* numbers of order unity.

This derivation has been based on the smallness of $1-\eta$ which, considering both real and imaginary parts, means that $\omega_c/\omega\approx(1+A_2)^{-1}$ and $\omega\tau$ much greater than 1. The contribution of (B16) to the surface admittance (A8) in this limit is

$$\Delta Y^{(2)}\equiv(MA^{-1}J)/ik_{01}l$$

$$=-\left(\frac{C}{V_F}\right)\left(\frac{A_2}{1+A_2}\right)^2\left(\frac{a_1+a_2(1-\eta)^{1/2}}{a_3+a_4(1-\eta)^{1/2}}\right)\left(\frac{\omega}{\omega_c}-1-\frac{-i}{\omega\tau}\right)^{-1} \tag{B17}$$

$$\approx\left(\frac{C}{V_F}\right)\left(\frac{A_2}{1+A_2}\right)^2\left(\frac{a_5+a_6(1-\eta)^{1/2}}{(\omega_c/\omega-1)-i/\omega\tau}\right) \quad , \tag{B18}$$

where again, a_5 and a_6 are real numbers of order unity. Equation (B18) appears as (4.21) in the text.

APPENDIX C: PROPERTIES OF KERNELS

Using the definition of the first two $(l,1)$ spherical harmonics below,

$$Y_{11}(\vec{\Omega})=(3/8\pi)^{1/2}\sqrt{(1-u^2)}e^{i\varphi} \quad , \tag{C1a}$$

$$Y_{21}(\vec{\Omega})=(15/8\pi)^{1/2}\sqrt{(1-u^2)}e^{i\varphi} \quad , \tag{C1b}$$

where $u=\cos\theta$, and the definition (3.2) of the kernels, we have

$$K_{ij}(x-x')=\int_0^1 u^{-1}\,du f_m(u)\exp[-a(x-x')/u], \quad x>x' \tag{C2a}$$

$$K_{ij}(x-x')=(-1)^m K_{ij}(x'-x), \quad x'>x \tag{C2b}$$

where $m=i+j-2$ (C3)

and $f_m(u)=\frac{3}{4}(5^{m/2}u^m)(1-u^2)$. (C4)

The single subscript m provides as much information as the double subscript i, j and will be used here. Since we use kernels $K_m(x-x')$ in which $m=0,1,2$, we are free to define kernel $K_3(x-x')$ by extending definitions (C2) and (C4) to $m=3$. The kernel, so defined, does not correspond to the definition (3.2). The utility of the extended definition is that using (C2) and (C4) one easily obtains

$$K_{m+1}(x-x')=a\sqrt{(5)}\int_x^\infty dx''\,K_m(x''-x')$$

$$-\theta(x'-x)[K_{m+1}(0^+)-K_{m+1}(0^-)] \tag{C5}$$

and

$$\int_0^{x'} K_m(x-x'')dx'' = \frac{1}{a\sqrt{(5)}}\{K_{m+1}(x-x') - K_{m+1}(x) + \theta(x'-x)[K_{m+1}(0^+) - K_{m+1}(0^-)]\}, \quad \text{(C6)}$$

where $\theta(y) = 1, \quad y > 0$
$= 0, \quad y < 0$. (C7)

By extending the definition, we can use (C5) and (C6) for $m = 2$ as well as for $m = 0, 1$. The use of (C5) and (C6) is invaluable in evaluating the integrals (A32).

Using the definition (C2), we can evaluate the H_{ij} integrals (A7) and the D_{ij} functions (B3) as

$$H_{ij}(p, q) = iD_{ij}(p, q)/(p+q) \quad ,$$
$$D_{ij}(p, q) = d_m(p) + (-1)^m d_m(q) \quad , \quad \text{(C8)}$$

where $d_m(p) = \frac{i}{p}\int_0^1 (u + ia/p)^{-1} f_m(u)\,du$, (C9)

where $m = i + j - 2$. It is also a simple matter to show that

$$D_{ij}(p, -p) = \int_{-\infty}^{\infty} K_{ij}(x)e^{ipx}dx \quad , \quad \text{(C10)}$$

i.e., that $D(p, -p)$ is the Fourier transform of the kernel. It is for this reason that setting $r = -p$ in the denominator of (B4) leads to an expression involving the Fourier transforms of the kernels – quantities which are normally associated with infinite-medium propagation studies. Having established this, the identity between zero of the denominator of (B4) and the CCM dispersion relation is trivial to demonstrate.

APPENDIX D: EVALUATION OF SOURCE INTEGRALS

We are concerned with the $S_{ij}(p)$ and $T_{ij}(r)$ integrals defined by (A32). We shall demonstrate here how these can be evaluated by carrying through that evaluation of enough of them to illustrate each of the steps involved:

$$T_{12}(r) \equiv ibh_2 \int_0^{\infty} dx \int_0^{\infty} dx'\, \psi_3^0(x) K_1(x-x')e^{irx'} . \quad \text{(D1)}$$

The subscript 1 on the kernel follows the notation convention of Appendix C, namely, $i, j \to i + j - 2$. Using the fact that K_1 is odd, this is also

$$T_{12}(r) = -ibh_2 \int_0^{\infty} dx \int_0^{\infty} dx'\, e^{irx} K_1(x-x')\psi_3^0(x') ,$$

or, using (C5),

$$T_{12}(r) = -ibh_2 a\sqrt{(5)} \int_0^{\infty} dx \int_0^{\infty} dx' \int_x^{\infty} dx'' \times e^{irx} K_0(x''-x')\psi_3^0(x') + 2ibh_2 K_1(0^+) \times \int_0^{\infty}\int_0^{\infty} dx\,dx'\, e^{irx}\theta(x'-x)\psi_3^0(x') . \quad \text{(D2)}$$

The equation governing ψ_3^0 is (A30). In this notation, this is

$$\frac{d^2}{dx^2}\psi_3^0(x) = -ib\int_0^{\infty} K_0(x-x')\psi_3^0(x')\,dx' \quad , \quad \text{(D3)}$$

wherein we have dropped the displacement current term $k_0^2 l^2$ as being numerically unimportant. Using (D3) in the first term of (D2) performs the x' integration and leaves $d^2\psi_3^0(x'')/d(x'')^2$ in the integrand. The x'' integration converts this to a first derivative with respect to x. The derivative with respect to x in the integral over x is removed by integration by parts. Hence, the first term in (D2) is

$$h_2 a\sqrt{(5)}[\psi_3^0(x=0) + ir\psi(r)] \quad , \quad \text{(D4)}$$

where $\psi(p) \equiv \int_0^{\infty} e^{ipx}\psi_3^0(x)\,dx$ (D5)

is the Fourier transform of the electric field $\psi_3^0(x)$. The initial value of this field is unity, and this value should appear in (D4). In the second term in (D2), we perform the x integration first:

$$\int_0^{\infty}\int_0^{\infty} dx\,dx'\, e^{irx}\theta(x'-x)\psi_3^0(x') = \int_0^{\infty} dx'\psi_3^0(x')\int_0^{x'} e^{irx}dx = \frac{1}{ir}[\psi(r) - \psi(0)] .$$

Thus, $T_{12}(r) = h_2 a\sqrt{(5)}[1 + ir\psi(r)] + 2bh_2 K_1(0^+)[\psi(r) - \psi(0)]/r$. (D6)

The S_{ij} integrals are somewhat more complicated, and contain some component parts which require use of an approximation which is best illustrated in the evaluation of S_{11}.

Using (D3) in (A32a),

$$S_{11}(p) = \frac{h_1}{ib}\int_0^{\infty}\int_0^{\infty} dx\,dx' \times e^{ipx} K_0(x-x')\left(\frac{d}{dx'}\right)^2\psi_3^0(x') . \quad \text{(D7)}$$

Recall that $\psi_3^0(x)$ is large only within the skin depth although it does have a weak, slowly varying, extremely long-range tail which extends to distances of the order of a mean free path. In this situation, $d\psi_3^0/dx$ will be concentrated even more within the skin depth than is ψ itself, in the sense that the long-range tail of $d\psi_3^0/dx$ is even smaller, relative to $d\psi_3^0/dx$ in the skin depth, than was true for ψ_3^0. The same situation holds *a fortiori* for $d^2\psi_3^0/dx^2$.

If $p < b^{1/3} \approx 1/\delta$, then the function of x' defined by

$$\int_0^{\infty} dx\, e^{ipx} K_0(x-x')$$

will be slowly varying over that region of x', $0 \leq x' \approx \delta$, where $d^2\psi_3^0/d(x')^2$ is large, and we can therefore approximate (D7) as

$$S_{11}(p) = \frac{h_1}{ib}\int_0^\infty dx\, e^{ipx} K_0(x-0)\int_0^\infty dx'\,\frac{d^2\psi_3^0}{d(x')^2} \qquad \text{(D8)}$$

$$= (ih_1/b)d_0(p)M_0 \quad . \qquad \text{(D9)}$$

Here, we have defined

$$d_m(p) \equiv \int_0^\infty dx\, e^{ipx} K_m(x) \quad , \qquad \text{(D10)}$$

deliberately choosing the same notation as used in (C8) because, upon evaluation, we find the definition (D10) equivalent to the evaluation (C9). Also in (D9), we have used

$$M_0 = [d\psi_3^0(x)/dx]_{x=0} \quad , \qquad \text{(D11)}$$

choosing the same notation as used in (A3).

The other integrals yield to the same techniques as have been illustrated here, including, in evaluation of one of the pieces which arise in S_{12} and S_{22}, use of (C6) in a manner similar to that in which (C5) was used above. For reference, we list all of the nonzero source integrals below.

$$T_{11}(r) = h_1[r^2\psi(r) + M_0 - ir] \quad ,$$

$$T_{12}(r) = h_2 a\sqrt{(5)}[1 + ir\psi(r)] + 2bh_2 K_1(0^+)[\psi(r) - \psi(0)]/r \quad ,$$

$$S_{11}(p) = [[ih_1 M_0 d_0(p)/b]] \quad ,$$

$$S_{21}(p) = [[ih_1 M_0 d_1(p)/b]] \quad ,$$

$$S_{12}(p) = \frac{h_2 a\sqrt{(5)}}{ib}\{[[d_1(p)]] - 2K_1(0^+)\psi(p)\} - [2h_2 K_1(0^+)/a\sqrt{(5)}]\psi(0)d_2(p) - \frac{4h_2 K_1^2(0^+)}{(ip)^2}\left[\psi(p) - \left(\psi(p) + p\frac{d}{dp}\psi(p)\right)_{p=0}\right],$$

$$S_{22}(p) = \frac{h_2 a\sqrt{(5)}}{ib} \times \left([[d_2(p)]] - \frac{2K_1(0^+)a\sqrt{(5)}}{ip}[\psi(p) - \psi(0)]\right) - \frac{2h_2 K_1(0^+)}{a\sqrt{(5)}}\psi(0)d_3(p) - \frac{4h_2 a\sqrt{(5)}K_1^2(0^+)}{(ip)^3} \times \left[\psi(p) - \left(\psi(p) + p\frac{d\psi}{dp} + \tfrac{1}{2}p^2\frac{d^2\psi}{dp^2}\right)_{p=0}\right] .$$

The double square bracket [[ ]] surrounds the terms in which we use the approximation that $d\psi/dx$ or $d^2\psi/dx^2$ is short ranged. The approximation is not valid for $p \approx b^{1/3}$, but is excellent for $p \ll b^{1/3}$. Clearly, the large size of b makes it possible to ignore T_{11}, and the first term in T_{12}, S_{12}, and S_{22}.

APPENDIX E: UNPERTURBED ELECTRIC FIELD

The source integrals given in Appendix D were all given in terms of the Fourier transform

$$\psi(p) = \int_0^\infty dx\, e^{ipx}\psi_3^0(x) \ , \qquad \text{(E1)}$$

where $\psi_3^0(p)$ satisfies

$$\frac{d^2\psi_3^0}{dx^2} = -ib\int_0^\infty K_0(x-x')\psi_3^0(x')\,dx' \qquad \text{(E2)}$$

subject to

$$\psi_3^0(x) = 1 \ \text{ at } \ x = 0 \ ,$$
$$= 0 \ \text{ at } \ x = \infty \ .$$

The Fourier transform can be calculated using Wiener-Hopf methods. The first solution to this problem was given by Reuter and Sondheimer.[13] The form given by Reuter and Sondheimer involved knowledge of those roots of the dispersion relation which happen to lie closer to the real axis than an inverse mean free path. It also involved knowledge of the dispersion function along the lines $\mathrm{Im}\,k = \pm 1$ between which the roots (if any) had to be known.

By using contour integration, Dingle[20] showed that the line along which the dispersion function must be known can be moved to the real axis and in doing so, it will pick up contributions at the roots of the dispersion relation which make unnecessary the specific knowledge of the roots lying between the original contour and the real axis.

If the reader will reverse Dingle's procedure and move the line of integration *away* from the real axis as far as possible, the line will encircle the cut in the dispersion function. In its way to that position, it will cross other roots of the dispersion function and will pick up contributions similar in form to those in the expression given by Reuter and Sondheimer. The net result will be an expression which explicitly involves *all* the upper half-plane roots and the discontinuity in the dispersion function across the cut; namely,

$$\psi(p) = \frac{i(\beta+p)^{N-1}}{\prod_i^N (k_i+p)} \exp\left(\frac{1}{2\pi i}\int_1^\infty \frac{\beta du}{\beta u + p}\ln G(u)\right). \qquad \text{(E3)}$$

The meaning of the terms in (E3) is as follows: Let $\Phi(k)$ be the dispersion function, the Fourier transform of the operators in (E1),

$$\Phi(k) = k^2 - ib\int_{-\infty}^\infty K_0(x)e^{-ikx}dx \quad . \qquad \text{(E4)}$$

This function has a branch point in the upper half-plane at

$$k = \beta = ia = (\omega - \omega_c)\tau + i$$

and a branch cut which goes from that branch point to infinity, most conveniently along the line $k = \beta u$,

$1 \leq u < \infty$. The roots of $\Phi(k)$ in the upper half-plane are at positions k_i, and there are either one or two of them. The product in (E3) involves all N of the upper half-plane roots:

$$\Phi(k_i) = 0, \quad i = 1, \ldots, N, \quad \mathrm{Im} k_i > 0 \ . \tag{E5}$$

We designate values of $\Phi(k)$ along the branch cut as $\Phi^{\pm}$; that is,

$$\Phi^{\pm}(\beta u) = \lim_{\epsilon \to 0^+} \Phi[k = \beta(u \pm i\epsilon)] \ .$$

Then $$G(u) \equiv \Phi^-(\beta u)/\Phi^+(\beta u) \ . \tag{E6}$$

The form (E3) can also be derived directly using the Weiner-Hopf factorization subject to the additional requirement that the function being factored have no poles or zeros, only branch cuts, in the complex plane.

We have given a rather detailed discussion of the function

$$I(u) \equiv \frac{1}{2\pi i} \ln G(u) \tag{E7}$$

and stated the location of the roots elsewhere.[21] We take those results for use here, making only those changes in notation which arise because, here, lengths are measured in units of mean free path. The results we need are these:

Let ξ be the phase of β; that is,

$$\beta = (\omega - \omega_c)\tau + i = |\beta| e^{i\xi} \ . \tag{E8}$$

Then there are three values of k which are potentially roots of (E5). These are at

$$k = (3\pi b/4)^{1/3} \exp(i\varphi_l) \ , \tag{E9a}$$

where $$\varphi_1 = \tfrac{1}{6}\pi, \quad \varphi_2 = \tfrac{1}{2}\pi, \quad \varphi_3 = \tfrac{5}{6}\pi \ . \tag{E9b}$$

Only one or two of these are, in fact, roots. Which of these are in fact roots depends on ξ as follows:

	Actual roots	Number of roots
$0 < \xi < \frac{1}{6}\pi$	$l = 2$	$N = 1$
$\frac{1}{6}\pi < \xi < \frac{1}{2}\pi$	$l = 2, \ l = 1$	$N = 2$
$\frac{1}{2}\pi < \xi < \frac{5}{6}\pi$	$l = 1$	$N = 1$
$\frac{5}{6}\pi < \xi < \pi$	$l = 1, \ l = 3$	$N = 2$

(E10)

The expression for $I(u)$ is

$$I(u) = \left[N - \frac{3}{2} + \frac{4}{2_u} + \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\frac{1}{u^5}\right] + \theta(\rho, u) + J(u, \xi) \ ,$$

$$\theta(y) = 1, \quad y > 0$$
$$= 0, \quad y < 0 \ . \tag{E11}$$

J is a function peaked more or less tightly about $u = \rho$ and having a peak whose area is of magnitude ρ,

$$J(u, \xi) \equiv \frac{1}{\pi} \sum_{n>0, \text{ odd}} \frac{(-1)^{(n-1)/2}}{n} \left(\frac{ue^{i\xi}}{\rho}\right)^{3n}, \quad u < \rho$$

$$= -\frac{1}{\pi} \sum_{n>0, \text{ odd}} \frac{(-1)^{(n-1)/2}}{n} \left(\frac{\rho e^{-i\xi}}{u}\right)^{3n}, \quad u > \rho \ . \tag{E12}$$

The quantity ρ which enters into (E11) and (E12) is

$$\rho \equiv (3\pi b/4)^{1/3}/|\beta| \ . \tag{E13}$$

We can now consider evaluation of $\psi(p)$ for $p < b^{1/3}$. Using (E11), the integral in the exponent is

$$\frac{1}{2\pi i}\int_1^\infty \frac{\beta du}{\beta u + p} \ln G(u) = \int_1^\infty \frac{\beta du}{\beta du + p} J(u, \xi)$$

$$+ (N - \tfrac{3}{2}) A_0 \frac{p}{\beta} + \frac{4}{\pi^2} A_1 \frac{p}{\beta} + \left(\frac{1}{2} - \frac{4}{\pi^2}\right) A_5 \frac{p}{\beta} \ , \tag{E14}$$

where $$A_n \frac{p}{\beta} = \int_1^\rho \frac{du}{u^n(u + p/\beta)} \ . \tag{E15}$$

Since $J(u)$ is concentrated about $u = \rho$, we can ignore the p in the denominator of the first term in the right-hand side of (E14). The integral then becomes

$$\int_1^\infty u^{-1} du J(u, \xi) = iF_0(\xi) + \mathcal{O}(1/b) \ , \tag{E16}$$

where

$$F_0(\xi) = \frac{1}{3i\pi} \sum_{n \text{ odd} > 0} \frac{(-1)^{(n-1)/2}}{n^2} (e^{3in\xi} - e^{3in\xi}),$$

$$F_0(\xi) = \tfrac{1}{2}\xi, \qquad -\tfrac{1}{6}\pi < \xi < +\tfrac{1}{6}\pi, \tag{E17a}$$

$$F_0(\xi) = \tfrac{1}{6}\pi - \tfrac{1}{2}\xi \ , \qquad \tfrac{1}{6}\pi < \xi < \tfrac{1}{2}\pi \ , \tag{E17b}$$

$$F_0(\xi) = F_0(\xi - 2\pi/3), \qquad \tfrac{1}{2}\pi < \xi < \pi \ . \tag{E17c}$$

In the denominator of (E3), the condition $p < b^{1/3}$ allows us to write

$$\Pi_i^N (k_i + p) \approx \Pi^N k_i = (3\pi b/4)^{N/3} \exp(i \textstyle\sum^N \varphi_i) \ . \tag{E18}$$

Note that, using (E10) and (E17),

$$F_0(\xi) - \sum^N \varphi_i = -\tfrac{1}{2}\pi - (N - \tfrac{3}{2}) \ . \tag{E19}$$

Finally, defining

$$U\frac{p}{\beta} = \exp\left[\frac{4}{\pi^2} A_1 \frac{p}{\beta} + \left(\frac{1}{2} - \frac{4}{\pi^2}\right) A_5 \frac{p}{\beta}\right] \tag{E20}$$

allows us to cast (E3) into the form

$$\psi(p) = \frac{(\beta + p)^{N-1} U(p/\beta)}{(3\pi b/4)^{N/3}}$$

$$\times \exp\{(N - \tfrac{3}{2})[A_0(p/\beta) - i\xi]\} \ . \tag{E21}$$

Note that the definition (E15) gives

$$A_0(p/\beta) = \ln(p+p/\beta) - \ln(1+p/\beta)$$
$$\approx \ln\rho - \ln(1+p/\beta) \ , \qquad \text{(E22)}$$

where the approximate equality arises for $p < b^{1/3}$. We can therefore rewrite (E21) as

$$\psi(p) = \left(\frac{\beta\rho e^{-i\xi}}{(3\pi b/4)^{1/3}}\right)^N \frac{U(p/\beta)}{\beta(\rho e^{-i\xi})^{3/2}} \exp[\tfrac{1}{2}\ln(1+p/\beta)] \ . \qquad \text{(E23)}$$

Combing definitions (E8) and (E13) simplifies (E23) to

$$\psi(p) = \frac{\beta^{1/2}U(p/\beta)}{(3\pi b 14)^{1/2}} \exp[\tfrac{1}{2}\ln(1+p/\beta)]$$
$$= \left(\frac{4\beta}{3\pi b}\right)^{1/2} U(p/\beta)(1+p/\beta)^{1/2} \ . \qquad \text{(E24)}$$

The branch cut for the square root must run along the negative value of its argument.

The function $U(p/\beta)$ is readily evaluated. Again, for $p \ll b^{1/3}$, we evaluate (E15) to get

$$A_1(x) = \frac{1}{x}\ln(1+x) \ ,$$

$$A_5(x) = \frac{1}{4x} - \frac{1}{3x^2} + \frac{1}{2x^3} - \frac{1}{x^4} + \frac{1}{x^5}\ln(1+x) \ ,$$

and $U(p/\beta)$ turns out to be of order unity throughout. Hence, the characteristic behavior of $\psi(p)$ is its $p^{1/2}$ growth in the range $p < b^{1/3}$.

In the range $p > b^{1/3}$, $\psi(p)$ exhibits the $1/p$ behavior common to all Fourier transforms. Details of this can be obtained with approximately the same labor as required for the study of the $p < b^{1/3}$ region.

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[1]An excellent review of this field may be found in E. A. Kaner and V. G. Skobov, Advan. Phys. 17, 605 (1968). See also the review article by W. M. Walsh, Jr., in *Solid State Physics, The Simon Fraser University Lectures* (Gordon and Breach, New York, 1968), Vol. 1.

[2]P. M. Platzman and W. M. Walsh, Jr., Phys. Rev. Letters 19, 54 (1967); 20, 89(E) (1968).

[3]P. M. Platzman, W. M. Walsh, Jr., and E-Ni Foo, Phys. Rev. 172, 689 (1968).

[4]P. M. Platzman and P. A. Wolff, Phys. Rev. Letters 18, 280 (1967).

[5]S. Schultz and G. Dunifer, Phys. Rev. Letters 18, 283 (1967).

[6]L. D. Landau, Zh. Eksperim. i Teor. Fiz. 30, 1058 (1956) [Soviet Phys. JETP 3, 920 (1956)].

[7]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 33, 495 (1957) [Soviet Phys. JETP 6, 387 (1958)].

[8]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 33, 1227 (1957) [Soviet Phys. JETP 6, 945 (1958)].

[9]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 35, 1243 (1958) [Soviet Phys. JETP 8, 870 (1958)].

[10]Y. C. Cheng, J. S. Clarke, and N. D. Mermin, Phys. Rev. Letters 20, 1486 (1968).

[11]V. F. Gantmakker and E. A. Kaner, Zh. Eksperim. i Teor. Fiz. 48, 1572 (1965) [Soviet Phys. JETP 21, 1055 (1965)].

[12]P. M. Platzman and K. D. Jacobs, Phys. Rev. 134, A974 (1964).

[13]G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1948).

[14]G. A. Baraff, C. C. Grimes, and P. M. Platzman, Phys. Rev. Letters 22, 590 (1969).

[15]G. A. Baraff, Phys. Rev. 187, 851 (1969).

[16]D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966).

[17]M. C. Jones and E. H. Sondheimer, Proc. Roy. Soc. (London) A278, 256 (1964).

[18]W. F. Brinkman, P. M. Platzman, and T. M. Rice, Phys. Rev. 174, 495 (1968).

[19]From now on, we replace $\omega_c = qB_0/m$ (a negative quantity when B_0 is along the $+Z$ direction because q, the charge on the electron is negative) by $-\omega_c = -|q|B_0/m$. This puts cyclotron resonance at $\omega_c = \omega$ instead of at $\omega_c + \omega = 0$, as earlier.

[20]R. B. Dingle, Physica 19, 311 (1953).

[21]G. A. Baraff, Phys. Rev. 167, 625 (1968).