

# Induced charge generated by a potential well in graphene

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The induced charge density,  $\rho_{ind}(\mathbf{r})$ , generated in graphene by the potential well of the finite radius  $R$  is considered. The result for  $\rho_{ind}(\mathbf{r})$  is derived for large distances  $r \gg R$ . We also obtained the induced charges outside of the radius  $r \gg R$  and inside of this radius for subcritical and supercritical regimes. The consideration is based on the convenient representation of the induced charge density via the Green's function of electron in the field.

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## I. INTRODUCTION

As known, the induced charge density,  $\rho_{ind}(\mathbf{r})$ , in the external electric field appears due to vacuum polarization. In the field of heavy nucleus, this important effect of quantum electrodynamics (QED) was investigated in detail in many papers, see, e.g., Refs. 1–4. New possibilities to study vacuum polarization in QED at large coupling constant have appeared after recent successful fabrication of a monolayer graphite (graphene), see Ref. 5 and recent Review.<sup>6</sup> The single-electron dynamics in graphene is described by a massless two-component Dirac equation<sup>7–10</sup> so that graphene represents a two-dimensional version of massless QED. On the one hand, this version is essentially simpler than conventional QED because effects of retardation are absent due to instant Coulomb interaction between electrons. On the other hand, the “fine-structure constant”  $\alpha = e^2/\hbar v_F$  is of the order of unity since the Fermi velocity  $v_F \approx 10^6$  m/s  $\approx c/300$  ( $c$  is the velocity of light), and therefore we have a strong-coupling version of QED. Below we set  $\hbar = c = 1$ .

Screening of charged impurity in graphene can also be treated in terms of vacuum polarization.<sup>11–22</sup> Investigation of impurity screening is important for understanding of the dependence of transport properties on the impurity concentration. There are two regimes for the Coulomb impurity in the gapless graphene, subcritical, and supercritical. In the subcritical regime, it is shown in the leading order in  $\alpha$  and exactly in the Coulomb potential that the induced charge is localized at the impurity position, see Refs. 16–18, 20, and 21. In the supercritical regime, vacuum polarization in the Coulomb field has been recently considered in Refs. 15 and 22. In this case, the induced charge density is not localized at the impurity position due to the effect similar to that of  $e^+e^-$  pair production in three-dimensional (3D) QED in the electric field of supercritical heavy nuclei. In the present paper, we answer to the question whether the phenomenon of the induced charge localization also exist in the potential well of finite size  $R$  and depth  $U_0$ . Namely, we calculate the asymptotics of  $\rho_{ind}(\mathbf{r})$  in the field of an azimuthally symmetric potential well at large distances  $r \gg R$ . We apply the method suggested in Ref. 23 for calculation of the finite-nuclear-size effect on the induced charge density at large distances in a strong Coulomb field in 3D QED. We show that there are also subcritical and supercritical regimes in this problem. However, the induced charge is not localized at  $r \lesssim R$  in the

subcritical regime and has power “tail” in contrast to the case of the Coulomb field. In the vicinity of transition from the subcritical regime to the supercritical one, small variation in the potential parameters drastically changes the induced charge density. We demonstrate that this fact is not related to the smoothness of the potential well. The attempt to calculate the induced charge distribution in the potential well in graphene was previously performed in Ref. 11. The authors of this paper used the method which akin to that used at calculation of conventional Friedel oscillations. However, our results for the induced charge density differ from that obtained in Ref. 11 mainly due to the mistake performed in Ref. 11 at the calculation of the phase shift.

The paper is organized as follows. In Sec. II we derive the general expression for the induced charge density convenient for calculation of the asymptotics at large distances. In Sec. III we consider the Green's function of electron in an azimuthally symmetric potential and use this function in calculations of  $\rho_{ind}(\mathbf{r})$  in Sec. IV. Critical values of  $g$  are discussed in Sec. V calculating the scattering phase shifts of electron wave function in the field of the potential well. The induced charges outside of the radius  $r \gg R$  and inside of this radius for subcritical and supercritical regimes are considered in Sec. VI. Finally, in Sec. VII the main conclusions of the paper are presented.

## II. GENERAL DISCUSSION

In graphene, the induced charge density in the potential  $U(r)$  have the form

$$\rho_{ind}(\mathbf{r}) = -ieN \int_C \frac{d\epsilon}{2\pi} \text{Tr}\{G(\mathbf{r}, \mathbf{r}|\epsilon)\}, \quad (1)$$

where  $N=4$  reflects the spin and valley degeneracies and the Green's function  $G(\mathbf{r}, \mathbf{r}'|\epsilon)$  satisfies the equation

$$[\epsilon - U(r) - v_F \boldsymbol{\sigma} \cdot \mathbf{p}]G(\mathbf{r}, \mathbf{r}'|\epsilon) = \delta(\mathbf{r} - \mathbf{r}')I. \quad (2)$$

Here  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ , and  $\sigma_i$  are the Pauli matrices;  $\mathbf{p} = (p_x, p_y)$  is the momentum operator,  $\mathbf{r} = (x, y)$ , and  $I = \text{diag}\{1, 1\}$ . The matrixes  $\boldsymbol{\sigma}$  act on the pseudospin variables and the spin degrees of freedom are taken into account in a factor  $N$ . According to the Feynman rules, the contour of integration over  $\epsilon$  goes below the real axis in the left half plane and above the real axis in the right half plane of the complex  $\epsilon$  plane. Using the

analytical properties of the Green's function, we deform the contour of integration with respect to  $\epsilon$  so that it coincides finally with the imaginary axis. Then we follow Ref. 23 and write the equation for the Green's function in the form

$$G(\mathbf{r}, \mathbf{r}' | i\epsilon) = G^{(0)}(\mathbf{r}, \mathbf{r}' | i\epsilon) + \int d\mathbf{r}_1 d\mathbf{r}_2 G^{(0)}(\mathbf{r}, \mathbf{r}_1 | i\epsilon) [U(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2) + U(\mathbf{r}_1) G(\mathbf{r}_1, \mathbf{r}_2 | i\epsilon) U(\mathbf{r}_2)] G^{(0)}(\mathbf{r}_2, \mathbf{r}' | i\epsilon), \quad (3)$$

where  $G^{(0)}(\mathbf{r}, \mathbf{r}' | i\epsilon)$  is the solution of Eq. (2) at zero external field.

It is convenient to represent  $\rho_{ind}(r)$  as a sum

$$\rho_{ind}(r) = \rho_{ind}^{(1)}(r) + \rho_{ind}^{(2)}(r), \quad (4)$$

where  $\rho_{ind}^{(1)}(r)$  is the linear in  $U(r)$  contribution and  $\rho_{ind}^{(2)}(r)$  is the contribution of high order in  $U(r)$  terms. It follows from Eqs. (1) and (3) that:

$$\rho_{ind}^{(1)}(r) = eN \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \int d\mathbf{r}_1 \text{Tr}\{G^{(0)}(\mathbf{r}, \mathbf{r}_1 | i\epsilon) U(\mathbf{r}_1) G^{(0)}(\mathbf{r}_1, \mathbf{r} | i\epsilon)\}, \quad (5)$$

$$\rho_{ind}^{(2)}(r) = eN \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \int d\mathbf{r}_1 d\mathbf{r}_2 \text{Tr}\{G^{(0)}(\mathbf{r}, \mathbf{r}_1 | i\epsilon) U(\mathbf{r}_1) \times G(\mathbf{r}_1, \mathbf{r}_2 | i\epsilon) U(\mathbf{r}_2) G^{(0)}(\mathbf{r}_2, \mathbf{r} | i\epsilon)\}. \quad (6)$$

Formulas (5) and (6) are very convenient for calculation of the induced charge density at large distances.

### III. GREEN'S FUNCTION IN AN AZIMUTHALLY SYMMETRIC POTENTIAL

The free Green's function  $G^{(0)}(\mathbf{r}, \mathbf{r}' | i\epsilon)$  is given by

$$G^{(0)}(\mathbf{r}, \mathbf{r}' | i\epsilon) = -\frac{i\epsilon}{2\pi} \left[ K_0(|\epsilon|\xi) - \text{sign}(\epsilon) \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\xi})}{\xi} K_1(|\epsilon|\xi) \right], \quad (7)$$

where  $\boldsymbol{\xi} = \mathbf{r} - \mathbf{r}'$  and  $K_{0,1}(x)$  are the modified Bessel functions of the third kind. Let us represent the electron Green's function  $G(\mathbf{r}, \mathbf{r}' | \epsilon)$  in an azimuthally symmetric potential  $U(r)$  in the form

$$G(\mathbf{r}, \mathbf{r}' | \epsilon) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \times \begin{pmatrix} A_m(r, r' | \epsilon) & -ie^{-i\phi'} B_m(r, r' | \epsilon) \\ ie^{i\phi} C_m(r, r' | \epsilon) & e^{i(\phi-\phi')} D_m(r, r' | \epsilon) \end{pmatrix}, \quad (8)$$

and use the relation

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')}{2\pi\sqrt{rr'}} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}. \quad (9)$$

Then, from Eq. (2) we obtain the equations

$$[\epsilon - U(r)]A_m - \frac{\partial C_m}{\partial r} - \frac{m+1}{r}C_m = \frac{\delta(r-r')}{\sqrt{rr'}},$$

$$[\epsilon - U(r)]C_m + \frac{\partial A_m}{\partial r} - \frac{m}{r}A_m = 0, \quad (10)$$

and the relations  $D_m = A_{-m-1}$  and  $B_m = -C_{-m-1}$ . Therefore, to find the Green's function in an azimuthally symmetric potential, it is sufficient to solve Eq. (10).

### IV. AN INDUCED CHARGE DENSITY AT LARGE DISTANCES

To calculate the asymptotics of the function  $\rho_{ind}^{(1)}$  at distances  $r \gg R$ , where  $R$  is a typical size of the potential, we can put  $r_1 = 0$  in the arguments of the free Green's functions in Eq. (5). After that we take the integral over  $\epsilon$  and obtain

$$\rho_{ind}^{(1)}(r) = \frac{eN}{16\pi^3} \int d\mathbf{r}' r' U(r'). \quad (11)$$

One can see that the induced charge density in the leading order in the external field goes to zero at large distances as  $1/r^3$ .

Let us consider the function  $\rho_{ind}^{(2)}(r)$  at  $r \gg R$ . We substitute Eqs. (7) and (8) to Eq. (6), put  $r_1 = 0$  and  $r_2 = 0$  in the arguments of the free Green's function, and take the integral over angles of the vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . Then we obtain

$$\rho_{ind}^{(2)}(r) = -\frac{eN}{2\pi^2} \int_{-\infty}^{\infty} d\epsilon \epsilon^2 [K_0^2(|\epsilon|r) - K_1^2(|\epsilon|r)] \times \int_0^\infty \int_0^\infty dr_1 dr_2 r_1 r_2 U(r_1) U(r_2) A_0(r_1, r_2 | i\epsilon). \quad (12)$$

Here  $A_0(r_1, r_2 | i\epsilon)$  is the solution of Eq. (10) at  $m=0$ . Note that Eq. (12) includes the contributions of the terms with  $m=0$  and  $-1$  in the Green's function Eq. (8), since  $D_{-1} = A_0$ . It is convenient to introduce the functions

$$a(r, \epsilon) = \int_0^\infty dr' r' U(r') A_0(r, r' | i\epsilon),$$

$$c(r, \epsilon) = \int_0^\infty dr' r' U(r') C_0(r, r' | i\epsilon). \quad (13)$$

Let us multiply both sides of the Eq. (10) by  $r' U(r')$ , and then take the integral over  $r'$  from zero to infinity. As a result we obtain the following equations for the functions  $a(r, i\epsilon)$  and  $c(r, i\epsilon)$

$$[i\epsilon - U(r)]a(r, \epsilon) - \frac{\partial c(r, \epsilon)}{\partial r} - \frac{c(r, \epsilon)}{r} = U(r),$$

$$[i\epsilon - U(r)]c(r, \epsilon) + \frac{\partial a(r, \epsilon)}{\partial r} = 0. \quad (14)$$

The boundary conditions for these equations are  $a(0, \epsilon), c(0, \epsilon) < \infty$ , and  $\lim_{r \rightarrow \infty} a(r, \epsilon) = \lim_{r \rightarrow \infty} c(r, \epsilon) = 0$ . In terms

of the function  $a(r, i\epsilon)$ , Eq. (12) has the form

$$\rho_{ind}^{(2)}(r) = -\frac{eN}{2\pi^2} \int_{-\infty}^{\infty} d\epsilon \epsilon^2 [K_0^2(|\epsilon|r) - K_1^2(|\epsilon|r)] \times \int_0^{\infty} dr' r' U(r') a(r', \epsilon). \quad (15)$$

Then we pass in this equation from the variable  $\epsilon$  to the variable  $E=r\epsilon$  and replace  $a(r', E/r)$  on  $a(r', 0)$  at  $r \gg R$ . We can do that because the integral over  $E$  converges at  $E \sim 1$  due to the properties of the  $K$  functions. After this replacement we take the integral over  $E$  and arrive at the following expression for the asymptotics of  $\rho_{ind}^{(2)}(r)$ :

$$\rho_{ind}^{(2)}(r) = \frac{eN}{16r^3} \int_0^{\infty} dr' r' U(r') a(r', 0). \quad (16)$$

Thus, the function  $\rho_{ind}^{(2)}(r)$  has the same behavior at large distances as  $\rho_{ind}^{(1)}(r)$ .

Let us consider a simple example of the potential,  $U(r) = -U_0 \theta(R-r)$ , where  $\theta(x)$  is the step function,  $R$  is the radius of the potential well. The solution  $a(r, 0)$  of Eq. (14) is

$$a(r, 0) = \begin{cases} \frac{J_0(U_0 r)}{J_0(U_0 R)} - 1, & r < R, \\ 0, & r > R, \end{cases} \quad (17)$$

where  $J_n(x)$  is the Bessel function. Using this solution, we find the sum of the contributions Eqs. (11) and (16),

$$\rho_{ind}(r) = -\frac{eN J_1(g) R}{16 J_0(g) r^3}, \quad (18)$$

where  $g = U_0 R$  is the effective dimensionless coupling constant. The induced charge density Eq. (18) is the odd function of the parameter  $g$ , which corresponds to the Furry theorem in QED. The formula (18) contains singularities at the critical values of  $g = g_c$  satisfying the equation  $J_0(g_c) = 0$ . In our case, the first three values are  $g_c \approx 2.41, 5.52, 8.65$ . Existence of such singularities is not related to strong variation in our potential around the point  $r = R$ . We found numerically the first three critical values of  $g = U_0 R$  for the smooth potentials  $U_1(r) = -U_0 e^{-r/R}$  and  $U_2(r) = -U_0 e^{-r^2/R^2}$ . In the first case,  $g_c \approx 2.87, 5.9, 9.0$ . In the second case,  $g_c \approx 2.7, 5.7, 8.0$ . We see that the corresponding numerical values of  $g_c$  are close to each other.

Actually, singularities in Eq. (18) have appeared as a result of substitution  $a(r', E/r) \rightarrow a(r', 0)$  in Eq. (16), which is not valid in the vicinity of  $g = g_c$  since Eq. (17) has no sense at  $g = g_c$ . In the vicinity of  $g = g_c$ , it is necessary to perform calculation of the integrals in Eq. (15) more accurately. For the steplike potential  $U(r) = -U_0 \theta(R-r)$ , the solution of Eq. (14) at  $\epsilon \neq 0$  has the form

$$a(r, \epsilon) = \begin{cases} \frac{1}{\gamma} J_0[(U_0 + i\epsilon)r] - U_0/(U_0 + i\epsilon), & r < R, \\ \beta K_0(|\epsilon|r), & r > R, \end{cases} \quad (19)$$

$$c(r, \epsilon) = \begin{cases} \frac{1}{\gamma} J_1[(U_0 + i\epsilon)r], & r < R, \\ -i\beta \text{sign}(\epsilon) K_1(|\epsilon|r), & r > R. \end{cases} \quad (20)$$

Taking into account continuity of the functions  $a(r, \epsilon)$  and  $c(r, \epsilon)$  at  $r = R$ , we obtain

$$\gamma = \left(1 + \frac{i\epsilon}{U_0}\right) \times \left\{ J_0[(U_0 + i\epsilon)R] - i \text{sign}(\epsilon) J_1[(U_0 + i\epsilon)R] \frac{K_0(|\epsilon|R)}{K_1(|\epsilon|R)} \right\}. \quad (21)$$

Then we substitute Eqs. (19) and (21) to Eq. (15) and take the integral over  $r'$ . As above, the main contribution to the integral over  $\epsilon$  at  $r \gg R$  is given by the region  $\epsilon \lesssim 1/r$ , so that we can use the relations  $\epsilon R \ll 1$  and  $\epsilon/U_0 \ll 1$ . Finally we find the expression for the sum of  $\rho_{ind}^{(1)}(r)$  and  $\rho_{ind}^{(2)}(r)$  at large distances,

$$\rho_{ind}(r) = \frac{eN J_0(g) J_1(g) R}{\pi^2} \int_0^{\infty} d\epsilon \epsilon^2 \frac{K_0^2(\epsilon r) - K_1^2(\epsilon r)}{J_0^2(g) + J_1^2(g) (\epsilon R)^2 \ln^2(\epsilon R)}. \quad (22)$$

This expression is valid at arbitrary value of the coupling constant  $g = U_0 R$ . If  $|J_0(g)| \gg (R/r) \ln(r/R)$ , then it is possible to neglect the second term in the denominator of the integrand, and we return to the expression (18). If  $g$  is close to some  $g_c$  so that  $|J_0(g)| \ll (R/r) \ln(r/R) \ll 1$ , we obtain

$$\rho_1(r) = -\frac{eN \text{sign}(g - g_c)}{2\pi r^2 \ln|g - g_c|}. \quad (23)$$

In this case the induced charge density diminishes as  $1/r^2$  and has opposite sign for  $g < g_c$  and  $g > g_c$ . In terms of distances, the asymptotics Eq. (23) is valid at

$$1 \ll r/R \ll -\ln|g - g_c|/|g - g_c|.$$

At  $r/R \gg -\ln|g - g_c|/|g - g_c|$  and  $|g - g_c| \ll 1$ , we have [see Eq. (18)]

$$\rho_2(r) = \frac{eNR}{16r^3(g - g_c)}. \quad (24)$$

In order to illustrate the transition from the asymptotics Eq. (23) to the asymptotics Eq. (24), we consider the ratio  $\rho_{ind}(r)/\rho_2(r)$  at  $r \gg R$  and  $|g - g_c| \ll 1$ . In this case this ratio depends only on the variable  $\eta = -r|g - g_c|/(R \ln|g - g_c|)$ . The dependence of  $\rho_{ind}(r)/\rho_2(r)$  on  $\eta$  is shown in Fig. 1. We see that  $\rho_{ind}(r) \approx \rho_2(r)$  already at  $\eta \approx 2$ .

## V. CRITICAL VALUES OF $g$ AND SCATTERING PROBLEM

It is possible to explain critical values of  $g$  using the approach based on the scattering problem, as it is usually performed at the consideration of Friedel oscillations, see Ref. 24. Writing the wave function of electron as

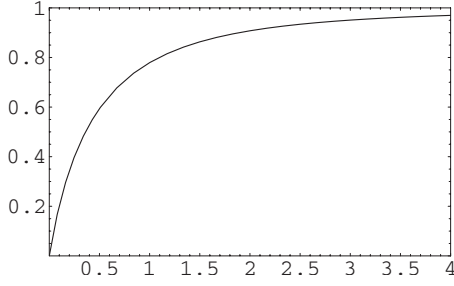


FIG. 1. Ratio  $\rho_{ind}(r)/\rho_2(r)$  at  $r \gg R$  and  $|g - g_c| \ll 1$  as a function of  $\eta = -r|g - g_c|/(R \ln|g - g_c|)$ . The asymptotics  $\rho_2(r)$  is given by Eq. (24) and  $\rho_{ind}$  by Eq. (22).

$$\psi(r) = \begin{pmatrix} u_m(r)e^{im\phi} \\ id_m(r)e^{i(m+1)\phi} \end{pmatrix}, \quad (25)$$

we obtain equations for the functions  $u_m(r)$  and  $d_m(r)$ , cf. Eq. (10),

$$\begin{aligned} [\epsilon - U(r)]u_m - \frac{\partial d_m}{\partial r} - \frac{m+1}{r}d_m &= 0, \\ [\epsilon - U(r)]d_m + \frac{\partial u_m}{\partial r} - \frac{m}{r}u_m &= 0. \end{aligned} \quad (26)$$

The solution of these equations in the the steplike potential has the form (common normalization factor is omitted)

$$\begin{aligned} u_m(r) &= \begin{cases} J_m(|U_0 + \epsilon|r), & r < R, \\ \mu_m J_m(|\epsilon|r) + \nu_m N_m(|\epsilon|r), & r > R, \end{cases} \\ d_m(r) &= \begin{cases} \text{sign}(U_0 + \epsilon)J_{m+1}(|U_0 + \epsilon|r), & r < R, \\ \text{sign}(\epsilon)[\mu_m J_{m+1}(|\epsilon|r) + \nu_m N_{m+1}(|\epsilon|r)], & r > R. \end{cases} \end{aligned} \quad (27)$$

Here  $N_m(x)$  are the Bessel functions of the second kind. From continuity of the functions  $u_m(r)$  and  $d_m(r)$  at  $r=R$ , we have

$$\begin{aligned} \mu_m &= -\frac{\pi|\epsilon|R}{2}[J_m(|U_0 + \epsilon|R)N_{m+1}(|\epsilon|R) \\ &\quad - J_{m+1}(|U_0 + \epsilon|R)N_m(|\epsilon|R)\text{sign}(\epsilon)\text{sign}(\epsilon + U_0)], \\ \nu_m &= -\frac{\pi|\epsilon|R}{2}[-J_m(|U_0 + \epsilon|R)J_{m+1}(|\epsilon|R) \\ &\quad + J_{m+1}(|U_0 + \epsilon|R)J_m(|\epsilon|R)\text{sign}(\epsilon)\text{sign}(\epsilon + U_0)]. \end{aligned} \quad (29)$$

Using the asymptotics of the Bessel functions at large value of argument, we find the phase shift  $\delta_m(\epsilon) = -\arctan(\nu_m/\mu_m)$ . Critical values of  $g$  are given by the solution of the equation  $J_m(g_c) = 0$  at  $m \geq 0$  and  $J_{|m|-1}(g_c) = 0$  at  $m < 0$ . Taking into account the asymptotics

$$N_0(x) \approx \frac{2 \ln x}{\pi}, \quad N_{|m|}(x) \approx -\frac{2^{|m|}(|m|-1)!}{x^{|m|}\pi}$$

at  $x \ll 1$ , we find for  $|\epsilon|R \ll 1$ ,  $|\epsilon| \ll U_0$ , and  $g = U_0 R$  close to  $g_c$

$$\delta_0(\epsilon) = \arctan \left[ \frac{\frac{\pi}{2}\epsilon R}{\epsilon R \ln(|\epsilon|R) - (g - g_c)} \right],$$

$$\begin{aligned} \delta_m(\epsilon) &= \\ &= -\arctan \left[ \frac{\pi(\epsilon R)^{2m+1}}{2^{2m}m!(m-1)![(2m+1)\epsilon R + 2m(g - g_c)]} \right] \\ &\text{at } m > 0, \end{aligned} \quad (30)$$

and  $\delta_{-|m|}(\epsilon) = \delta_{|m|-1}(\epsilon)$ . If  $\epsilon < 0$ , which corresponds to electrons inside Fermi surface, and  $g < g_c$ , then  $\delta_m$  is always small. For  $g > g_c$ , the phase shift  $\delta_m(\epsilon)$  can be equal to  $\pm \pi/2$  at some  $\epsilon < 0$ . That means the appearance at  $g = g_c$  of the additional quasibound state on the Fermi surface.

Calculation of the phase shift in the steplike potential was previously performed in Ref. 11. However, the coefficients corresponding to  $\mu_m$  and  $\nu_m$ , Eq. (29), were found in Ref. 11 by matching the function  $u_m(r)$  and its first derivative at  $r=R$ , instead of matching the functions  $u_m(r)$  and  $d_m(r)$ . It is easy to check that the first derivative of  $u_m(r)$  is not a continuous function in the point  $r=R$ . As a consequence, the asymptotics of the induced charge density at large distances obtained in Ref. 11 is not correct.

## VI. AN INDUCED CHARGE

Let us consider the induced charge  $Q_>(r)$  outside of the radius  $r \gg R$ ,

$$\begin{aligned} Q_>(r) &= 2\pi \int_r^\infty dr' r' \rho_{ing}(r') \\ &= -\frac{eN J_0(g) J_1(g) R r^2}{\pi} \int_0^\infty d\epsilon \epsilon^2 \\ &\quad \times \frac{K_0^2(\epsilon r) + K_0(\epsilon r) K_2(\epsilon r) - 2K_1^2(\epsilon r)}{J_0^2(g) + J_1^2(g)(\epsilon R)^2 \ln^2(\epsilon R)}. \end{aligned} \quad (31)$$

For  $|J_0(g)| \gg (R/r) \ln(r/R)$ , we have

$$Q_>(r) = -\frac{eN\pi R J_1(g)}{8J_0(g)r}. \quad (32)$$

In the case  $|g - g_c| \ll (R/r) \ln(r/R)$ , we find with logarithmic accuracy

$$Q_>(r) = eN \text{sign}(g - g_c) \left( 1 + \frac{\ln(r/R)}{\ln|g - g_c|} \right). \quad (33)$$

Since  $N=4$ , then  $Q_>(r)/e$  tends to the integer number  $N \text{sign}(g - g_c)$  at  $g \rightarrow g_c$ , having opposite sign for  $g < g_c$  and  $g > g_c$ .

Let us discuss the induced charge  $Q_<(r)$  inside of the radius  $r \gg R$ . Since the total induced charge  $Q_{tot} = Q_>(r) + Q_<(r)$  is zero for the potential well at  $g$  less than the minimal  $g_c$ , we have  $Q_<(r) = -Q_>(r)$  for such value of  $g$ . Note that  $Q_{tot}$  is not zero for massless electron in graphene in the Coulomb potential  $U_c(r) = -Z\alpha/r$  even in the subcritical regime  $Z\alpha < 1/2$ , see Ref. 20, due to zero mass of



a particle and slow decreasing of a Coulomb potential at large distances. For  $g$  larger than the minimal  $g_c$ , the total induced charge is already not equal to zero due to the effect similar to  $e^+e^-$  pair production in the electric field of super-heavy nucleus.<sup>4,25</sup> In this case  $Q_{tot}=eNM$ , where  $M$  is a number of  $g_c$  less than  $g$ , so that  $Q_<(r)=-Q_>(r)+eNM$ . The quantity  $M$  is nothing but the number of the quasibound states at a given value of  $g$ , see discussion in Sec. V. The explicit values of  $g_c$  are given by zeros of the Bessel functions, as it is pointed out in Sec. V.

In Sec. IV and this section we have considered the contributions of the angular momenta  $m=0$  and  $-1$  in the Green's function Eq. (8) to induced charge density and  $Q_>(r)$  at large distances. Of course, the contributions of  $m>0$  and  $m<-1$  are not zero though they are strongly suppressed by some power of  $R/r$  even in the vicinity of the corresponding critical points. However,  $M$  in  $Q_{tot}=eNM$  includes numbers of  $g_c$  coming from  $m>0$  and  $m<-1$ .

In order to illustrate behavior of the induced charge in the vicinity of some critical point  $g_c$ , it is convenient to represent  $Q_>(r)$  and  $Q_<(r)$  at  $r\gg R$  as follows:

$$Q_>(r) = eN[\text{sign}(g - g_c) + F(g, r)],$$

$$Q_<(r) = eN[\text{sign}(g_c - g) - F(g, r) + M\theta(g - g_c)], \quad (34)$$

where  $F(g, r)$  is some continuous function of  $g$ . The dependence of this function on  $g$  at  $R/r=0.1$ , obtained from Eq. (31) in the vicinity of minimal value of  $g_c$ , is shown in Fig. 2 (solid line), as well as its asymptotics, obtained with the use of Eq. (32) (dashed line).

It is seen that the region, where Eq. (32) is not applicable, is very narrow.

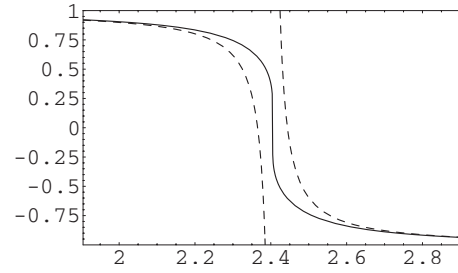


FIG. 2. Dependence of the function  $F(g, r)$ , defined in Eq. (34), on  $g$  at  $R/r=0.1$  in the vicinity of a minimal value of  $g_c$ . Exact result obtained from Eq. (31) is shown as a solid line, the asymptotics, obtained with the use of Eq. (32), as a dashed line.

## VII. CONCLUSION

In this paper we have calculated the induced charge density generated by the potential well in graphene at large distances. Besides, we have obtained the induced charges outside of the radius  $r\gg R$  and inside of this radius for subcritical and supercritical regimes. Small variation in the potential parameters drastically changes the induced charge density in the vicinity of the critical values of  $g$ .

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