

Statistics of voltage fluctuations in resistively shunted Josephson junctions

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The intrinsic nonlinearity of Josephson junctions converts Gaussian current noise in the input into non-Gaussian voltage noise in the output. For a resistively shunted Josephson junction with white input noise we determine numerically exactly the properties of the few lowest cumulants of the voltage fluctuations, and we derive analytical expressions for these cumulants in several important limits. The statistics of the voltage fluctuations is found to be Gaussian at bias currents well above the Josephson critical current but Poissonian at currents below the critical value. In the transition region close to the critical current the higher-order cumulants oscillate and the voltage noise is strongly non-Gaussian. For colored input noise we determine the third cumulant of the voltage.

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I. INTRODUCTION

Resistively shunted Josephson junctions, described in the frame of the so-called RSJ model, have been widely studied for many years for the analysis of the high-frequency response, noise, nonlinear dynamics, and other properties of Josephson junctions.^{1,2} Analytical expressions for the current-voltage characteristics³ and noise,^{1,4,5} as well as numerous numerical results for this model are known. In the present paper we analyze the non-Gaussian statistics of the voltage fluctuations in the RSJ model, which has not been addressed in the literature before. Our work is partly motivated by recent progress with experimental techniques, which made it possible to detect non-Gaussian corrections to the statistics of current fluctuations in a normal tunnel junction.^{6–9} It should be possible to apply similar techniques to Josephson junctions. Another motivation is to study the effects of the intrinsic nonlinearity inherent to the RSJ, which is absent in normal tunnel junctions. One of its manifestations are the Shapiro steps in the I - V characteristic of an ac-biased RSJ.^{1,2} The nonlinearity also converts high-frequency input current noise to low-frequency voltage noise. The latter effect allowed Koch *et al.* to measure the high-frequency quantum noise of an Ohmic resistor.¹⁰ In this paper we will study the effect of the Josephson nonlinearity on the statistics of voltage fluctuations, and we find strong deviations of the latter from a simple Gaussian form.

We proceed using the formalism of the full-counting statistics (FCS),^{11,12} which is the appropriate tool for analyzing non-Gaussian stochastic processes. The concept of FCS proved to be very useful in the transport theory of mesoscopic systems. So far most efforts focused on the FCS of electron transport through tunnel junctions, quantum dots, metallic wires, and similar mesoscopic structures.^{12,13} The first two cumulants—current and zero-frequency current noise—are routinely measured in experiments. Higher cumulants provide information about deviations from Gaussian statistics, and are therefore of great importance. The FCS formalism has also been applied to the charge transport

through a voltage biased Josephson junction,^{14–16} where it was shown that the statistics of current fluctuations at bias voltages below the superconducting energy gap is determined by the processes of multiple Andreev reflections.

On the experimental side, a direct measurement of the third cumulant for a tunnel junction between two normal metals has been first performed by Reulet *et al.*⁶ and good agreement with theory¹⁷ was found. Subsequently it has been realized^{18,19} that a current biased Josephson junction can be used as threshold detector of large current fluctuations, which makes it an effective tool of measuring higher-order cumulants of non-Gaussian noise. This idea has been implemented in experiments^{7–9} where the third cumulant of the current noise of a tunnel junction was measured. In several studies^{20–22} the theory of a Josephson junction, serving as detector of non-Gaussian noise, has been developed in detail.

In this paper we treat the resistively shunted Josephson junction as a source, rather than a detector, of non-Gaussian noise. In contrast to Refs. 14–16 we consider a current biased junction and study the statistics of the resulting voltage fluctuations. After briefly introducing the RSJ model of an overdamped Josephson junction and the concept of FCS for voltage noise, we will study the FCS of voltage fluctuations caused by white Gaussian input current noise. Instead of counting transferred electrons, we study the statistical distribution of the Josephson phase. We find that the statistics of voltage fluctuations in a RSJ is Gaussian at bias currents far above the critical current $I \gg I_c$, and Poissonian in the opposite limit, $I \lesssim I_c$. In the transition region, $I \sim I_c$, the voltage cumulants oscillate, and at low temperature the statistics may become strongly non-Gaussian. In the last part of the paper we turn to the case of colored input noise and derive approximate expressions for the third cumulant of the voltage noise.

II. MODEL

An overdamped resistively shunted Josephson junction is described by the equation^{1,2}

$$\frac{1}{R} \frac{\hbar \dot{\varphi}}{2e} + I_c \sin \varphi = I + \xi(t). \quad (1)$$

Here φ is the phase difference of the superconducting order parameters across the junction, R is the parallel shunt resistance, I_c and I are the critical current and bias current due to a current source, respectively, and $\xi(t)$ is the input Gaussian noise. In equilibrium $\xi(t)$ is just the noise of the resistance R fixed by the fluctuation-dissipation theorem,

$$\langle \xi(t_1) \xi(t_2) \rangle = \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} \frac{\hbar \omega}{R} \coth \frac{\hbar \omega}{2T}. \quad (2)$$

At high temperature, $T \gtrsim 2eI_c R$, the equilibrium noise $\xi(t)$ is white with correlator

$$\langle \xi(t_1) \xi(t_2) \rangle = \frac{2T}{R} \delta(t_1 - t_2). \quad (3)$$

Out of equilibrium the noise spectrum may be arbitrary but often it is sufficiently well described by Eq. (3) with an enhanced effective temperature T^* . At low frequencies, $1/f$ noise usually dominates but we will not consider it here. In Sec. IV we will also consider colored noise with an arbitrary spectrum requiring only that it does not diverge in the limit of low frequencies.

We note that Eq. (1) is not exact since it ignores charge accumulation on the capacitance C of the junction and the quantum dynamics derived from this effect. At high resistance, namely, for $R \gtrsim \pi \hbar / 2e^2$, this effect is not important at sufficiently high temperatures, $T \gtrsim \sqrt{2\hbar e I_c / C}$. In the opposite limit $R \ll \pi \hbar / 2e^2$ quantum effects may be ignored at any temperature (see, e.g., Ref. 23 for more details).

For single-electron tunneling in normal junctions the FCS is contained in the probability $W(t, N)$ for N electrons to be transferred through the system during time t . The central object of FCS theory is the cumulant generating function

$$S(\chi) = \ln \left[\sum_N e^{iN\chi} W(t, N) \right], \quad (4)$$

where χ is the so-called counting field. Derivatives of this function determine the zero-frequency cumulants of the electric current flowing through the system,

$$C_n^{(I)} = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{1}{i^n} \frac{\partial^n S(\chi)}{\partial \chi^n} \Big|_{\chi=0}. \quad (5)$$

Proceeding in analogy to standard FCS theory of electron transport, we first introduce the probability $P(t, \varphi)$ for the Josephson phase to change by φ during time t . The phase φ can take any real value, i.e., it is not restricted to the interval $[0, 2\pi]$. The corresponding cumulant generating function reads

$$F(t, k) = \ln[\mathcal{P}], \quad (6)$$

$$\mathcal{P}(t, k) = \int_{-\infty}^{+\infty} d\varphi e^{ik\varphi} P(t, \varphi), \quad (7)$$

where we have introduced the counting field k and the Fourier-transformed distribution function $\mathcal{P}(t, k)$. The zero-

frequency cumulants of the voltage $V = \hbar \dot{\varphi} / 2e$ are given by the derivatives

$$C_n^{(V)} = \frac{\hbar^n}{(2e)^n} \lim_{t \rightarrow \infty} \frac{(-i)^n}{t} \frac{\partial^n F(t, k)}{\partial k^n} \Big|_{k=0}. \quad (8)$$

Note that in contrast to the usual cumulant generating function in Eq. (4) the function $F(t, k)$ is not periodic in k . The reason for that is the fact that the transferred charge is discrete while the phase φ is continuous.

To proceed we rewrite Eq. (1) in dimensionless form by defining the dimensionless parameters $\tau = 2eI_c R t / \hbar$, $j = I / I_c$, $\zeta = \xi / I_c$, and obtain

$$\frac{d\varphi}{d\tau} + \sin \varphi = j + \zeta(\tau). \quad (9)$$

The correlator of the dimensionless noise $\zeta(\tau)$ reads

$$\langle \zeta(\tau_1) \zeta(\tau_2) \rangle = \int \frac{d\nu}{2\pi} e^{-i\nu(\tau_1-\tau_2)} S(\nu), \quad (10)$$

where $\nu = \hbar \omega / 2eI_c R$ is the dimensionless frequency, and

$$S(\nu) = \frac{2eR}{\hbar I_c} \int dt \exp\left(i \frac{2eI_c R \nu}{\hbar} t\right) \langle \xi(t) \xi(0) \rangle \quad (11)$$

is the dimensionless noise spectrum. In the same way we normalize the cumulants. The zero-frequency n th cumulant of the dimensionless voltage $v = d\varphi / d\tau$ becomes

$$C_n = \lim_{\tau \rightarrow \infty} \frac{(-i)^n}{\tau} \frac{\partial^n F(\tau, k)}{\partial k^n} \Big|_{k=0}. \quad (12)$$

The cumulant in Eq. (8), with dimensions restored, is then given by

$$C_n^{(V)} = \left(\frac{\hbar}{2e} \right)^{n-1} I_c R C_n.$$

The two lowest cumulants are directly related to measurable physical quantities. The first is the average voltage $C_1 \equiv \langle v \rangle$, while the second is the zero-frequency voltage noise $C_2 \equiv \int d\tau \langle \delta v(\tau) \delta v(0) \rangle$, with $\delta v(\tau) = v(\tau) - \langle v \rangle$. In the following we will also use the explicit expression for the zero-frequency third cumulant

$$C_3 \equiv \int d\tau_1 d\tau_2 \langle \delta v(\tau_1) \delta v(\tau_2) \delta v(0) \rangle. \quad (13)$$

III. WHITE NOISE

In this section we consider white noise with local pair correlator, Eq. (3). In this regime the Langevin Eq. (9) is equivalent to the Smoluchowski equation for the distribution function $P(\tau, \varphi)$ (see, i.e., Ref. 24 for the derivation),

$$\frac{\partial P}{\partial \tau} + \frac{\partial}{\partial \varphi} [(j - \sin \varphi) P] - \gamma \frac{\partial^2 P}{\partial \varphi^2} = 0. \quad (14)$$

Here we introduced the dimensionless constant $\gamma = 2eT / \hbar I_c$ characterizing the noise strength.

The average voltage \mathcal{C}_1 in this case is known exactly and reads³

$$\mathcal{C}_1 = \frac{2\pi\gamma}{\int_{-\pi}^{\pi} d\varphi_1 \int_{-\infty}^{\varphi_1} d\varphi_2 e^{[j(\varphi_1 - \varphi_2) + \cos \varphi_1 - \cos \varphi_2]/\gamma}}. \quad (15)$$

In order to access higher-order cumulants we have solved Eq. (14) numerically and, in some limiting cases, analytically. Let us first present the numerical solution. The Fourier-transformed distribution function $\mathcal{P}(t, k)$ satisfies the following equation

$$\frac{\partial \mathcal{P}(t, k)}{\partial \tau} = ik \left[j\mathcal{P}(t, k) - \frac{\mathcal{P}(t, k+1) - \mathcal{P}(t, k-1)}{2i} \right] - \gamma k^2 \mathcal{P}(t, k). \quad (16)$$

In order to solve this equation we define a square matrix $\hat{L}(k)$ with elements

$$L_{\alpha\beta}(k) = [ij(k + \alpha) - \gamma(k + \alpha)^2] \delta_{\alpha,\beta} - \frac{k + \alpha}{2} (\delta_{\alpha,\beta-1} - \delta_{\alpha,\beta+1}), \quad (17)$$

and truncate it restricting the indices α and β as follows: $-N/2 \leq \alpha, \beta \leq N/2$, where N is sufficiently large. For the results presented below we have chosen $N=200$, which guarantees sufficient accuracy.

Next we note that the long-time behavior of $F(\tau, k)$ is given by a simple formula²⁵

$$F(\tau, k) = \lim_{\tau \rightarrow \infty} \ln \mathcal{P}(\tau, k) = \tau \Lambda(k), \quad (18)$$

where $\Lambda(k)$ is the eigenvalue of the matrix $\hat{L}(k)$ with the biggest real part. In order to compute the cumulants, we first numerically diagonalized the matrix $L_{\alpha\beta}(k)$ for several values of k close to zero and then took the numerical derivative of $\Lambda(k)$ at $k=0$. To check the convergence of this procedure we have varied the step size in k by two orders of magnitude. The numerically exact results for the first three cumulants obtained in this way are shown in Figs. 1–3, and the first five cumulants—in Fig. 4.

Before discussing these results further we first consider in the following section the limits of high and low temperatures, where we can derive analytical results.

A. High-temperature regime, $\gamma \gg 1$

In the limit of high temperature, $\gamma \gg 1$, we use a simple perturbation theory which allows us to find an analytical expression for the cumulant generating function (18) at any bias current. Treating the off-diagonal elements of the matrix, Eq. (17), as weak perturbation we find the lowest-order correction to the dominating eigenvalue $\Lambda(k)$. This simple procedure relies on perturbation theory in the critical current I_c in the original RSJ Eq. (1). For this approximation to be justified, I_c should be small compared to bias current I or some scale set by the temperature T . We will discuss this point in more detail below. Thus we arrive at the following

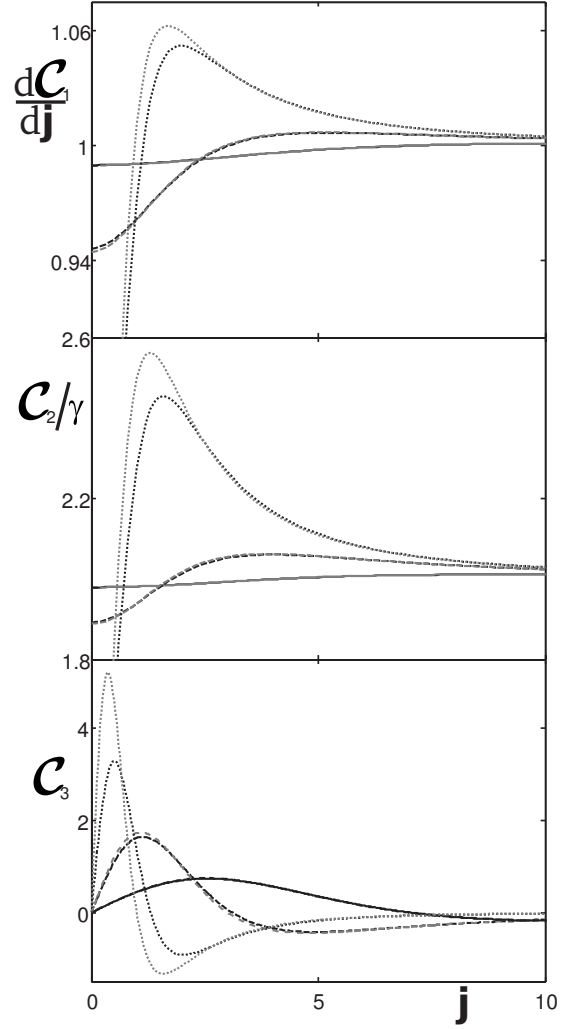


FIG. 1. The derivative of the normalized voltage $d\mathcal{C}_1/dj$, the second cumulant normalized by the temperature \mathcal{C}_2/γ and the third cumulant \mathcal{C}_3 as a function of the bias current j . (Full lines) $\gamma=7$, (dashed lines) $\gamma=3$, and (dotted lines) $\gamma=1$. (Black) numerical results and (gray) analytical results, Eq. (20).

expression for the cumulant generating function

$$F(\tau, k) = \left[ijk - \gamma k^2 + \frac{1}{2} \frac{ijk - \gamma k^2}{(ij - 2\gamma k)^2 - \gamma^2} \right] \tau. \quad (19)$$

The first three cumulants read

$$\begin{aligned} \mathcal{C}_1 &= j \left[1 - \frac{1}{2(j^2 + \gamma^2)} \right], \\ \mathcal{C}_2 &= \gamma \frac{2j^4 + 3j^2 + 4j^2\gamma^2 + 2\gamma^4 - \gamma^2}{(j^2 + \gamma^2)^2}, \\ \mathcal{C}_3 &= 24\gamma^2 j \frac{\gamma^2 - j^2}{(j^2 + \gamma^2)^3}. \end{aligned} \quad (20)$$

We observe that the voltage fluctuations become purely Gaussian ($\mathcal{C}_n \rightarrow 0$, $n \geq 3$) at $\gamma \rightarrow \infty$ or $j \rightarrow \infty$ because in this regime the nonlinear Josephson current may be ignored.

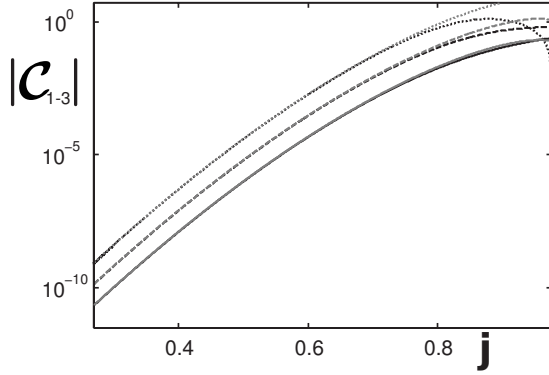


FIG. 2. The absolute value of the first three cumulants $|C_{1-3}|$, as a function the bias current j for $j < 1$ and $\gamma = 0.05$. (Full line) the first cumulant C_1 , (dashed line) the second cumulant C_2 , and (dotted line) the third cumulant C_3 . (Black) numerical results and (gray) analytical results [see Eq. (27)].

Higher order cumulants ($n \geq 3$) can be written in a compact form

$$C_n = (-1)^n \frac{n!}{2} (2\gamma)^{n-2} \text{Im} \frac{j^2 + \gamma^2}{(j - i\gamma)^{n+1}}. \quad (21)$$

Let us discuss the validity range of the results, Eqs. (19)–(21). The approximate expression for the cumulant generating function (19), and hence Eqs. (20) and (21), is valid if the correction induced by the nonlinearity (the second term in the brackets) is much smaller than the dominating contribution (the first term in the brackets). At $|j| > 1/\gamma$ the absolute value of the correction has a maximum at $k=0$ and comparing the two contributions we arrive at the condition $j^2 + \gamma^2 \gtrsim 1$. Since in this section we have assumed $\gamma \gtrsim 1$, this condition is always satisfied. At small currents, $|j| \ll 1/\gamma$, the correction in Eq. (19) diverges at $k=1/2$ and hence the whole expression may be used only at sufficiently small k . More precisely, one should require $|k| \leq k_c$, where the boundary value $k_c = \sqrt{1/4 - 1/8\gamma^2}$ is defined through $F(\tau, k_c)|_{j=0} = 0$. Therefore the cumulants of the order higher than $n_{\max} \sim \gamma^2$, which determine the k dependence of the exact cumulant generating function $F(\tau, k)$ at $|k| > k_c$, cannot be reliably

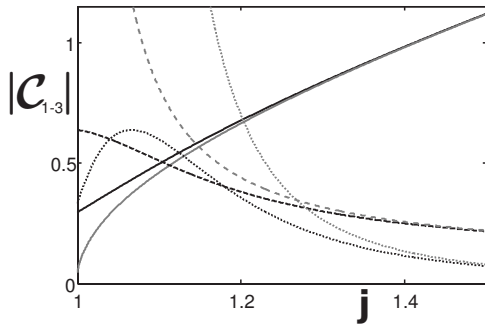


FIG. 3. The absolute value of the first three cumulants $|C_{1-3}|$, as a function the bias current j for $j > 1$ and $\gamma = 0.05$. (Full line) the first cumulant C_1 , (dashed line) the second cumulant C_2 , and (dotted line) the third cumulant C_3 . (Black) numerical results and (gray) analytical results [see Eqs. (36)–(38)].

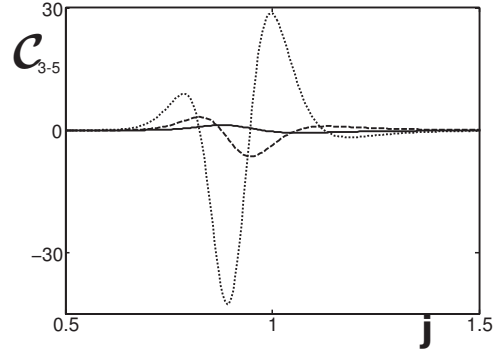


FIG. 4. Numerical solutions for the third, fourth, and fifth cumulant C_{3-5} , as a function the bias current j for $\gamma = 0.05$. (Full line) the third cumulant C_3 , (dashed line) the fourth cumulant C_4 , and (dotted line) the fifth cumulant C_5 .

described by Eq. (19). In summary: the expression (21) for the cumulant of the n th order is valid either at $|j| \gtrsim 1/\gamma$ and $\gamma \gtrsim 1$, or at $|j| \lesssim 1/\gamma$ and $\gamma \gtrsim \sqrt{n}$.

In Fig. 1 we compare analytical results, Eq. (20), with the exact numerical simulations. While analytics and numerics agree well for the first and second cumulant, even for rather small γ , the analytical approximation for the third cumulant breaks down for $\gamma \approx 1$. The most interesting observation is the sign change in C_3 . It is positive at low bias, negative at high bias and equals zero at $j = \gamma$. According to Eq. (21) the higher-order cumulants oscillate even stronger: C_4 , C_5 change their sign twice, C_6 , C_7 —three times, C_8 , C_9 —four times, etc. An exact numerical solution confirms this fact and shows that the oscillations of the high order cumulants occur even at $\gamma \ll 1$, where the Eq. (19) is no longer valid (see Fig. 4). As it has been realized recently,²⁶ such an oscillatory behavior of the cumulants is a very general phenomenon related to the analytic properties of the cumulant generating function. In our case it is associated with the shift of the poles of the function $F(\tau, k)$ in the complex plane caused by the change in the bias current j .

The third cumulant C_3 , Eq. (20), has two extrema at $j = \gamma(4 \pm \sqrt{13})^{1/2}/\sqrt{3}$. The extremal values of C_3 are proportional to $1/\gamma$, $C_3^{\max/\min} \propto 1/\gamma$. As γ becomes smaller, C_3 increases while $C_2 \propto \gamma$ decreases. At $\gamma \sim 1$, where the approximation, Eq. (19), breaks down, we have $C_2 \sim C_3$. This result indicates significant deviation from the simple Gaussian statistics of voltage fluctuations. We will explore this region now coming from the opposite limit of small γ .

B. Low-temperature regime, $\gamma \lesssim 1$

At low temperature, $\gamma \lesssim 1$, we will distinguish low bias regime, $(1-j)^{3/2} \gtrsim \gamma$, and high bias regime, $(j-1)^{3/2} \gtrsim \gamma$. In the vicinity of the critical current, namely, at $|j-1|^{3/2} \lesssim \gamma$, we restrict ourselves to numerics.

Let us first consider low bias currents $(1-j)^{3/2} \gtrsim \gamma$. In this case the phase is strongly localized at the minima $\varphi_n = \arcsin j + 2\pi n$ of the effective potential $U(\varphi) = -j\varphi - \cos \varphi$.²⁷ Defining the occupation probability $P_n(\tau)$ of the n th minimum one can express the function $P(\tau, \varphi)$ in the form

$$P(\tau, \varphi) = \sum_n P_n(\tau) \frac{(1-j^2)^{1/4} e^{-(\varphi - \varphi_n)^2 \sqrt{1-j^2}/2\gamma}}{\sqrt{2\pi\gamma}}. \quad (22)$$

$P_n(\tau)$ satisfies the following master equation²⁷

$$\frac{\partial P}{\partial \tau} = -(\Gamma_+ + \Gamma_-)P_n + \Gamma_+ P_{n-1} + \Gamma_- P_{n+1}, \quad (23)$$

where

$$\Gamma_{\pm} = \frac{\sqrt{1-j^2}}{2\pi} \exp \left[-2 \frac{j \arcsin j + \sqrt{1-j^2}}{\gamma} \pm \frac{\pi j}{\gamma} \right]. \quad (24)$$

The Fourier-transformed distribution function $\mathcal{P}(\tau, k)$ in this case acquires the form

$$\mathcal{P}(\tau, k) = e^{i \arcsin j} e^{-\gamma k^2/2 \sqrt{1-j^2}} \sum_n e^{2\pi i k n} P_n(\tau) \quad (25)$$

and satisfies the equation

$$\frac{\partial \mathcal{P}}{\partial \tau} = [\Gamma_+(e^{2\pi i k} - 1) + \Gamma_-(e^{-2\pi i k} - 1)] \mathcal{P}. \quad (26)$$

Thus the cumulant generating function takes the form

$$F(\tau, k) = \Gamma_+ \tau (e^{2\pi i k} - 1) + \Gamma_- \tau (e^{-2\pi i k} - 1). \quad (27)$$

The first three cumulants in this regime read

$$\mathcal{C}_1 = 2\sqrt{1-j^2} \sinh(\pi j/\gamma) e^{-2(j \arcsin j + \sqrt{1-j^2})/\gamma},$$

$$\mathcal{C}_2 = 2\pi \coth(\pi j/\gamma) \mathcal{C}_1,$$

$$\mathcal{C}_3 = 4\pi^2 \mathcal{C}_1. \quad (28)$$

The cumulant generating function (27) describes the Poissonian statistics of bidirectional phase slip tunneling. The voltage fluctuations in this case are determined by random switches between the wells of the tilted washboard potential. In Fig. 2 we compare analytical results with numerical ones and find an excellent agreement between both as long as $(1-j)^{3/2} \gtrsim \gamma$, i.e., as long as the height of the potential barrier between the neighboring minima is greater than the temperature. At low currents the tunneling rate between the minima becomes exponentially small and all cumulants are strongly suppressed.

Next we assume that the bias current is bigger than the critical one but not too close to it, i.e., $j > 1$ and $(j-1)^{3/2} \gtrsim \gamma$. To find the eigenvalue Λ of the operator L , Eq. (17), we rewrite it back in-phase representation and arrive at the following equation

$$-\frac{\partial}{\partial \varphi} [(j - \sin \varphi) \Psi] + \gamma \frac{\partial^2 \Psi}{\partial \varphi^2} = \Lambda \Psi, \quad (29)$$

where the eigenfunction Ψ is quasiperiodic: $\Psi(\varphi + 2\pi) = \exp[-2\pi i k] \Psi(\varphi)$. Our aim is to find the eigenvalue Λ as a function of the counting field k . We express the function Ψ as follows

$$\Psi(\varphi) = \exp \left[\frac{1}{\gamma} \int_0^\varphi d\varphi' W(k, \varphi') \right]. \quad (30)$$

Equation (29) then takes the form

$$\gamma \frac{\partial W}{\partial \varphi} = (j - \sin \varphi) W - W^2 - \gamma \cos \varphi + \gamma \Lambda. \quad (31)$$

The quasiperiodicity of the function Ψ translates into the following condition

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi W(\varphi) = -i\gamma k. \quad (32)$$

In the limit $\gamma \ll 1$ and $(j-1)^{3/2} \gtrsim \gamma$, which we are considering, the solution of Eq. (31) reads

$$W(\varphi) = \frac{j - \sin \varphi - \sqrt{(j - \sin \varphi)^2 + 4\gamma\Lambda}}{2}. \quad (33)$$

Combining this result with Eq. (32) we find the eigenvalue Λ and obtain the cumulant generating function in the form

$$F(\tau, k) = \frac{f(j, 2i\gamma k)}{4\gamma} \tau, \quad (34)$$

where the function $f(j, x)$ is implicitly defined by the equation

$$-j + \frac{1}{2\pi} \int_0^{2\pi} d\phi \sqrt{f(j, x) + (j - \sin \phi)^2} = x. \quad (35)$$

Having derived the cumulant generating functions (34) and (35) we can find the first three cumulants,

$$\mathcal{C}_1 = \sqrt{j^2 - 1}, \quad (36)$$

$$\mathcal{C}_2 = \frac{2j^2 + 1}{j^2 - 1} \gamma, \quad (37)$$

$$\mathcal{C}_3 = -\frac{3}{2} \frac{16j^2 + 1}{(j^2 - 1)^{5/2}} \gamma^2. \quad (38)$$

The first cumulant Eq. (36) is just the usual voltage-current dependence of an RSJ junction,^{1,2} while the second one Eq. (37) follows from the well-known result by Likharev and Semenov.^{1,4} In the vicinity of the critical current the n th ($n \gtrsim 10$) cumulant can be roughly estimated as follows:

$$\mathcal{C}_n \approx (-1)^n \beta \sqrt{j-1} \left[\alpha(n-1) \frac{\gamma}{(j-1)^{3/2}} \right]^{n-1}, \quad (39)$$

where $\alpha \approx 1.71$, $\beta \approx 0.061$. The divergence close to $j=1$ is due to the fact that our approach is only valid for $(j-1)^{3/2} \gtrsim \gamma$.

The full numerical solution for the first three cumulants is shown in Fig. 3. It demonstrates that the approximate analytical results Eqs. (36)–(38) work well at sufficiently high bias current. We also observe that close to the critical current the analytics is less accurate for the third cumulant.

In the transition region $|j-1|^{3/2} \lesssim \gamma$ the higher-order cumulants oscillate as illustrated in Fig. 4. The amplitude of

these oscillations increases with the order of the cumulant. Since the cumulants oscillate out of phase, one can tune the ratio between them varying the bias current. We observe, for example, that $C_4=0$ at the bias current where $|C_3|$ and $|C_5|$ are close to their maximum values.

IV. COLORED NOISE

In this section we consider the effect of frequency dependent, or colored noise. It is well known that due to the non-linearity of an RSJ the high-frequency part of the current noise $\xi(t)$ contributes to low-frequency voltage noise C_2 .^{4,10} In this section we show that the third cumulant C_3 is sensitive to the high-frequency current noise as well. Since the Smoluchowski Eq. (14) is no longer valid, we apply perturbation theory in the Josephson critical current I_c and current noise $\xi(t)$ deriving analytical expressions for C_3 valid in the limits of strong and weak current noise. As an application of our general results, we study the behavior of the third cumulant in the vicinity of a Shapiro step.

A. Strong noise

In case of strong noise we can apply the perturbation theory in critical current I_c while solving Eq. (1). Keeping the terms $\propto I_c^2$ and switching to the dimensionless parameters, from Eq. (9) we find

$$v(\tau) \equiv \dot{\varphi}(\tau) = j + \zeta(\tau) - \sin \left[j\tau + \int_0^\tau ds \zeta(s) \right] + \cos \left[j\tau + \int_0^\tau ds \zeta(s) \right] \int_0^\tau ds_1 \sin \left[js_1 + \int_0^{s_1} ds \zeta(s) \right].$$

The first cumulant is given by the average of this expression over the fluctuations of the Gaussian noise $\zeta(\tau)$ and has a well-known form

$$C_1 = j - \frac{1}{2} \int_0^\infty d\tau e^{-\mathcal{F}(\tau)} \sin j\tau, \quad (40)$$

where

$$\mathcal{F}(\tau) = \int \frac{d\nu}{2\pi} S(\nu) \frac{1 - \cos \nu\tau}{\nu^2}. \quad (41)$$

The second cumulant in this regime reads

$$C_2 = S(0) + \int_0^\infty d\tau [1 - S(0)\tau] e^{-\mathcal{F}(\tau)} \cos j\tau. \quad (42)$$

This expression can also be written in the form

$$C_2 = S(0) \left(2 \frac{\partial C_1}{\partial j} - 1 \right) + \frac{1}{2} \int d\tau e^{-\mathcal{F}(\tau)} \cos j\tau. \quad (43)$$

The third cumulant, which is of primary interest for us here, is defined by Eq. (13). After averaging of the latter over noise fluctuations, we arrive at the following result

$$C_3 = \frac{3}{2} S^2(0) \int_0^\infty d\tau \tau^2 e^{-\mathcal{F}(\tau)} \sin j\tau - 3S(0) \int_0^\infty d\tau \tau e^{-\mathcal{F}(\tau)} \sin j\tau. \quad (44)$$

One can verify that in case of white noise, $S(\nu)=2\gamma$, Eqs. (40), (42), and (44) reduce to expression (20) derived in Sec. III A. In the regime of strong noise, considered here, the third cumulant can be related the first and second ones as follows:

$$C_3 = 3S(0) \left(\frac{\partial C_2}{\partial j} - S(0) \frac{\partial^2 C_1}{\partial j^2} \right). \quad (45)$$

The results in Eqs. (40), (42), and (44) are valid if the correction to differential resistance at zero bias remains small, i.e., if

$$\int_0^\infty d\tau e^{-\mathcal{F}(\tau)} \tau \ll 1. \quad (46)$$

Let us now investigate the behavior of the third cumulant in the vicinity of a Shapiro step. Shapiro steps appear on the voltage-current characteristics of an RSJ subject to an ac bias.^{1,2} In this case the noise $\xi(t)$ in Eq. (1) is the sum of the white noise and an ac signal with the frequency ω_0 , the line-width δ and the amplitude \sqrt{a} . The corresponding noise spectrum reads

$$S(\nu) = 2\gamma + \frac{a\delta}{(\nu - \omega_0)^2 + \delta^2} + \frac{a\delta}{(\nu + \omega_0)^2 + \delta^2}, \quad (47)$$

and the function $\mathcal{F}(\tau)$ takes the form

$$\mathcal{F}(\tau) = \gamma|\tau| + a \frac{(\omega_0^2 - \delta^2)(1 - e^{-\delta|\tau|} \cos[\omega_0\tau])}{(\omega_0^2 + \delta^2)^2} - a \frac{2\delta\omega_0 e^{-\delta|\tau|} \sin[\omega_0|\tau|]}{(\omega_0^2 + \delta^2)^2}. \quad (48)$$

Figure 5 shows the behavior of the first three cumulants for an RSJ subject to a strong Gaussian noise with the spectrum in Eq. (47). In the strong noise limit considered here the Shapiro steps occur at bias currents $j_n = n\omega_0$. They are seen as dips in differential resistance $\partial C_1 / \partial j$. We observe that the second cumulant C_2 roughly follows the differential resistance, while C_3 exhibits beatings and changes its sign in the vicinity of every Shapiro step.

B. Weak noise

In this section we assume the noise $\zeta(\tau)$ to be weak. More precisely, we require

$$S(\nu_0) \ll \nu_0^2, \quad (49)$$

where $\nu_0 = \sqrt{j^2 - 1}$ is the frequency of Josephson oscillations.

Below the critical current, i.e., at $j < 1$, we expect the Poissonian statistics of the voltage noise defined by Eq. (27) with modified tunneling rates Γ_\pm . It is in general not possible to derive a closed formula for these rates, although for certain noise spectra $S(\nu)$ it can be done.^{27,28}

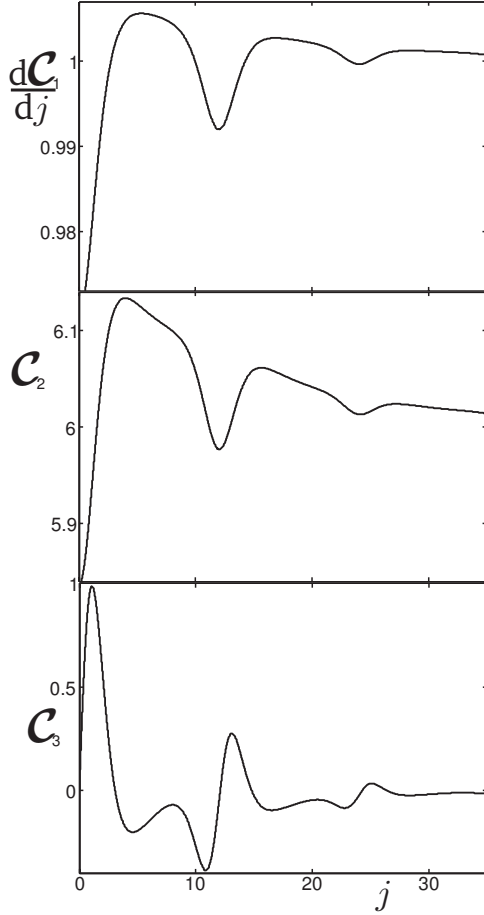


FIG. 5. The normalized differential resistance $\partial C_1/\partial j$, the second cumulant (noise) C_2 and the third cumulant C_3 of an RSJ subject to strong microwave irradiation as a function of the bias current j . C_1 , C_2 , and C_3 are defined by Eqs. (40), (42), and (44), respectively, while the noise spectrum by Eq. (47). We have chosen the following values for the parameters characterizing the noise spectrum: $\gamma=3$, $a=100$, $\delta=0.01$, and $\omega_0=12$.

Above the critical current, i.e., for $j > 1$, we solve Eq. (9) treating the noise $\zeta(\tau)$ perturbatively. The first step of our approach is to put $\zeta(\tau)=0$ in Eq. (9) and find the zeroth order solution of this equation. It can be expressed as follows

$$\varphi_0(\tau) = \nu_0 \tau + \Delta\varphi_0(\tau), \quad (50)$$

where $\Delta\varphi_0(\tau)$ is a periodic function.

Next we search for a solution of Eq. (9) in the form $\varphi(\tau) = \varphi_0[\theta(\tau)]$, where $\theta(\tau)$ is an unknown phase satisfying the equation

$$\dot{\theta} = 1 + \frac{j + \sin \nu_0 \theta}{\nu_0^2} \zeta(\tau). \quad (51)$$

This equation provides a convenient starting point for the perturbation theory in the noise $\zeta(\tau)$. To the second order in $\zeta(\tau)$ its solution reads

$$\begin{aligned} \theta = \tau + \int_0^\tau d\tau' \frac{j + \sin \nu_0 \tau'}{\nu_0^2} \zeta(\tau') \\ + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \frac{\cos \nu_0 \tau' (j + \sin \nu_0 \tau'')}{\nu_0^3} \zeta(\tau') \zeta(\tau''). \end{aligned} \quad (52)$$

Before we turn to the derivation of zero-frequency cumulants of the voltage, let us discuss the time averaging procedure. Since $\Delta\varphi_0(\tau)$ is a periodic function of time, it varies within certain finite limits. Therefore the time derivative $(d/d\tau)\Delta\varphi_0[\theta(\tau)]$ vanishes after time averaging for any realization of the noise $\zeta(\tau)$. That means, in turn, that we can replace $\varphi(\tau) \equiv \varphi_0[\theta(\tau)] \rightarrow \nu_0 \theta(\tau)$ as long as we are interested only in zero-frequency cumulants of the voltage fluctuations.

Thus, deriving $\nu_0 \dot{\theta}$ from Eq. (52) and averaging it over the realizations of the noise $\zeta(\tau)$ as well as over time, we find the first cumulant C_1

$$C_1 = \nu_0 - \frac{1}{2\nu_0^2} \int_0^\infty d\tau S_\zeta(\tau) \sin \nu_0 \tau, \quad (53)$$

where $S_\zeta(\tau) = \langle \zeta(\tau) \zeta(0) \rangle$ is the pair correlator of dimensionless noise.

Next we evaluate the second cumulant (noise) in the lowest nonvanishing order in $\zeta(\tau)$, namely, in the order $\zeta^2(\tau)$. We reproduce the known result^{1,4}

$$\begin{aligned} C_2 &= \frac{\nu_0}{2\pi} \int d\tau \int_{-\pi/\nu_0}^{\pi/\nu_0} ds \langle \delta v(s + \tau/2) \delta v(s - \tau/2) \rangle \\ &= \int d\tau S_\zeta(\tau) \left(\frac{j^2}{\nu_0^2} + \frac{\cos \nu_0 \tau}{2\nu_0^2} \right) = \frac{j^2 S(0)}{\nu_0^2} + \frac{S(\nu_0)}{2\nu_0^2}. \end{aligned} \quad (54)$$

Finally we turn to the third cumulant C_3 . Here we use the following definition of C_3

$$C_3 = \nu_0^3 \lim_{t \rightarrow \infty} \frac{\nu_0}{2\pi} \int_{t-\pi/\nu_0}^{t+\pi/\nu_0} \frac{d\tau}{\tau} \langle [\theta(\tau) - \langle \theta \rangle]^3 \rangle. \quad (55)$$

One can demonstrate that it is equivalent to the original formula (13). Since the noise $\zeta(\tau)$ is Gaussian, the first nonvanishing contribution to C_3 comes from the terms $\propto \zeta^4(\tau)$. Making use of the explicit expression for θ Eq. (52), we arrive at the result

$$\begin{aligned} C_3 &= \frac{6}{\nu_0^4} \lim_{t \rightarrow \infty} \frac{\nu_0}{2\pi} \int_{t-\pi/\nu_0}^{t+\pi/\nu_0} d\tau \frac{1}{\tau} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \int_0^\tau d\tau_3 \int_0^{\tau_3} d\tau_4 \\ &\quad \times (j + \sin \nu_0 \tau_1) S_\zeta(\tau_1 - \tau_3) \cos \nu_0 \tau_3 (j + \sin \nu_0 \tau_2) \\ &\quad \times S_\zeta(\tau_2 - \tau_4) (j + \sin \nu_0 \tau_4). \end{aligned} \quad (56)$$

Evaluating the integral we arrive at the main result of this section

$$C_3 = -\frac{3}{8} \frac{8j^2 S(0)[S(0) + S(\nu_0)] + S^2(\nu_0)}{\nu_0^5} + \frac{3}{4} \frac{2j^2 S(0) + S(\nu_0)}{\nu_0^4} \frac{dS(\nu_0)}{d\nu_0}. \quad (57)$$

We observe that C_3 is a sum of two terms. The first one, given by the first line of Eq. (57), is always negative. The second contribution is proportional to the derivative of the noise spectrum and may have any sign.

In a trivial case of white noise Eqs. (53), (54), and (57) reduce respectively to Eqs. (36)–(38). For further illustration let us again assume that the noise spectrum contains a narrow Lorentzian line and defined by Eq. (47) with $\delta \ll \gamma\omega_0^2/a$. In the limit of high frequency, $\omega_0 \gg 1$, and in the vicinity of the first Shapiro step, i.e., at j close to ω_0 , the cumulants Eqs. (53), (54), and (57) take the form

$$C_1 = j - \frac{1}{2j} + \frac{a}{4\omega_0^2} \frac{j - \omega_0}{(j - \omega_0)^2 + \delta^2}, \quad (58)$$

$$C_2 = 2\gamma + \frac{a\delta}{2\omega_0^2} \frac{1}{(j - \omega_0)^2 + \delta^2}, \quad (59)$$

$$C_3 = \frac{3\gamma a\delta}{\omega_0^2} \frac{\omega_0 - j}{[(j - \omega_0)^2 + \delta^2]^2}. \quad (60)$$

Equation (58) illustrates the formation of the Shapiro step on the I - V curve at $j = \omega_0$, while the voltage noise Eq. (59) reproduces the Lorentzian line of the input noise spectrum Eq. (47). The third cumulant Eq. (60) changes its sign at $j = \omega_0$ and in this regime can be related to the second one by means of a simple formula $C_3 = 6\gamma\delta C_2 / \partial j$.

At low temperatures quantum interference effects become important and the results Eqs. (54) and (57), derived from a classical Eq. (1), are in general not valid. However, in the specific case of the small shunt resistance, namely, at $R \ll \pi\hbar/2e^2$, the quantum interference is suppressed and the Langevin Eq. (1) with the equilibrium noise Eq. (2) remains valid down to $T=0$. In this case the cumulants C_2, C_3 are given respectively by Eqs. (54) and (57) with the noise spectrum $S(\nu_0) = (4e^2 R/\hbar)|\nu_0|$. The validity of the expression (54) for the second cumulant in this regime has been established in Refs. 4 and 10 and later confirmed by the exact solution valid for any value of shunt resistance and junction capacitance.⁵

V. DISCUSSION AND SUMMARY

We have demonstrated that the intrinsic nonlinearity of a resistively shunted Josephson-junction converts Gaussian current noise into non-Gaussian voltage noise. The non-Gaussian properties of the latter are strongest at low temperature and at bias current below or slightly above I_c . It is in this range of parameters where the nonlinearity of the RSJ is most important. It leads, in particular, to the nonlinear voltage-current characteristics with well resolved critical current.

We have shown that well below I_c the statistics of voltage fluctuations is Poissonian and the fluctuations are caused by

random-phase slip events. In contrast, at high bias current, $I \gg I_c$, the statistics becomes Gaussian. In this case the nonlinear properties of the RSJ can be ignored, and it effectively operates as a linear resistor.

The transition between the two limiting cases turns out to be nontrivial. With the decrease in the bias current the statistics gradually deviates from a Gaussian one, and the third and higher-order cumulants of the voltage noise grow. Close to the critical current, namely, for $|I - I_c| \lesssim (2eT/\hbar I_c)^{2/3}$, the statistics of voltage fluctuations strongly deviates from both Gaussian and Poissonian. In this regime the phase slips overlap in time and are therefore poorly defined, while, at the same time, the nonlinear effects are very strong. Under these conditions the voltage cumulants of order higher than three strongly oscillate with the bias current. Finally, at $I \lesssim I_c - (2eT/\hbar I_c)^{2/3}$ the statistics becomes Poissonian.

Thus the noise properties of a RSJ are very different from those of a tunnel junction, a normal diffusive wire and other nonsuperconducting mesoscopic structures. In the latter case the voltage cumulants monotonously (usually linearly) grow with the bias current. The unusual properties of a RSJ open up an interesting opportunity of tuning the statistics of voltage noise by changing the bias current. Sweeping the current from $I \ll I_c$ to $I \gg I_c$ one can cover a wide range of statistical distributions. It may be useful, for example, for testing the properties of Josephson detectors of non-Gaussian noise.

The general properties of the voltage noise statistics, which we have just outlined, are found for a weakly frequency-dependent input current noise spectrum. These properties change if the noise spectrum contains narrow lines. The latter situation corresponds to an ac-biased RSJ which is known to have Shapiro steps on the I - V curve. It is also known^{1,2} that the shape of every Shapiro step basically repeats that of the I - V curve of the RSJ in the absence of an external ac signal. We have shown that the same is true for the third cumulant of the voltage noise, i.e., the current dependence of C_3 close to a Shapiro step qualitatively repeats the dependence $C_3(j)$ of a dc-biased RSJ, see Fig. 5. In particular, depending on the parameters, C_3 may strongly increase and change its sign in the vicinity of a Shapiro step. This means, in turn, that under sufficiently strong microwave irradiation the statistics of voltage fluctuations in a RSJ may significantly deviate from Gaussian one even at high bias current $I \gg I_c$ provided the closest Shapiro step on the I - V curve is well resolved. We expect that not only the third, but all higher-order cumulants should oscillate in the vicinity of the step in the same way as they do in a dc-biased RSJ close to I_c .

In summary, we have investigated the statistics of voltage fluctuations in a resistively shunted Josephson junction. For white input current noise we have analyzed the FCS of these fluctuations numerically. In addition we have derived approximate analytical expressions for the FCS in the limit of strong white noise ($\gamma \gg 1$) for arbitrary bias currents, as well as, in the limit of weak noise ($\gamma \ll 1$), for currents below the critical one $[(1-j)^{3/2} \gtrsim \gamma]$ and above it $[(j-1)^{3/2} \gtrsim \gamma]$. In the intermediate range of currents ($|j-1|^{3/2} \lesssim \gamma$) we observed the transition from high bias Gaussian voltage fluctuations to low bias Poissonian statistics of phase slips. In this range the voltage cumulants of higher than second-order oscillate with

the bias current, and the voltage noise may be strongly non-Gaussian.

We have also derived approximate analytical expressions for the third cumulant C_3 of the voltage in case of colored input noise. We have considered two limits: (i) the limit the strong input noise, and (ii) the limit of weak input noise and bias current higher than the Josephson critical current. We have demonstrated that all cumulants C_1 , C_2 , C_3 are sensitive to high-frequency input noise. We have also shown that C_3 is enhanced in the vicinity of a Shapiro step caused by an

ac bias current, and may even change sign if the ac amplitude is sufficiently large.

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