Inserting this into (1) yields

$$x=\frac{E}{e}\left[\left(1-\frac{\theta}{\pi}\right)^{-2}-1\right],\quad \theta=\pi\left[1-\left(1+\frac{ex}{E}\right)^{-\frac{1}{2}}\right]. \tag{20}$$

Upon substituting (20) into (11), we obtain

$$v=\exp\frac{1}{\pi}\int_0^w \frac{1}{\sqrt{w'}} \times\int_0^{\pi[1-(1+ew'/E)^{-\frac{1}{2}}]} \frac{d\theta}{\{w-(E/e)[(1-\theta/\pi)^{-2}-1]\}^{\frac{1}{2}}}dw'. \tag{21}$$

After evaluating the θ integral, we have

$$v=\exp\int_0^w \frac{(e/E)dw}{1+(ew/E)} =\exp\left[\log\frac{1}{1+(ew/E)}\right]=\frac{1}{1+(ew/E)}. \tag{22}$$

Now from (8), $w=u^2v^{-1}$, so (22) yields

$$v=1-(e/E)u^2. \tag{23}$$

Then, using (8) again, we have

$$V=E(1-v)=eu^2=e/r^2. \tag{24}$$

The inverse square potential (24) is just the one from which the cross section (19) was originally obtained.

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Calculation of the Scattering Potential from Reflection Coefficients*

H. E. Moses
New York University, New York, New York
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It is shown how the scattering potential can be calculated from suitably defined reflection coefficients in the case of the one- and three-dimensional reduced wave equations by means of a formal series expansion. The more general problem of calculating scattering potentials from elements of the scattering operator is also discussed and it is shown that to calculate the scattering potential it is often sufficient to prescribe the representation in which it is to be diagonal.

1. INTRODUCTION AND SUMMARY

Jost and Kohn[1] have developed a procedure for finding spherically symmetric potentials from scattering phases. This problem is simplified by the fact that the solutions of the radial equation are not degenerate.

It has been found possible to generalize their procedure to cases where the outgoing eigenfunctions are degenerate. One is able to show that in many cases the scattering potential can be obtained from certain elements of the scattering operator, provided one specifies the representation in which V is to be diagonal.

In the present paper the method is applied to the one- and three-dimensional scattering problems, where the scattering potential V is assumed to be a function (not necessarily symmetric) of the space variables. It is shown that in the one-dimensional case, the potential can be obtained from the reflection coefficient at one end. The potential is calculated explicitly to the first two orders in the reflection coefficient where it is shown that the results are the same as those obtained using the Gelfand-Levitan procedure.[2,3]

In the three-dimensional case it is shown that the potential can be obtained from the amplitudes of the spherical waves reflected back along the rays on which the incident plane waves are sent, the totality of such rays being those pointing out at right angles to a hemisphere whose center is at the origin.

The general procedure for obtaining the scattering potential from the scattering operator is also discussed. It is shown that one must specify the representation in which V is diagonal to get a unique answer.

We restrict our discussion to cases in which the unperturbed and perturbed Hamiltonians have purely continuous spectra which coincide. However, it is possible to generalize the results to cases where the

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[1] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).

[2] I. Kay, "On the Determination of a Linear System from the Reflection Coefficient," Research Report No. EM-74, Institute of Mathematical Science, Division of Electromagnetic Research, New York University, 1955 (unpublished).

[3] I. Kay and H. E. Moses, "The Determination of the Scattering Potential from the Spectral Measure Function. Part III. Calculation of the Scattering Potential from the Scattering Operator for the One-Dimensional Schrödinger Equation," Nuovo cimento (to be published).

total Hamiltonian has point eigenvalues in addition to the continuous spectrum.

2. REFLECTION COEFFICIENT AND OUTGOING WAVE SOLUTION: ONE-DIMENSIONAL CASE

Let us consider the one-dimensional reduced equation:

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\Psi(x|k)=k^2\Psi(x|k), \tag{2.1}$$

where the scattering potential $V(x)$ vanishes as $|x|\to\infty$.

For $k>0$, let $\Psi(x|k)$ have the asymptotic form:

$$\begin{aligned}\Psi(x|k)&\to\frac{e^{ikx}}{(2\pi)^{\frac{1}{2}}}+b(k)\frac{e^{-ikx}}{(2\pi)^{\frac{1}{2}}}, \quad \text{as} \quad x\to-\infty;\\ \Psi(x|k)&\to c(k)e^{ikx}, \quad \text{as} \quad x\to+\infty.\end{aligned} \tag{2.2}$$

Then we define $b(k)$ to be the reflection coefficient. The reflection coefficient gives the amplitude of the reflected wave moving toward the left when the incident waves moves from $-\infty$ toward the origin.

It is our objective to show how the potential $V(x)$ can be calculated from the reflection coefficient $b(k)$.

The "outgoing wave solution" $\Psi_-(x|k)$ is the solution of (2.1) which is the sum of an incident wave and an outgoing scattered wave. It satisfies the integral equation

$$\Psi_-(x|k)=\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}-\frac{i}{2|k|}\int_{-\infty}^{+\infty}e^{i|k|\,|x-x'|}V(x')\Psi_-(x'|k)dx'. \tag{2.3}$$

If we take $k>0$ and let x approach $-\infty$, we see that $\Psi_-(x|k)$ takes the form (2.2), where

$$b(k)=-\frac{i\pi}{k}\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}V(x)\Psi_-(x|k)dx, \quad k>0. \tag{2.4}$$

Though $b(k)$ has been defined for $k>0$ only, we can continue this function analytically to negative values of k. From (2.3) and (2.4) it can be shown that

$$b(-k)=b^*(k), \quad k>0. \tag{2.5}$$

The asterisk means complex conjugate.

It will be useful to express $\Psi_-(x|k)$ in terms of the representation of the operator $-d^2/dx^2$. If we write

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}e^{-ik'x}\Psi_-(x|k)dx=u_-(k'|k), \tag{2.6}$$

it can be shown that the integral equation (2.3) can be written

$$u_-(k'|k)=\delta(k'-k)+\gamma_-(k^2-k'^2)T_-(k'|k), \tag{2.7}$$

where

$$T_-(k'|k)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}e^{-ik'x}V(x)\Psi_-(x|k)dx=\int_{-\infty}^{+\infty}V(k'|k'')u_-(k''|k)dk''. \tag{2.8}$$

In (2.8), $V(k|k')$ is just the operator $V(x)$ in the representation of the operator $-d^2/dx^2$, i.e.,

$$V(k|k')=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ikx}V(x)e^{ik'x}dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}V(x)e^{-i(k-k')x}dx. \tag{2.9}$$

The function $\gamma_-(\xi)$ is defined by

$$\gamma_-(\xi)=-i\pi\delta(\xi)+\frac{\mathrm{P}}{\xi}=\lim_{\epsilon\to+0}\frac{1}{\xi+i\epsilon}, \tag{2.10}$$

where the symbol P means that in integrations over ξ, the principal part of the integral is to be used.

From (2.4) and (2.8), we see that

$$b(k)=\frac{-i\pi}{k}T_-(-k|k), \quad k>0. \tag{2.11}$$

The eigenfunctions $\Psi_-(x|k)$ or equivalently $u_-(k'|k)$ can be shown to form an orthonormal set. They satisfy

$$\int_{-\infty}^{+\infty}\Psi_-^*(x|k)\Psi_-(x|k')dx=\delta(k-k'), \tag{2.12}$$

$$\int_{-\infty}^{+\infty}u_-^*(k''|k)u_-(k''|k')dk''=\delta(k-k'). \tag{2.12a}$$

We shall assume that there are no bound states. Thus the following completeness relations are valid:

$$\int_{-\infty}^{+\infty}\Psi^*(x|k)\Psi(x'|k)dk=\delta(x-x'), \tag{2.13}$$

$$\int_{-\infty}^{+\infty}u_-^*(k|k'')u_-(k'|k'')dk''=\delta(k-k'). \tag{2.13a}$$

We shall now show how $V(x)$ and $V(k|k')$ can be obtained in terms of $V(-k|k)$. Let us write

$$W(k)=V(-k|k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}V(x)e^{2ikx}dx. \tag{2.14}$$

From (2.14), a relation which will later prove useful is obtained, namely,

$$W(-k)=W^*(k). \tag{2.15}$$

From (2.14), it is seen that

$$V(x)=2\int_{-\infty}^{+\infty} W(k)e^{-2ikx}dk. \tag{2.16}$$

On using (2.9) and (2.16), we find

$$V(k|k')=W\left(\frac{k'-k}{2}\right). \tag{2.17}$$

Equation (2.17) shows that any element of $V(k|k')$ can be obtained from the knowledge of the elements $V(-k|k)=W(k)$. This observation is, in fact, the principal departure from the original Jost-Kohn treatment.

3. BASIC EQUATIONS

We shall now set up two equations which, with Eq. (2.17), will permit us to solve for the three unknowns $W(k)$, $V(k|k')$, and $T_-(k|k')$ in terms of the reflection coefficient $b(k)$. Having obtained $W(k)$, we can find $V(x)$ from (2.16).

Our first equation is that for $T_-(k|k')$. From (2.8) and (2.7), we have

$$T_-(k|k')=V(k|k')+\int_{-\infty}^{+\infty} V(k|k'')\gamma_-(k'^2-k''^2)T_-(k''|k')dk''. \tag{3.1}$$

We now want to find $V(k|k'')$ in terms of $T_-(k|k')$. In reference 1 this task is accomplished by solving (3.1) for $T_-(k|k')$ in terms of $V(k|k')$ through the use of a series. The series is then inverted to get $V(k|k')$ in terms of $T_-(k|k')$. Rather than go through this complicated procedure, we shall use the completeness relation to find $V(k|k')$ in terms of $T_-(k|k')$. From (2.8) and (2.13a), we have

$$V(k|k')=\int_{-\infty}^{+\infty} T_-(k|k'')u_-^*(k'|k'')dk''. \tag{3.2}$$

However, on using (2.7), we see that

$$V(k|k')=T_-(k|k')+\int_{-\infty}^{+\infty} T_-(k|k'')\gamma_-^*(k''^2-k'^2)T_-^*(k'|k'')dk''. \tag{3.3}$$

In particular,

$$V(-k|k)=T_-(-k|k)+\int_{-\infty}^{+\infty} T_-(-k|k'')\gamma_-^*(k''^2-k^2)T_-^*(k|k'')dk''. \tag{3.4}$$

If $k>0$ we have, on using (2.14) and (2.11),

$$W(k)=\frac{ik}{\pi}b(k)+\int_{-\infty}^{+\infty} T_-(-k|k'')\gamma_-^*(k''^2-k^2)\times T_-^*(k|k'')dk'',\quad k>0. \tag{3.5}$$

To extend this result for $k<0$, let $k=-p$, where $p>0$. Then from (2.15) and (2.5),

$$W(k)=W^*(p)=-\frac{ip}{\pi}b^*(p)+\int_{-\infty}^{+\infty} T_-^*(-p|k'')\gamma_-(k''^2-p^2)T_-(p|k'')dk'', \tag{3.6}$$

or

$$W(k)=\frac{ikb(k)}{\pi}+\int_{-\infty}^{+\infty} T_-(-k|k'')\gamma_-(k''^2-k^2)\times T_-^*(k|k'')dk'',\quad k<0. \tag{3.7}$$

We can combine (3.5) and (3.7) as

$$W(k)=\frac{ikb(k)}{\pi}+\int_{-\infty}^{+\infty} T_-(-k|k')[\eta(k)\gamma_-^*(k'^2-k^2)+\eta(-k)\gamma_-(k'^2-k^2)]T_-^*(k|k')dk', \tag{3.8}$$

where $\eta(k)$ is the Heaviside step function defined by

$$\eta(k)=1,\quad k>0;\qquad \eta(k)=0,\quad k<0. \tag{3.9}$$

Equations (3.1), (3.8), and (2.17) are our basic equations for $T_-(k|k')$, $W(k)$, and $V(k|k')$. To solve these equations we replace $b(k)$ in (3.8) by $\epsilon b(k)$, where ϵ is a smallness parameter. Furthermore, we write

$$W(k)=\sum_{n=1}^{\infty}\epsilon^n W^{(n)}(k),\quad T_-(k|k')=\sum_{n=1}^{\infty}\epsilon^n T_-^{(n)}(k|k'),\quad V(k|k')=\sum_{n=1}^{\infty}\epsilon^n V^{(n)}(k|k'), \tag{3.10}$$

substitute into (3.1), (3.8), and (2.17), and equate coefficients of the same power of ϵ. The lowest orders can be expressed quite simply in terms of $b(k)$. It is seen that

$$W^{(1)}(k)=ikb(k)/\pi,$$
$$V^{(1)}(k|k')=\frac{i(k'-k)}{2\pi}b\left(\frac{k'-k}{2}\right),$$
$$T_-^{(1)}(k|k')=V^{(1)}(k|k')=\frac{i(k'-k)}{2\pi}b\left(\frac{k'-k}{2}\right), \tag{3.11}$$
$$W^{(2)}(k)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} b\left(\frac{k''+k}{2}\right)b^*\left(\frac{k''-k}{2}\right)dk''=\frac{2}{(2\pi)^2}\int_{-\infty}^{+\infty} b(k'')b(k-k'')dk'',$$

etc. It is easily seen by induction that all orders of the unknown can be obtained in terms of a knowledge of $b(k)$.

We can obtain $V(x)$ in terms of the lowest orders of $b(k)$ quite simply. Writing

$$V(x)=\sum_{n=1}^{\infty}\epsilon^n V^{(n)}(x), \tag{3.12}$$

we see, from (2.17), that

$$V^{(n)}(x)=2\int_{-\infty}^{+\infty}W^{(n)}(k)e^{-2ikx}dk. \tag{3.13}$$

On using (3.11), we obtain

$$V^{(1)}(x)=\frac{2i}{\pi}\int_{-\infty}^{+\infty}kb(k)e^{-2ikx}dk=-2\frac{d}{dx}F(2x), \tag{3.14}$$

where $F(x)$ is the Fourier transform of $b(k)$,

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}b(k)e^{-ikx}dx. \tag{3.15}$$

[It is to be noted, that as a consequence of (2.5), $F(x)$ is real.] Also,

$$\begin{aligned}V^{(2)}(x)&=\frac{4}{(2\pi)^2}\int_{-\infty}^{+\infty}dke^{-2ikx}\int_{-\infty}^{+\infty}b(k'')b(k-k'')dk''\\&=\frac{4}{(2\pi)^2}\int_{-\infty}^{+\infty}e^{-2ik''x}b(k'')dk''\\&\qquad\times\int_{-\infty}^{+\infty}e^{-2ix(k-k'')}b(k-k'')dk\\&=4[F(2x)]^2.\end{aligned} \tag{3.16}$$

The expressions for $V^{(1)}(x)$ and $V^{(2)}(x)$ are precisely those which one would obtain using the Gelfand-Levitan procedure of references 2 and 3.

An alternative, but entirely equivalent procedure for obtaining the potential $V(x)$, is obtained when one substitutes $W(k)$, as given by (3.8), into (2.16). On using (2.8) and the fact that

$$\frac{1}{2\pi}\int_{\infty}^{+\infty}e^{ikx}[\eta(k)\gamma_-^*(k'^2-k^2)+\eta(-k)\gamma_-(k'^2-k^2)]dk=-\eta(-x)\frac{\sin k'x}{k'}, \tag{3.17}$$

it is seen that

$$\begin{aligned}V(x)&=\frac{2i}{\pi}\int_{-\infty}^{+\infty}kb(k)e^{-2ikx}dk\\&\quad-2\int_{-\infty}^{+\infty}\frac{dk}{k}\int_{-\infty}^{+\infty}V(x'')\Psi_-^*(x''|k)dx''\\&\quad\times\int_{-\infty}^{2x-x''}V(x')\Psi_-(x'|k)\sin k(x'+x''-2x)dx'.\end{aligned} \tag{3.18}$$

Equation (3.18) together with (2.3), which we rewrite as

$$\begin{aligned}\Psi_-(x|k)&=\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}\\&\quad-\frac{i}{2|k|}\int_{-\infty}^{+\infty}e^{i|k|\,|x-x'|}V(x')\Psi_-(x'|k)dx',\end{aligned} \tag{3.19}$$

are sufficient to solve for $\Psi_-(x|k)$ and $V(x)$. As before, one replaces $b(k)$ by $\epsilon b(k)$ and writes

$$V(x)=\sum_{n=1}^{\infty}\epsilon^n V^{(n)}(x), \tag{3.20}$$

$$\Psi_-(x|k)=\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}+\sum_{n=1}^{\infty}\epsilon^n\Psi_-^{(n)}(x|k). \tag{3.21}$$

On substituting (3.21) and (3.20) into (3.18) and (3.19) and equating coefficients of ϵ, one again sees how one may obtain $V^{(n)}(x)$, $\Psi_-^{(n)}(x|k)$ from a knowledge of $b(k)$ alone. On looking at (3.18), it is seen that as $x\to-\infty$, the second term on the right becomes small compared to first term. Hence, when $x\to-\infty$, $V(x)$ is approximated by the first term which is just $V^{(1)}(x)$, i.e.,

$$V(x)\cong V^{(1)}(x)=\frac{2i}{\pi}\int_{-\infty}^{+\infty}kb(k)e^{-2ikx}dk,\qquad\text{for}\quad x\to-\infty. \tag{3.22}$$

It is also useful to note that even to the lowest order approximation,

$$\Psi_-(x|k)\cong\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}+\Psi_-^{(1)}(x|k)$$

will reproduce the *exact* reflection coefficient $b(k)$. For

$$\begin{aligned}\Psi_-^{(1)}(x|k)&\cong-\frac{i}{2k}\frac{e^{-ikx}}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}V^{(1)}(x')e^{2ikx'},\\&\qquad\text{when}\quad k>0\quad\text{and}\quad x\to-\infty,\\&=\frac{e^{-ikx}}{(2\pi)^{\frac{1}{2}}}b(k).\end{aligned} \tag{3.23}$$

It can further be shown that $\Psi^{(n)}(x|k)\to0$ when $x\to-\infty$, $k>0$. Hence, when $V(x)$ and $\Psi_-(x|k)$ are solved to *any* order in ϵ, the *exact* reflection coefficient $b(k)$ will be reproduced.

4. SIMPLE EXAMPLE

Let us consider the case where the scattering potential is

$$V(x)=A\delta(x). \tag{4.1}$$

From the direct problem, it can be shown that

$$\Psi_-(x|k)=\frac{1}{(2\pi)^{\frac{1}{2}}}e^{ikx}+\frac{1}{(2\pi)^{\frac{1}{2}}}e^{i|k|\,|x|}C(|k|),\quad (4.2)$$

where

$$C(|k|)=-\tfrac{1}{2}iA\left(\frac{1}{|k|+\frac{1}{2}A}\right).\quad (4.3)$$

From (2.2), it is clear that the reflection coefficient is given by

$$b(k)=-\tfrac{1}{2}iA\left(\frac{1}{k+\frac{1}{2}iA}\right)=C(k).\quad (4.4)$$

From the inverse point of view, we must show that with $b(k)$ given by (4.4), Eqs. (3.18) and (3.19) for $V(x)$ and $\Psi_-(x|k)$ are satisfied by (4.1) and (4.2). That this is the case is easily proved by substitution.

It is, perhaps, interesting to evaluate the lowest two orders for $V(x)$ using the expansion scheme. We have

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}b(k)e^{-ikx}dk=-\frac{A}{2}\eta(x)e^{-Ax/2}.\quad (4.5)$$

Hence, from (3.14),

$$V^{(1)}(x)=-2\frac{d}{dx}F(2x)=A\delta(x)-A^2\eta(x)e^{-Ax};\quad (4.6)$$

while from (3.16),

$$V^{(2)}(x)=4[F(2x)]^2=A^2\eta(x)e^{-2Ax}.\quad (4.7)$$

Hence to the two lowest orders in the smallness parameter, after setting $\epsilon=1$.

$$V(x)=A\delta(x)-A^2\eta(x)[e^{-Ax}-e^{-2Ax}].\quad (4.8)$$

The second term on the right of (4.8) represents the error arising from the fact that we are working only to the second approximation.

5. REFLECTION COEFFICIENT AND OUTGOING WAVE SOLUTION: THREE-DIMENSIONAL CASE

In order to define the reflection coefficient for the three-dimensional wave equation, it is useful to introduce the "ray" or "optical" description of vectors which pass through the origin of coordinates.

Let us introduce a rectangular coordinate system. Let us also introduce a system of directed rays. These rays are described by unit vectors whose origin is the origin of coordinates. These vectors may point in any directions so long as the angle θ which this vector makes with the positive z-axis is less than $\pi/2$. An arbitrary vector, say $\mathbf{k}$, which passes through the origin may then be written

$$\mathbf{k}=k\mathbf{r},\quad (5.1)$$

where $\mathbf{r}$ is the unit vector describing the direction of the ray on which the vector $\mathbf{k}$ lies and

$$k=|\mathbf{k}|\ \mathrm{sign}\ k_z,\quad (5.2)$$

where k_z is the z component of $\mathbf{k}$.

Any function of the vector $\mathbf{k}$, $f(\mathbf{k})$ may be written

$$f(\mathbf{k})=f(k,\theta,\phi),\quad 0\leq\phi<2\pi,\quad 0\leq\theta<\pi/2,\quad -\infty<k<\infty,\quad (5.3)$$

if one wishes to show explicitly the dependence of the function $f(\mathbf{k})$ on k, and the angles θ (which is the angle made by the ray with the positive z axis) and ϕ (which is the angle which the projection of the ray on the x-y plane makes with the positive x axis). Vector integrals, written in terms of ray coordinates, have the following form:

$$\int f(\mathbf{x})d\mathbf{x}=\int_{-\infty}^{+\infty}dkk^2\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta d\theta f(k,\theta,\phi).\quad (5.4)$$

The integral on the right of (5.4) is the same as that obtained using the usual spherical coordinates except the limits of the integral in the variable θ now range from 0 to $\pi/2$, and the integral in the variable k ranges from $-\infty$ to $+\infty$.

In terms of the ray description, we can now define a reflection coefficient along a ray in three dimensions analogous to the reflection coefficient for one dimension.

Let us consider the solutions $\Psi_-(\mathbf{x}|\mathbf{k})$ of the equation

$$[-\nabla^2+V(\mathbf{x})]\Psi_-(\mathbf{x}|\mathbf{k})=|\mathbf{k}|^2\Psi_-(\mathbf{x}|\mathbf{k}),\quad (5.5)$$

which satisfy the condition that they are the sum of an incident wave and an outgoing spherical wave. Such solutions satisfy the integral equation:

$$\Psi_-(\mathbf{x}|\mathbf{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}-\frac{1}{4\pi}\int_{-\infty}^{+\infty}e^{i|\mathbf{k}|\,|\mathbf{x}-\mathbf{x}'|}V(\mathbf{x}')\Psi_-(\mathbf{x}'|\mathbf{k})d\mathbf{x}'.\quad (5.6)$$

We shall define the reflection coefficient $b(\mathbf{k})=b(k,\theta,\phi)$ as being the amplitude of the spherical wave which one obtains from (5.6) when the optical coordinate k is taken positive and when $|\mathbf{x}|$ approaches infinity in the negative direction on the ray on which $\mathbf{k}$ lies. Generally, when $|\mathbf{x}|\rightarrow\infty$ we have, from (5.6),

$$\Psi_-(\mathbf{x}|\mathbf{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}-\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{i|\mathbf{k}|\,|\mathbf{x}|}}{|\mathbf{x}|}\times\int\frac{1}{(2\pi)^{\frac{3}{2}}}e^{-i\mathbf{k}'\cdot\mathbf{x}'}V(\mathbf{x}')\Psi_-(\mathbf{x}'|\mathbf{k})d\mathbf{x}',\quad (5.7)$$

where

$$\mathbf{k}'=|\mathbf{k}|\mathbf{x}\frac{\mathbf{x}}{|\mathbf{x}|}.$$

On setting $k>0$ and $\mathbf{k}'=-\mathbf{k}$ we have, by definition of $b(\mathbf{k})$,

$$\Psi_-(\mathbf{x}|\mathbf{k})=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}+\frac{e^{i|\mathbf{k}|\,|\mathbf{x}|}}{|\mathbf{x}|}b(\mathbf{k}), \quad (5.8a)$$

where

$$b(\mathbf{k})=b(k,\theta,\phi)=-\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\int\frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}'} \times V(\mathbf{x}')\Psi_-(x'|\mathbf{k})d\mathbf{x}', \quad k>0. \quad (5.8b)$$

It is our objective to show how the scattering potential can be obtained from $b(\mathbf{k})$, where $\mathbf{k}$ is such that it makes an angle less than $\pi/2$ radians with the positive z-axis. The development is almost identical with the one-dimensional case.

We note first that though $b(\mathbf{k})=b(k,\theta,\phi)$ is defined for $k>0$ only, we can analytically continue $b(k,\theta,\phi)$ (for fixed θ,ϕ) to negative values of k. It can be shown that

$$b(-\mathbf{k})=b^*(\mathbf{k}), \quad k>0 \quad (5.9)$$

or

$$b(-k,\theta,\phi)=b^*(k,\theta,\phi), \quad k>0. \quad (5.9a)$$

It is convenient to express $\Psi_-(\mathbf{x}|\mathbf{k})$ in terms of the representation of the operator $-\nabla^2$. If we write

$$u_-(\mathbf{k}'|\mathbf{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{-i(\mathbf{k}'\cdot\mathbf{x})}\Psi_-(\mathbf{x}|\mathbf{k})d\mathbf{x}, \quad (5.10)$$

it can be shown that the integral equation (5.6) yields the following equation for $u_-(\mathbf{k}|\mathbf{k}')$:

$$u_-(\mathbf{k}|\mathbf{k}')=\delta(\mathbf{k}-\mathbf{k}')+\gamma_-(k'^2-k^2)T_-(\mathbf{k}|\mathbf{k}') \quad (5.11)$$

where

$$T_-(\mathbf{k}|\mathbf{k}')=\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{-i\mathbf{k}\cdot\mathbf{x}}V(\mathbf{x})\Psi_-(\mathbf{x}|\mathbf{k}')dx = \int V(\mathbf{k}|\mathbf{k}'')u_-(\mathbf{k}''|\mathbf{k}')d\mathbf{k}'' \quad (5.12)$$

and

$$V(\mathbf{k}|\mathbf{k}')=\frac{1}{(2\pi)^3}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}V(\mathbf{x})d\mathbf{x}. \quad (5.13)$$

If we assume that there are no bound states, we have the equations

$$\int u_-^*(k|k')u_-(k|k'')dk=\delta(k'-k''), \qquad \int u_-^*(k'|k)u_-(k''|k)dk=\delta(k'-k''). \quad (5.14)$$

From (5.8b) it is seen that

$$b(\mathbf{k})=-(\pi/2)^{\frac{1}{2}}T_-(-\mathbf{k}|\mathbf{k}), \quad k>0. \quad (5.15)$$

6. BASIC EQUATIONS IN THREE DIMENSIONS

Since the procedure for finding the basic equations is almost identical to that for the one-dimensional case, we shall merely summarize our results below.

The first basic equation is

$$T_-(\mathbf{k}|\mathbf{k}')=V(\mathbf{k}|\mathbf{k}') +\int V(\mathbf{k}|\mathbf{k}'')\gamma_-(k'^2-k''^2)T_-(\mathbf{k}''|\mathbf{k}')d\mathbf{k}. \quad (6.1)$$

The second is

$$W(\mathbf{k})=-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}b(\mathbf{k})+\int T_-(-\mathbf{k}|\mathbf{k}')[\eta(k)\gamma_-^*(k'^2-k^2) +\eta(-k)\gamma_-(k'^2-k^2)]T_-^*(\mathbf{k}|\mathbf{k}')d\mathbf{k}', \quad (6.2)$$

while the third is

$$V(\mathbf{k}|\mathbf{k}')=W\left(\frac{\mathbf{k}'-\mathbf{k}}{2}\right). \quad (6.3)$$

One solves these three equations as in the one-dimensional case. One replaces $b(\mathbf{k})$ by $\epsilon b(\mathbf{k})$ and writes

$$\begin{aligned} T_-(\mathbf{k}|\mathbf{k}')&=\sum_{n=1}^{\infty}\epsilon^nT_-^{(n)}(\mathbf{k}|\mathbf{k}'),\\ W(\mathbf{k})&=\sum_{n=1}^{\infty}\epsilon^nW^{(n)}(\mathbf{k}),\\ V(\mathbf{k}|\mathbf{k}')&=\sum_{n=1}^{\infty}\epsilon^nW^{(n)}\left(\frac{\mathbf{k}'-\mathbf{k}}{2}\right).\end{aligned} \quad (6.4)$$

Upon substituting (6.4) into (6.1)–(6.3), one sees that $T^{(n)}$ and $W^{(n)}$ can be obtained from a knowledge of $b(\mathbf{k})$ alone. To obtain $V(\mathbf{x})$ from $W(\mathbf{k})$, one uses

$$V(\mathbf{x})=8\int W(\mathbf{k})e^{-2i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}. \quad (6.5)$$

In the special case where $V(\mathbf{x})$ is a function of $|\mathbf{x}|$ alone, i.e., where $V(\mathbf{x})$ is a spherically symmetric potential, $b(\mathbf{k})$ will be independent of the direction of the ray. In principle, one could calculate such a potential by observing the amplitude for all energies of the spherical wave reflected backward along a single ray upon which an incident plane wave was sent. This datum is quite different from that of reference 1, where one requires knowledge of the value of a scattering phase for all energies for a single angular momentum. It also differs from the data required in the work of Wheeler.[4] Here knowledge of all phases is required for a single energy. The classical analog of our result for spherically symmetric potentials is given by Keller *et al.*[5]

[4] J. A. Wheeler, Phys. Rev. **99**, 630 (1955).
[5] Keller, Kay, and Shmoys, Phys. Rev. **102**, 557 (1956).

One can write the basic equations in the $\mathbf{x}$ representation in analogy to the one-dimensional case. Our basic equations are then

$$V(\mathbf{x})=-\left(\frac{128}{\pi}\right)^{\frac{1}{2}}\int b(\mathbf{k})e^{-2i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}$$
$$+8\int d\mathbf{k}\int V(\mathbf{x}')\Psi_-(\mathbf{x}'|\mathbf{k})d\mathbf{x}',$$
$$\times\int V(\mathbf{x}'')\Psi_-^*(\mathbf{x}''|\mathbf{k})d\mathbf{x}''g_{\mathbf{k}}(\mathbf{x}'+\mathbf{x}''-2\mathbf{x}), \quad (6.6)$$

and

$$\Psi_-(\mathbf{x}|\mathbf{k})=\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}}-\frac{1}{4\pi}\int e^{i|\mathbf{k}|\,|\mathbf{x}-\mathbf{x}'|}V(\mathbf{x}')\Psi_-(\mathbf{x}'|\mathbf{k})d\mathbf{x}'. \quad (6.7)$$

In (6.6), the function $g_{\mathbf{k}}(\mathbf{x})$ is a Green's function for the operator $(\nabla^2+\mathbf{k}^2)$ given by

$$g_{\mathbf{k}}(\mathbf{x})=\frac{1}{(2\pi)^3}\int e^{i\mathbf{p}\cdot\mathbf{x}}[\eta(p)\gamma_-^*(k^2-p^2)+\eta(-p)\gamma_-(k^2-p^2)]d\mathbf{p}$$
$$=\frac{-\eta(-z)}{(2\pi)^2}\int_{-\infty}^{+\infty}d\lambda\int_{-\infty}^{+\infty}d\nu e^{i\lambda x}e^{i\nu y}\times\frac{\sin[(k^2-\lambda^2-\nu^2)^{\frac{1}{2}}x]}{(k^2-\lambda^2-\nu^2)^{\frac{1}{2}}}. \quad (6.8)$$

In (6.8), k and p are the optical coordinates of $\mathbf{k}$ and $\mathbf{p}$; x, y, and z are the Cartesian coordinates of the vector $\mathbf{x}$.

To solve (6.6) and (6.8) simultaneously, one replaces $b(\mathbf{k})$ by $\epsilon b(\mathbf{k})$ and writes

$$V(\mathbf{x})=\sum_{n=1}^{\infty}\epsilon^nV^{(n)}(\mathbf{x}),$$
$$\Psi_-(\mathbf{x}|\mathbf{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}+\sum_{n=1}^{\infty}\epsilon^n\Psi_-^{(n)}(\mathbf{x}|\mathbf{k}). \quad (6.9)$$

Upon substituting into (6.6) and (6.8), one obtains an iteration procedure for finding $V^{(n)}(\mathbf{x})$ and $\Psi_-^{(n)}(\mathbf{x})$. In analogy to the one-dimensional case, it can be shown that, to any approximation, $\Psi_-(\mathbf{x}|\mathbf{k})$ reproduces the reflection coefficient $b(\mathbf{k})$.

7. GENERAL PROCEDURE

There is a more general procedure which may often be used to obtain the scattering potential from appropriate elements of the scattering operator provided one has specified the representation in which the scattering potential is diagonal. In fact, the writer originally obtained the results for the one- and three-dimensional case in terms of this procedure.

We shall now discuss the problem in detail. We shall use the notation and equations of a previous paper.[6]

Let H and H_0 be the total and unperturbed Hamiltonians respectively, and let V denote the perturbation. Then we have

$$H=H_0+V. \quad (7.1)$$

Let the eigenfunctions of H_0 be denoted by $\omega_0(E,\alpha)$, where E is the quantum number associated with the operator H_0 $(0\leq E<\infty)$ and α labels the degeneracy. For example, α might denote the direction or angular momentum in the case of a single particle. We shall assume that H has the same spectrum as H_0 (more general Hamiltonians can be treated similarly). There is one class of eigenfunctions of H which is of interest, namely the "outgoing" eigenfunctions. We shall call them $\omega_-(F,\beta)$. The variable F denotes the eigenvalues of H $(0\leq F<\infty)$. The variable β is the degeneracy label for H. Since we have assumed the spectrum of H to be the same as that of H_0, β will have the same range as α.

It is convenient to work in the H_0 representation. The variables corresponding to ω_- in the H_0 representation, denoted by $u_-(E,\alpha|F,\beta)$, are

$$u_-(E,\alpha|F,\beta)=(\omega_0(E,\alpha),\omega_-(F,\beta)); \quad (7.2)$$

the right-hand side of (7.2) indicates the inner product of the eigenvectors involved. The eigenfunctions ω_- are characterized by the fact that the corresponding variables in the H_0 representation satisfy the integral equations

$$u_-(E,\alpha|F,\beta)=\delta(F-E)\delta(\beta,\alpha)+\gamma_-(F-E)T_-(E,\alpha|F,\beta), \quad (7.3)$$

where $\delta(\alpha,\beta)$ is a suitable generalization of the Dirac or Kronecker δ function, and where

$$\gamma_-(x)=-i\pi\delta(x)+\mathrm{P}/x \quad (7.4)$$

(P means Cauchy principal value), and

$$T_-(E,\alpha|F,\beta)=\int\int V(E,\alpha|E',\alpha')u_-(E',\alpha'|F,\beta)d\alpha'dE'. \quad (7.5)$$

In (7.5) $V(E,\alpha|E',\alpha')$ is the "matrix" of V in the H_0-representation. Explicitly,

$$V(E,\alpha|E',\alpha')=(\omega_0(E,\alpha),V\omega_0(E',\alpha')). \quad (7.6)$$

It can be shown that the scattering operator in the H_0-representation is given by

$$S(E,\alpha|E',\alpha')=\delta(E-E')\delta(\alpha,\alpha')-2\pi i\delta(E-E')T_-(E',\alpha|E',\alpha'). \quad (7.7)$$

[6] H. E. Moses, Nuovo cimento 1, 103 (1955).

Now, our problem is to find $V(E,\alpha|E',\alpha')$ in terms of a given $S(E,\alpha|E',\alpha')$, or equivalently $T_-(E',\alpha|E',\alpha')$. From (7.3) and (7.5) it is clear that

$$T_-(E,\alpha|F,\beta)=V(E,\alpha|F,\beta)+\int\int V(E,\alpha|E',\alpha')\times\gamma_-(F-E')T_-(E',\alpha'|F,\beta)dE'd\alpha'. \quad (7.8)$$

Since we have assumed that the spectrum of H is the same as that of H_0, the completeness condition

$$\int\int u_-^*(E,\alpha|F,\beta)u_-(E',\alpha'|F,\beta)dFd\beta=\delta(E-E')\delta(\alpha,\alpha') \quad (7.9)$$

must be satisfied. Using (7.9) and (7.5), we have

$$\int\int T_-(E,\alpha|F,\beta)u_-^*(E'',\alpha''|F,\beta)dFd\beta$$
$$=\int\int V(E,\alpha|E',\alpha')d\alpha'dE'\times\int\int u_-(E'',\alpha''|F,\beta)u_-^*(E',\alpha'|F,\beta)dFd\beta$$
$$=V(E,\alpha|E'',\alpha'').$$

Using (7.3) we have finally

$$V(E,\alpha|E',\alpha')=T_-(E,\alpha|E',\alpha')+\int\int T_-(E,\alpha|F,\beta)\times\gamma_-^*(F-E')T_-^*(E',\alpha'|F,\beta)dFd\beta. \quad (7.10)$$

On first thought it might seem that Eqs. (7.8) and (7.10) can be considered as two equations in two unknowns, namely $V(E,\alpha|E',\alpha')$ and $T_-(E,\alpha|F,\beta)$. However, this conjecture is false, for if $V(E,\alpha|E',\alpha')$ as given by (7.10) is substituted in (7.8) so as to obtain an equation for T_- only, and if the orthogonality relations

$$\int\int u_-^*(E,\alpha|F,\beta)u_-(E,\alpha|F',\beta')dEd\alpha=\delta(F-F')\delta(\beta,\beta')$$

are used, the result is a trivial identity. Hence (7.8) and (7.10) do not contain enough information to solve for V. The additional information needed is obtained, for example, by specifying the representation in which V is to be diagonal. It will be shown that with such a specification it is possible to solve for $T_-(E,\alpha|E',\alpha')$ in terms of $T_-(E',\alpha|E',\alpha')$ (i.e., in terms of the scattering operator). Then we can find $V(E,\alpha|E',\alpha')$ from (7.10).

Let q denote collectively the eigenvalues of the complete set of commuting variables in which we wish V to be diagonal. Furthermore, let $\Psi(q|E,\alpha)$ (assumed known) be the eigenfunctions $\omega_0(E,\alpha)$ of H_0 as given in the q-representation. Then we have

$$V(E,\alpha|E',\alpha')=\int\Psi^*(q|E,\alpha)V(q)\Psi(q|E',\alpha')dq, \quad (7.11)$$

where $V(q)$ is the potential as expressed in the q-representation. In particular,

$$V(E,\alpha|E,\alpha')=\int\Psi^*(q|E,\alpha)\Psi(q|E,\alpha')V(q)dq$$
$$=\int L(E,\alpha,\alpha'|q)V(q)dq. \quad (7.11\text{a})$$

The kernel L is defined by the second equation (7.11a). Equation (7.11a) is to be considered as an integral equation for $V(q)$ which is to be solved in terms of $V(E,\alpha|E,\alpha')$.

$$V(q)=\int\int\int L^{-1}(E,\alpha,\alpha'|q)\times V(E,\alpha|E,\alpha')dEd\alpha d\alpha', \quad (7.12)$$

where the operator L^{-1} is the inverse of L. The existence of this operator will depend on the choice of q. From (7.11) we see that we can write $V(E,\alpha|E',\alpha')$ in terms of $V(E,\alpha|E,\alpha')$ as follows:

$$V(E,\alpha|E',\alpha')$$
$$=\int\int\int\int\Psi^*(E,\alpha|q)\Psi(E',\alpha'|q)\times L^{-1}(F',\beta,\beta'|q)V(F',\beta|F',\beta')dqdF'd\beta d\beta'$$
$$=\int\int\int K(E,\alpha|E',\alpha'||F',\beta|F',\beta')\times V(F',\beta|F',\beta')dF'd\beta d\beta', \quad (7.13)$$

where $K(E,\alpha|E',\alpha'||F',\beta|F',\beta')$ is a known function of its arguments and depends on the representation to be chosen for the diagonalization of V. Equation (7.13) provides the additional information needed to solve for $T_-(E,\alpha|F,\beta)$ in terms of $T_-(E,\alpha|E,\alpha')$. Equation (7.13) shows how all of the matrix elements $V(E,\alpha|E',\alpha')$ may be obtained from a knowledge of some of the elements $V(E,\alpha|E,\alpha')$. Equations (2.17) and (6.3) are the forms that Eq. (7.13) takes in the one- and three-dimensional cases. As will be shown below, those elements of $V(E,\alpha|E,\alpha')$ which are used to determine the remainder of the elements $V(E,\alpha|E',\alpha')$ determine also which elements of the scattering operator $T_-(E,\alpha|E,\alpha')$ are needed to find V.

Using (7.13) and (7.10), we have

$$V(E,\alpha|E',\alpha')=\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|F,\beta')dFd\beta d\beta'$$
$$+\int\int\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|G,\gamma)\gamma_-{}^*(G-F)$$
$$\times T_-{}^*(F,\beta'|G,\gamma)dFd\beta d\beta' dGd\gamma, \quad (7.14)$$

which we substitute into (7.18) to obtain

$$T_-(E,\alpha|E',\alpha')=\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|F,\beta')dFd\beta d\beta'$$
$$+\int\int\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|G,\gamma)\gamma_-{}^*(G-F)$$
$$\times T_-{}^*(F,\beta'|G,\gamma)dFd\beta d\beta' dGd\gamma$$
$$+\int\int\int\int\int K(E,\alpha|D,\delta||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|F,\beta')\gamma_-(E'-D)$$
$$\times T_-(D,\delta|E',\alpha')dFd\beta d\beta' dDd\delta$$
$$+\int\int\int\int\int\int\int K(E,\alpha|D,\delta||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|G,\gamma)\gamma_-{}^*(G-F)$$
$$\times T_-{}^*(F,\beta'|G,\gamma)\gamma_-(E'-D)T_-(D,\delta|E',\alpha')$$
$$\times dFd\beta d\beta' dGd\gamma dDd\delta. \quad (7.15)$$

When this equation is solved for $T_-(E,\alpha|E',\alpha')$, one can substitute in (7.10) to obtain $V(E,\alpha|F,\beta)$. The Jost-Kohn procedure for solving (7.15) is to replace $T_-(F,\beta|F,\beta')$ by $\epsilon T(F,\beta|F,\beta')$, where ϵ is a smallness parameter. One then expands $T_-(E,\alpha|E',\alpha')$ in an infinite series in ϵ as follows:

$$T_-(E,\alpha|F,\beta)=\sum_{n=1}^{\infty}\epsilon^n T_-{}^{(n)}(E,\alpha|F,\beta), \quad (7.16)$$

and, substituting (7.16) into (7.15) and equating coefficients of equal powers ϵ, one obtains a series of expressions in which the $T_-{}^{(n)}(E,\alpha|F,\beta)$ are given in terms of $T_-{}^{(m)}(E,\alpha|F,\beta)$, $(m<n)$. One then can solve for $T_-{}^{(n)}$ by induction and substitute into (7.16) to obtain $T_-(E,\alpha|F,\beta)$. For the lowest two orders, for example, we find

$$T_-{}^{(1)}(E,\alpha|E',\alpha')$$
$$=\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-(F,\beta|F,\beta')dFd\beta d\beta', \quad (7.17)$$

$$T_-{}^{(2)}(E,\alpha|E',\alpha')$$
$$=\int\int\int\int\int K(E,\alpha|E',\alpha'||F,\beta|F,\beta')$$
$$\times T_-{}^{(1)}(F,\beta|G,\gamma)\gamma_-{}^*(G-F)$$
$$\times T_-{}^{*(1)}(F,\beta|G,\gamma)dFd\beta d\beta' d\gamma$$
$$+\int\int\int\int\int K(E,\alpha|D,\delta||F,\beta|F,\beta')$$
$$\times T_-{}^{(1)}(F,\beta|F,\beta')\gamma_-(E'-D)$$
$$\times T_-{}^{(1)}(D,\delta|E',\alpha')dFd\beta d\beta' dDd\delta. \quad (7.18)$$