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Analysis of Cyclotron Absorption in Bismuth*

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Classical magnetoionic theory is used for calculating the absorption of microwaves in bismuth in the presence of a dc magnetic field. The detailed calculations are based on the models of the energy surfaces in momentum space for holes and electrons obtained from de Haas-van Alphen and galvanomagnetic experiments. Expressions are derived for the conductivities and resonance masses. Numerical computations of absorption *vs* magnetic field are made using the data from Shoenberg's de Haas-van Alphen experiments for the masses of the electrons. Since some of the experimental data for the holes is incomplete, the analysis takes this into account through the use of several choices of hole masses. The results are compared with those of preliminary microwave experiments. The limitations of the analysis, which neglects the anomalous skin effect and possible anisotropy of scattering, are discussed.

I. INTRODUCTION

THE de Haas-van Alphen experiments[1] and recent work on galvanomagnetic effects[2] in bismuth have indicated that the energy surfaces for holes and electrons in bismuth are two families of ellipsoids. Recent microwave measurements on cyclotron absorption in bismuth[3,4] have presented preliminary data which have not as yet been interpreted quantitatively. It is the object of this paper to take two sets of energy surfaces adapted from the de Haas-van Alphen and galvanomagnetic results and, using reasonable assumptions of scattering, attempt to fit the data of the microwave experiments. We shall endeavor to point out the difficulties encountered in applying the classical magneto-ionic theory to account for the resonance values reported by the microwave experiments.

The simplest model, adopted by Abeles and Meiboom[2] to explain their results on Hall measurements and magnetoresistance, has been that of an ellipsoid of revolution for holes with the major axis along the trigonal axis and a set of three ellipsoids for the electrons. One electron ellipsoid has a principal axis along a binary axis and another principal axis along the trigonal axis; the other two ellipsoids are generated from this one by 120° rotations about the trigonal axis. This model is essentially a combination of the types of energy surfaces proposed by Jones[5] and Blackman.[1] We shall refer to it as the A-M model.

A second model consists of a combination of a single ellipsoid of revolution for holes and a set of three inclined ellipsoids for the electrons as proposed by Shoenberg[1] to explain his de Haas-van Alphen experiments. One of the inclined ellipsoidal surfaces can be represented in k-space by the energy-momentum relation

$$E=\tfrac{1}{2}\hbar(\alpha_{11}k_1^2+\alpha_{22}k_2^2+2\alpha_{23}k_2k_3+\alpha_{33}k_3^2)/m, \qquad (1)$$

where subscripts 1, 2, and 3 refer to the binary, bisectrix, and trigonal axes, respectively. The α's are dimensionless constants and m is the free electron mass. Again there are two more such ellipsoids to give 120° rotational symmetry about the trigonal axis. This second model of Jones and Shoenberg will be referred to as the J-S model.

Although the A-M model explains the galvanomagnetic measurements satisfactorily, an interpretation based on the J-S model is still possible. Since Abeles and Meiboom do not report longitudinal magnetoresistance measurements along the trigonal axis, the J-S model, which would predict the presence of longitudinal magnetoresistance in this direction, is not ruled out. It should be added that the two models do not differ very greatly from one another since the ellipsoids of Shoenberg are tilted out of the trigonal plane only 6°.

II. THEORY

A. Electromagnetic Problem

In the microwave experiments, a polished face of a single-crystal surface was effectively made an integral part of one wall of a cavity. It can be shown that the absorption of energy by the sample in the cavity can be treated by evaluating the real part of the Poynting vector of a plane wave normally incident on a semi-infinite slab of metal. To do this one must solve the appropriate boundary value problem. Since this is not the object of this paper, and the problem has been treated elsewhere,[6] we shall simply give the result for a metal:

$$P/P_0=4\beta\beta_0/(\alpha^2+\beta^2), \qquad (2)$$

where P is the power absorbed in the metal and P_0 is

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[1] M. Blackman, Proc. Roy. Soc. (London) **A166**, 1 (1938); D. Shoenberg, Proc. Roy. Soc. (London) **A170**, 341 (1939); Trans. Roy. Soc. (London) **A245**, 1 (1952).

[2] B. Abeles and S. Meiboom, Phys. Rev. **101**, 544 (1956).

[3] R. N. Dexter and B. Lax, Phys. Rev. **100**, 1216 (1955).

[4] Galt, Yager, Merritt, Cetlin, and Dail, Phys. Rev. **100**, 748 (1955).

[5] H. Jones, Proc. Roy. Soc. (London) **A147**, 396 (1934); **A155**, 653 (1936).

[6] B. Lax and Laura M. Roth (to be published).

the incident power. The propagation constant in the metal is $\Gamma=\alpha+j\beta$ and β_0 is the value of the propagation constant in free space. The components of Γ can be readily evaluated from the general dispersion equation[6,7]

$$\Gamma^2=-\omega^2\epsilon\mu_0+j\omega\mu_0\sigma_{\text{eff}}, \tag{3}$$

where ω is the angular frequency of the rf field and ϵ is the dielectric permittivity of the sample.

σ_{eff} is a combination of the conductivity tensor components appropriate to the particular configuration. For convenience, we shall restrict our detailed treatment to conductivity tensors of the form

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & 0\\ \sigma_{yx} & \sigma_{yy} & 0\\ 0 & 0 & \sigma_{zz}\end{bmatrix},$$

where a dc magnetic field is applied along the z-direction. When the magnetic field is perpendicular to the surface of the sample, σ_{eff} takes the form for a metal

$$\sigma_{\perp}=\frac{\sigma_{xx}+\sigma_{yy}}{2}\pm\left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2+\sigma_{xy}\sigma_{yx}\right]^{\frac{1}{2}}, \tag{4}$$

where the positive and negative signs refer to the two contrarotating elliptically polarized plane waves. When the magnetic field is parallel to the surface of the metal, the effective conductivity is given by

$$\sigma_{\parallel}=\sigma_{xx}-(\sigma_{xy}\sigma_{yx}/\sigma_{yy}), \tag{5}$$

where the x-axis lies in the plane of the surface. This particular value of $\sigma_{\parallel}$ corresponds to the elliptically polarized wave where the plane of polarization is perpendicular to the magnetic field.

Since the conduction current is much greater than the displacement current in a metal at reasonable values of magnetic field, Eq. (3) can be rewritten

$$\Gamma\approx j\omega\mu_0(\sigma_r-j\sigma_i), \tag{6}$$

where σ_r and σ_i are the real and imaginary parts of σ_{eff}. If we substitute the result of Eq. (6) into Eq. (2), we can show that the power absorbed is proportional to the following (neglecting factors which do not depend on the magnetic field):

$$\frac{P}{P_0}\propto\left[\frac{1}{(\sigma_i^2+\sigma_r^2)^{\frac{1}{2}}}-\frac{\sigma_i}{\sigma_r^2+\sigma_i^2}\right]^{\frac{1}{2}}. \tag{7}$$

B. Conductivity Tensors

We shall derive the conductivity tensor components for the single and the three ellipsoid models using the tilted ellipsoids of Shoenberg for the electrons. The mass tensor of one of the tilted ellipsoids, normalized by the mass of the free electron, may be represented by

$$m_a/m_0=\begin{bmatrix}m_1 & 0 & 0\\ 0 & m_2 & m_4\\ 0 & m_4 & m_3\end{bmatrix}, \tag{8}$$

where axes 1, 2, and 3 are the binary, bisectrix, and trigonal axes, respectively. The A-M model can be represented by letting $m_4=0$. The other two ellipsoids have mass tensors m_b/m_0 and m_c/m_0 derived from Eq. (8) by the two transformation matrices

$$S_{b,\,c}=\tfrac{1}{2}\begin{bmatrix}-1 & \pm\sqrt{3} & 0\\ \mp\sqrt{3} & -1 & 0\\ 0 & 0 & 2\end{bmatrix}. \tag{9}$$

If one carries out the solution of the equation of motion using the resultant mass tensors, one obtains the following expressions for the conductivities (see Appendices I and II) when the magnetic field B is taken along one of the three principal axes:

B Along Trigonal Axis (Axis 3)

$$\begin{aligned}&\sigma_{11}/\sigma_0=\sigma_{22}/\sigma_0=(1/2\Delta)[(m_1+m_2)m_3-m_4^2],\\ &\sigma_{12}/\sigma_0=-\sigma_{21}/\sigma_0=bm_3/\Delta,\quad \sigma_{33}=(1/\Delta)(m_1m_2+b^2),\\ &\Delta_a=\Delta_b=\Delta_c=\Delta=m_1m_2m_3-m_1m_4^2+m_3b^2.\end{aligned}\tag{10}$$

B Along Binary Axis (Axis 1)

$$\begin{aligned}&\frac{\sigma_{11}}{\sigma_0}=1/3m_1+(2/3\Delta_c)\left[\frac{(3m_1+m_2)m_3-m_4^2}{4}+b^2\right],\\ &\frac{\sigma_{22}}{\sigma_0}=\frac{m_1m_3}{3\Delta_a}+\frac{2}{3\Delta_c}\left[\frac{(m_1+3m_2)m_3-3m_4^2}{4}\right],\\ &\frac{\sigma_{33}}{\sigma_0}=\frac{m_1m_2}{3\Delta_a}+\frac{2m_1m_2}{3\Delta_c},\\ &\sigma_{23}=\frac{m_1(b-m_4)}{3\Delta_a}+\left(\frac{2}{3\Delta_c}\right)\left[\frac{(m_1+3m_2)}{4}b+\frac{m_1m_4}{2}\right],\\ &\sigma_{32}=-\frac{m_1(b+m_4)}{3\Delta_a}+\left(\frac{2}{3\Delta_c}\right)\left[\frac{m_1m_4}{2}-\frac{(m_1+3m_2)}{4}b\right],\\ &\Delta_a=m_1(m_2m_3-m_4^2+b^2),\\ &\Delta_b=\Delta_c=m_1m_2m_3-m_1m_4^2+\tfrac{1}{4}b^2(m_1+3m_2).\end{aligned}\tag{11}$$

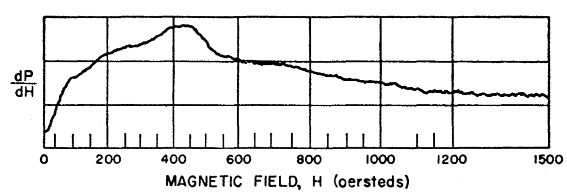

FIG. 1. Experimental trace of the derivative of absorption *vs* B, when the magnetic field is applied in the trigonal plane parallel to the binary (axis 1). (From the data of R. N. Dexter taken at 1.3°K and 23 000 Mc/sec.)

[7] B. Lax and Laura M. Roth, Phys. Rev. 98, 548 (1955).

B Along Axis 2

$$\frac{\sigma_{11}}{\sigma_0}=\frac{1}{3\Delta_a}(m_2m_3-m_4^2)+\left(\frac{2}{3\Delta_c}\right)\left[\frac{(3m_1+m_2)m_3-m_4^2}{4}\right],$$

$$\frac{\sigma_{22}}{\sigma_0}=\left(\frac{1}{3\Delta_a}\right)(m_1m_3+b^2)$$

$$+\left(\frac{2}{3\Delta_c}\right)\left[\frac{(m_1+3m_2)m_3-3m_4^2}{4}+b^2\right],$$

$$\frac{\sigma_{33}}{\sigma_0}=\frac{m_1m_2}{3\Delta_a}+\frac{2m_1m_2}{3\Delta_c}, \tag{12}$$

$$\sigma_{13}=-\sigma_{31}=-\frac{bm_2}{3\Delta_a}+\frac{2b}{3\Delta_c}\left(\frac{3m_1+m_2}{4}\right),$$

$$\sigma_{12}=-\sigma_{12}=-bm_4/\Delta_c,$$

$$\sigma_{23}=\sigma_{32}=\frac{m_1m_4}{3\Delta_a}+\frac{m_1m_4}{3\Delta_c},$$

$$\Delta_a=m_1m_2m_3-m_1m_4^2+m_2b^2,$$

$$\Delta_b=\Delta_c=m_1m_2m_3-m_1m_4^2+\tfrac{1}{4}(3m_1+m_2)b^2,$$

where $b=eB/[(\nu+j\omega)m_0]$ and $\sigma_0=ne^2/[m_0(\nu+j\omega)]$, n is the electron density, m_0 the mass of the free electron, ν the collision frequency, e the electron charge, and ω the angular frequency of the rf field.

To obtain the corresponding tensor components for the holes we simply let $m_4=0$, take only the terms involving Δ_a with the factor of $\frac{1}{3}$ omitted, and change the sign of b.

C. Resonances

One of the simplest applications of the above results is to obtain numerical values of the resonance masses from the values of the effective masses given by Shoenberg. This can be done simply by taking $\nu=0$ in the expression for b and setting the determinants in the denominators of the conductivity expressions equal to zero, i.e., $\Delta_a=\Delta_b=\Delta_c=0$. When one uses Shoenberg's values[1] for bismuth of $m_1=2.4\times10^{-3}$, $m_2=2.5$, $m_3=0.05$, $m_4=-0.25$, the results become

$B\|$axis 3,

$$m^*=m_0[(m_2m_3-m_4^2)m_1/m_3]^{\frac{1}{2}}=0.055m_0; \tag{13a}$$

$B\|$axis 1,

$$m_a^*=m_0(m_2m_3-m_4^2)^{\frac{1}{2}}=0.25m_0,$$

$$m_b^*=m_c^*=2m_0\left[\frac{m_1(m_2m_3-m_4^2)}{m_1+3m_2}\right]^{\frac{1}{2}}=0.009m_0; \tag{13b}$$

$B\|$axis 2,

$$m_a^*=m_0[(m_2m_3-m_4^2)m_1/m_2]^{\frac{1}{2}}=0.008m_0,$$

$$m_b^*=m_c^*=2m_0\left[\frac{m_1(m_2m_3-m_4^2)}{3m_1+m_2}\right]^{\frac{1}{2}}=0.016m_0. \tag{13c}$$

Using the foregoing values of effective masses, it is not possible to explain all of the lines reported by Dexter and Lax,[3] assuming that the experimental peaks of Fig. 1 correspond to resonance values. Figure 1 shows a trace of the derivative of absorption dP/dB *vs* B when the magnetic field is in the trigonal plane of the sample and parallel to the binary or axis 1. The estimated locations of the derivative peaks are at approximately 100, 220, and 450 oersteds. From the theoretical work on the line shapes of the derivative curves[6] one can estimate from the low field line that $\omega\tau\approx2$ for these carriers. The theory also states that the inflection point in the absorption curve or peak in the derivative curve is given by

$$\omega_c=\omega+K/\tau, \tag{14}$$

where $\omega_c=eB/m^*$ is the cyclotron frequency, τ is the collision time, and K is a number of the order of unity. Under the conditions of the experiment, the observed peaks of Fig. 1 are shifted about 50% above their true cyclotron resonance values. Hence, the cyclotron resonance values for the peaks correspond to about 70, 140, and 300 oersteds, respectively. At 24 000 Mc/sec, the data then yield effective masses of $0.008m_0$, $0.017m_0$, and $0.035m_0$. These values lie within the range obtained for electrons from Shoenberg's data.[8]

Dexter and Lax have also reported preliminary observations for B perpendicular to the surface where the surface of the crystal is a trigonal plane. The two peaks reported at 150 and 500 oersteds correspond to effective masses of approximately $0.01m_0$ and $0.04m_0$. The larger value is of the order of magnitude predicted for electrons by Eq. (13a). However, agreement in this single case could be fortuitous. Moreover, the reported values obtained by linear polarization[3] and circular polarization[4] methods in this configuration have not yet been reconciled. A combination of the circular polarization and inflection point techniques should prove useful in separating the contributions of the holes and electrons to cyclotron absorption and should simplify the interpretation of the data.

D. B Perpendicular to the Surface

Using circular polarization with B perpendicular to the surface of a sample containing electrons only, one obtains a single, nonresonant absorption curve similar

[8] M. Tinkham, Phys. Rev. **101**, 902 (1956) has attempted to interpret the peaks of Fig. 1 along lines similar to those outlined here. However, his speculation on the orientation of the magnetic field with respect to the crystalline axes was incorrect.

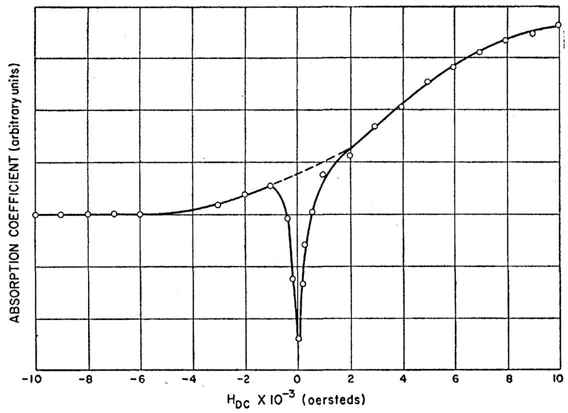

FIG. 2. Absorption *vs* dc magnetic field for circularly polarized radiation incident on the trigonal plane of bismuth at 4.2°K. These data were taken at 24 000 Mc/sec. The vertical scale is only approximately linear. The magnetic field is normal to the trigonal plane. Zero absorption is somewhere below the axis of abscissas (after Galt *et al.*).

to those discussed by Dresselhaus, Kip, and Kittel,[9] Anderson,[10] and Dexter and Lax[3] with an inflection point at a negative value of B. Similarly, for holes, one obtains a nonresonant absorption curve with an inflection point at a positive value of B. For an intrinsic sample with equal numbers of holes and electrons, the combination of carriers produces a minimum in the vicinity of $B=0$. In impure samples having a majority carrier, the situation is more complex, as indicated by the results of Galt and co-workers (see Fig. 2) who used a sample having an excess of holes. Their interpretation of a minimum near $B=0$ as a true cyclotron resonance is questionable.

To analyze such an experimental curve on a quantitative basis, we have carried out a series of calculations for the appropriate configuration, using the J-S model, and the results of Eqs. (7) and (10). The expression for the complex effective conductivity for the combined holes and electrons becomes

$$\frac{\sigma}{\sigma_h}=\frac{\frac{1}{2}(r_1+r_2-r_4^2/r_3)(1+j\omega\tau)-j\omega_c\tau}{r_m^2(1+j\omega\tau)^2+\omega_c^2\tau^2}R+\frac{1}{1+j(\omega-\omega_c)\tau}. \tag{15}$$

Here it has been assumed that the scattering time τ is isotropic and the same for both holes and electrons. This assumption may not be justified but it will be used as a reasonable first approximation. In Eq. (15), $\sigma_h=n_he^2\tau/m_h$, where n_h is the hole concentration and m_h the effective mass of holes in the trigonal plane; $\omega_c=eB/m_h$; $R=n_e/n_h$ is the ratio of electron to hole concentration; r_1, r_2, r_3, and r_4 are the ratios of the electron mass tensor components to m_h and $r_m=m_0\{[(m_1/m_3)(m_2m_3-m_4^2)]\}^{\frac{1}{2}}/m_h$ is the ratio of the resonance mass for electrons, with B along the trigonal axis, to that of holes. The real and imaginary parts of the effective conductivity evaluated from Eq. (15) become

$$\frac{\sigma_r}{\sigma_h}=\frac{1}{1+(N-x)^2}+R\frac{r'[x^2+r_m^2(1+N^2)]-2r_m^2Nx}{[x^2+r_m^2(1-N^2)]^2+4r_m^4N^2},$$

$$\frac{\sigma_i}{\sigma_h}=\frac{N-x}{1+(N-x)^2}+R\frac{r'N[r_m^2(1+N^2)-x^2]+r_m^2(1-N^2)x+x^3}{[x^2+r_m^2(1-N^2)]^2+4r_m^4N^2}, \tag{16}$$

where $x=\omega_c\tau$, $N=\omega\tau$ and $r'=\frac{1}{2}[r_1+r_2-(r_4^2/r_3)]$.

Since we are going to plot the power absorption as a function of magnetic field for both negative and positive values of x, the adjustable parameters that remain are the relative concentration R, the value of N which depends on scattering, and the mass of the hole. The mass of the hole is taken to be one of the adjustable parameters because no completely reliable measurement of m_h has been made for B along the trigonal direction. The masses of the electrons are those of Shoenberg.

Figure 3 shows a plot of the power absorption *vs* $\omega_c\tau$ for several values of R, with $\omega\tau=2$ and $m_h=0.035m_0$. For $R=1$, which corresponds to intrinsic bismuth, the

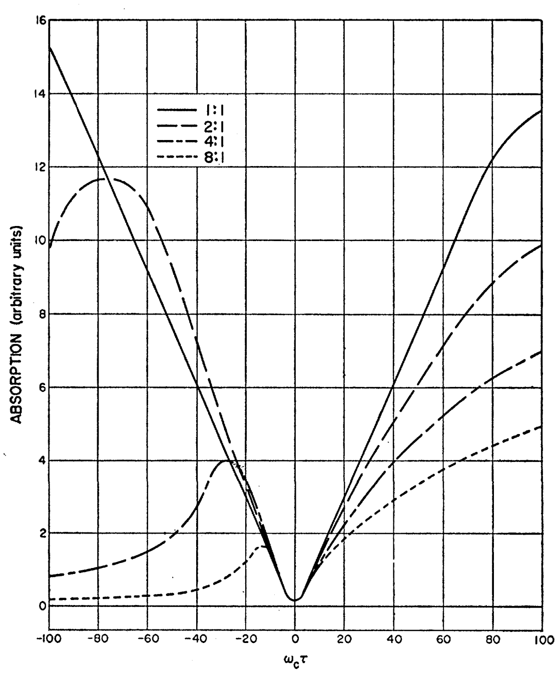

FIG. 3. Absorption *vs* $\omega_c\tau$ for $\omega\tau=2$ with B perpendicular to the surface of the sample [computed from Eqs. (7) and (16)]. The isotropic hole mass was chosen to be $0.035m_0$ and the masses of the electrons are those of Shoenberg. The solid curve shows the result for an intrinsic sample and the others represent an excess of holes to electrons in the ratios indicated.

[9] Dresselhaus, Kip, and Kittel, Phys. Rev. **100**, 618 (1955).
[10] P. W. Anderson, Phys. Rev. **100**, 749 (1955).

absorption curve shows the typical increase as a function of magnetic field, the holes being primarily responsible for the absorption for positive x and the electrons for negative x. When the concentration is unbalanced as for $R=\frac{1}{8}$, we obtain a curve similar to that obtained by Galt and co-workers with a peak for negative values of the magnetic field, corresponding to a minority concentration of electrons. Curves of this type were obtained with the use of the A-M model and also with two isotropic carriers of opposite charge. Figure 4 shows another family of curves for different values of $\omega\tau$ with $R=\frac{1}{4}$ and the same value of m_h. The peak of the minority carrier shifts to the left with increasing $\omega\tau$. Figure 5 illustrates the effect of changing the value of m_h. The general appearance of these curves resembles that of Fig. 2.

E. *B* Parallel to the Surface

When the magnetic field is parallel to the surface of the sample, then the effective conductivity is given by Eq. (5). The particular orientation which has been found to be most convenient experimentally is that which exposes the trigonal plane. If we then select the magnetic field parallel to the binary axis, the effective conductivity, using Eq. (11), becomes

$$\sigma_{11}=\sigma_{22}-(\sigma_{32}\sigma_{23}/\sigma_{33}) \tag{17}$$

$$=\frac{\frac{1}{2}m_2[(m_1+m_2)m_3-m_4^2]+\frac{1}{4}(m_1+3m_2)b^2}{m_2[m_1(m_2m_3-m_4^2)+\frac{1}{4}(3m_1+m_2)b^2]}\sigma_0. \tag{18}$$

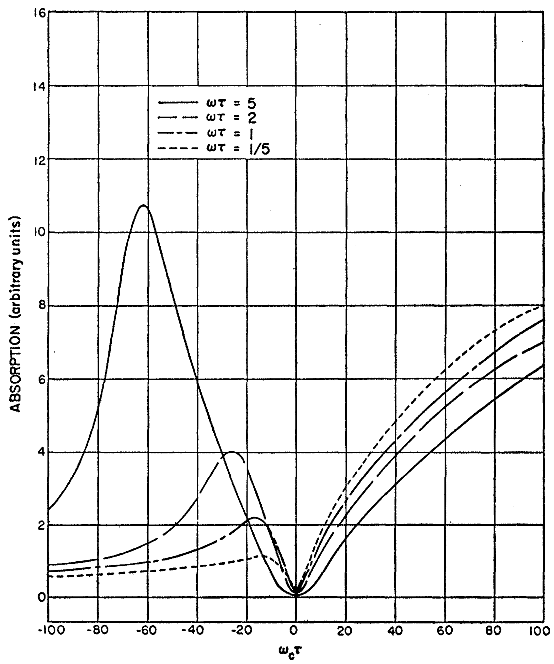

FIG. 4. Theoretical curves of absorption *vs* $\omega_c\tau$ for different values of $\omega\tau$ and a relative concentration of holes to electrons of 4 to 1. The other parameters are the same as in Fig. 3.

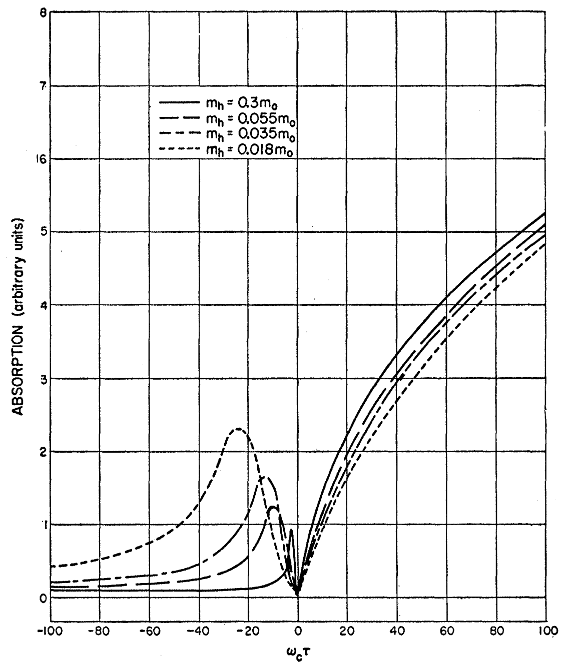

FIG. 5. Theoretical curves of absorption *vs* $\omega_c\tau$ with the mass of the isotropic hole as a parameter. Computations were carried out with $\omega\tau=2$ and an excess of holes of 8 to 1.

This result indicates that an extrinsic sample, containing primarily electrons, will show only one singularity at $m^*=2m_0[m_1(m_2m_3-m_4^2)/(3m_1+m_2)]^{\frac{1}{2}}=0.016m_0$, rather than the two one would expect in the usual resonance experiment.

The situation becomes even more involved when one considers both holes and electrons simultaneously. The problem has been carried out for a pure intrinsic sample using the J-S model. The resonance denominator, when equated to zero, can be written approximately as

$$b^4+(4/m_2)[(m_2m_3-m_4^2)(m_1+\tfrac{3}{4}m_2')+\tfrac{1}{4}m_2m_2'm_3']b^2$$
$$+(4m_2'm_1/m_2^2)(m_2m_3-m_4^2)(2m_2m_3-m_4^2)=0 \tag{19}$$

where the primed quantities refer to holes.[11]

This is obtained by making use of the fact that $m_2\gg m_1$, m_2'. This makes it convenient to evaluate the resonance masses which become approximately $0.06m_0$ and $0.01m_0$. The value of $(m_2'm_3')^{\frac{1}{2}}\approx 0.04$ was estimated from the microwave measurements[12] and $m_2'=0.02$ was estimated from Abeles and Meiboom assuming τ isotropic for holes. Since a reliable experimental value of m_2' is unavailable, it was estimated in this manner. The values of the effective masses in this case do not agree with those obtained previously from Eq. (13). In addition, instead of *three* resonance masses, we obtain *two*. This appears to contradict the results of the microwave experiments for pure bismuth.

[11] m_2' is the same as m_h/m_0 used previously.
[12] Walsh, Zeiger, Foner, and Powell (unpublished data).

III. DISCUSSION

If the classical electromagnetic theory developed in this paper and elsewhere[6,9,10] is used to analyze the experimental results of cyclotron absorption in bismuth, inconsistencies arise even for the preliminary observations made to date.[3,4,12] The authors of this paper recognize the limitations of the simple theory which has been presented. However, since the field of cyclotron absorption in metals is relatively new, it is useful to see just what general features of the experimental results can be described adequately by the classical treatment.

One of the criticisms of the present approach is that it ignores the anomalous skin effect. The existence of this difficulty at microwave frequencies and low temperatures in bismuth, has been recognized by both Anderson[10] and Dexter and Lax.[3] For B parallel to the surface, one may speculate on the influence of the anomalous skin effect on the results predicted by the classical theory. The anomalous skin effect occurs when the mean free path of carriers becomes comparable to or greater than the rf skin depth. If we assume that the electric field has a constant average value over a skin depth, then we can approximate the conductivity tensor of the carriers by a point relation. Carriers which have random velocities nearly parallel to the dc magnetic field will interact with the electric field for a normal mean free path or mean free time, but those which have increasingly larger components transverse to the magnetic field will react for a shorter than normal mean free time with the electric field. In essence, the effective mean free time becomes a distributed quantity. Hence, the conductivity must be appropriately integrated over these values of the effective time. The principal consequence is that the effective conductivity of Eq. (5) for the parallel absorption will no longer cancel the resonance denominators of the components of the conductivity tensor. In this case it is possible that the original resonance denominators of Eq. (13) should still be retained. Nevertheless, these considerations do not appear to help in reconciling the microwave data with those of the de Haas-van Alphen experiments.

Another consequence of the anomalous skin effect is a change of line shape and width of the absorption and derivative curves. It seems clear that the width will be increased and the peak of the derivative curve will probably shift accordingly. Perhaps the effect on the absorption line shape accounts for the difficulty in reproducing quantitatively the experimental curve of Fig. 2, when the classical theory is used. A solution, as given by Azbel' and Kaganov,[13] for the anomalous skin effect with the dc magnetic field perpendicular to the surface of a metal may be helpful in this situation.

One more aspect of the present theory which is unsatisfactory is the assumption of isotropic scattering. If one assumes an isotropic scattering time and calculates the relative value of the principle effective masses for electrons from the galvanomagnetic data of A-M, then one obtains values $m_1:m_2:m_3 \approx 1:40:2$. The results of the de Haas-van Alphen data, when one ignores the slight tilt, give for the principle mass ratios $m_1:m_2:m_3 \approx 1:1000:10$. Since the tilt is only 6°, these numbers can be reasonably compared in order of magnitude to the ratio obtained from the measurements of Abeles and Meiboom. Inasmuch as those ratios do not agree very well and the galvanomagnetic measurements give results in terms of mobility, i.e., $e\tau/m$, rather than masses directly, it is perhaps of some interest to use the apparent difference in the mass ratios to estimate the ratio of scattering times for the three principle directions. The result becomes $\tau_1:\tau_2:\tau_3 \approx 1:25:5$. Although perhaps these numbers are very approximate, they do indicate anisotropic scattering with crystal direction in the dc measurements. For the microwave case this situation is further complicated by the added anisotropy of the effective mean free time imposed by the anomalous skin effect.

In summary, one must conclude that neither the theory nor the experiments on cyclotron absorption are in a sufficiently satisfactory state to obtain a quantitative analysis of the data on bismuth which have been presented so far. Experimental observations of anisotropy and experiments using circular polarization, preferably on intrinsic samples, would do a great deal to clarify the situation. A theory taking into account both the anomalous skin effect and the anisotropic scattering should permit a better comparison between theory and experiment. Another possible approach is to attempt an experiment in which infrared frequencies would be used at higher magnetic fields. This would have the advantage that perhaps $\omega\tau$ greater than one could be achieved at room temperature and the possible difficulties of interpretation due to the anomalous skin effect could be avoided. In such a case the classical theory outlined here would be directly applicable.

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APPENDIX I

To obtain the effective-mass tensors of ellipsoids b and c, one has to transform the effective-mass tensor of ellipsoid a by the use of the matrices of Eq. (9), namely,

$$\mathbf{m}_{b,c}=(\mathbf{S}^{-1})_{b,c}\cdot\mathbf{m}_a\cdot\mathbf{S}_{b,c}$$
$$=m_0/4\begin{vmatrix}(m_1+3m_2) & \mp\sqrt{3}(m_1-m_2) & \mp 2\sqrt{3}m_4\\ \mp\sqrt{3}(m_1-m_2) & (3m_1+m_2) & -2m_4\\ \mp 2\sqrt{3}m_4 & -2m_4 & 4m_3\end{vmatrix}. \quad (20)$$

APPENDIX II

The equation of motion for a carrier which is moving on an ellipsoidal constant energy surface, is given by

$$\mathbf{m}\cdot\frac{d\mathbf{v}}{dt}+\frac{q\mathbf{B}}{m_0}\times\mathbf{v}+\nu\mathbf{m}\cdot\mathbf{v}=q\mathbf{E}/m_0. \quad (21)$$

Since we are interested only in the ac components of the velocity $d\mathbf{v}(t)/dt=j\omega\mathbf{v}$, one is then able to solve Eq. (21) for the velocity to obtain

$$\mathbf{v}=(\mathbf{m}+\mathbf{b}\times 1)^{-1}\cdot\frac{q\mathbf{E}/m_0}{\nu+j\omega}, \quad (22)$$

where $\mathbf{b}=[q\mathbf{B}/(\nu+j\omega)m_0]$.

The conductivity tensor is then simply evaluated from the relation for the current density, $\mathbf{J}=\boldsymbol{\sigma}\cdot\mathbf{E}=nq\mathbf{v}$, to give

$$\boldsymbol{\sigma}/\sigma_0=(\mathbf{m}+\mathbf{b}\times\mathbf{1})^{-1},$$

where $\sigma_0=nq^2/[m_0(\nu+j\omega)]$ and $\mathbf{1}$ is the identity matrix. Using these results, the conductivity tensor for the three ellipsoids takes the form

$$\boldsymbol{\sigma}/\sigma_0=\Delta^{-1}\times\begin{vmatrix}(m_{22}m_{33}-m_{23}^2+b_1^2) & (m_{13}m_{23}-m_{12}m_{33}+b_1m_{13}+b_2m_{23}+b_3m_{33}+b_1b_2) & (m_{12}m_{23}-m_{13}m_{22}-b_1m_{12}-b_2m_{22}-b_3m_{23}+b_1b_3)\\ (m_{13}m_{23}-m_{12}m_{33}-b_1m_{13}-b_2m_{23}-b_3m_{33}+b_1b_2) & (m_{11}m_{33}-m_{13}^2+b_2^2) & (m_{13}m_{12}-m_{11}m_{23}+b_1m_{11}+b_2m_{12}+b_3m_{13}+b_2b_3)\\ (m_{12}m_{23}-m_{13}m_{22}+b_1m_{12}+b_2m_{22}+b_3m_{23}+b_1b_3) & (m_{13}m_{12}-m_{11}m_{23}-b_1m_{11}-b_2m_{12}-b_3m_{13}+b_2b_3) & (m_{11}m_{22}-m_{12}^2+b_3^2)\end{vmatrix},$$

where $\Delta=m_1m_2m_3-m_1m_4^2+\mathbf{b}\cdot\mathbf{m}\cdot\mathbf{b}$.