Interpretation of the Phase Shifts in Pion-Nucleon Scattering*

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Wigner has shown that there is a lower limit on the derivative of the scattering phase shift with respect to energy, the limit being expressed in terms of the range of the interaction. Orear has pointed out that this condition is not satisfied by the pion-nucleon S-wave phase shifts given by Bethe and de Hoffmann for energies near 200 Mev. Orear's alternate solution, satisfying Wigner's criterion, fits the data equally well. The use of the Wigner condition for pion-nucleon scattering is somewhat questionable because the concept of interaction radius is not well defined. It is shown here that a similar condition may be obtained without the use of this concept. Orear's argument is strengthened on this basis. A simple interpretation of the S-wave phase shifts is given in terms of an $I=\frac{1}{2}$ resonance at very high energy, the only parameter being the ratio of level width to resonance energy. A simple two-parameter interpretation of the P-wave phase is also presented.

I. INTRODUCTION

WIGNER[1] has recently demonstrated that there exists a lower limit on the derivative of a scattering phase shift with respect to the momentum of the scattered particle. His argument is based on the concept of a finite interaction region in configuration space such as is used in the Wigner-Eisenbud theory[2] of nuclear reactions. The result has been applied by Orear[3] to the interpretation of the phase shifts in pion-nucleon scattering. Orear argued that some of the S-wave phase shifts obtained by Bethe and de Hoffmann[4] are not consistent with the Wigner condition; hence he sought other solutions and found one that was equally consistent with the experimental cross section and, at the same time, satisfies Wigner's condition.

There is some doubt as to the applicability of Wigner's condition to this problem because it involves directly the size, a, of the region of interaction. Since the pions presumably do not interact with the nucleons via a potential, it is not clear just what meaning is to be attached to a. Orear argues that a cannot be much larger than one pion Compton wavelength, but, as Wigner[1] has pointed out, one cannot be sure that two Compton wavelengths would be an excessive value for a. Such a large value of a would cause Orear's argument against the Bethe-de Hoffmann solution to collapse.

The real difficulty here lies in the application of the notion of a finite interaction region to a process which is essentially relativistic because it involves the annihilation and creation of pions. It is desirable to have a description of the scattering process, including possible resonance effects, which does not invoke the notion of a finite range of interaction. A method which seems to meet this need has recently been described.[5] It is the purpose of the present note to show that the method leads to a condition on the derivative of the scattering phase shift which is very similar to Wigner's condition, but stronger when applied to Orear's problem. However, the new condition is subject to restrictions concerning the location and width of the resonances so that final conclusions concerning the validity of the Bethe-de Hoffmann solution require some interpretation of the phase shifts throughout the low-energy region. To this end, we provide a reinterpretation of the phase shifts in terms of the aforementioned resonance theory. As a consequence of the increased accuracy in the experimental values of the phase shifts, the present interpretation is far simpler than that presented in reference 5. Our conclusion concerning the S-wave phase shift agrees with that of Orear, namely, that the Bethe-de Hoffmann solution implies a much less realistic nucleon size than does Orear's solution.

II. CONDITION ON THE DERIVATIVE OF THE PHASE SHIFT

The form of the phase shift for pion-nucleon scattering is[5]

$$\alpha_{lJI}=\eta_l+\delta_{lJI}, \tag{1}$$

where l is the orbital angular momentum of the pion which is coupled with the nucleon to form a state of total angular momentum J and total isotopic spin I. The "orthogonality phase shift" η_l is introduced because the problem is formulated in terms of the interaction of the nucleon with the pion waves orthogonal to the bound pion states of the (fully dressed) nucleon. The form of the energy dependence of η_l has been shown to be

$$\tan\eta_l=-\pi p p_0(p/p_0)^{2l}q_l^{-1}(p), \tag{2}$$

where p is the momentum, p_0 the energy of the pion, and the units are such that $\hbar=c=m=1$. The function q_l is known to be positive for $p=0$ and to cross the axis at least once as a function of the energy.

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[1] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
[2] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[3] J. Orear, Phys. Rev. **100**, 288 (1955).
[4] H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol II.
[5] R. G. Sachs, Phys. Rev. **95**, 1065 (1954).

The resonance phase shift δ_{lJI} satisfies an equation of the form

$$\tan\delta_{lJI}=\pi p p_0 (p/p_0)^{2l}\sum_\lambda g_\lambda^2(p)/(E_\lambda-p_0), \tag{3}$$

where the coupling $g_\lambda(p)$ is expected to be a slowly varying function of p except in the neighborhood of any cutoff occurring in the interaction.

We shall direct our attention to the behavior of $\alpha_{lJI}'=d\alpha_{lJI}/dp$ for small and moderate value of p. According to Eq. (3)

$$\delta_{lJI}'=\cos^2\delta_{lJI}\left\{\pi p p_0\left(\frac{p}{p_0}\right)^{2l}\sum_\lambda\left[\frac{g_\lambda^2 p p_0^{-1}}{(E_\lambda-p_0)^2}+\frac{2g_\lambda g_\lambda'}{E_\lambda-p_0}\right]+\frac{2l+1+2p^2}{pp_0^2}\tan\delta_{lJI}\right\}.$$

Therefore,

$$\delta_{lJI}'>2\pi p p_0\left(\frac{p}{p_0}\right)^{2l}\cos^2\delta_{lJI}\sum_\lambda\frac{g_\lambda g_\lambda'}{E_\lambda-p_0}+\frac{2l+1+2p^2}{2pp_0^2}\sin 2\delta_{lJI}. \tag{4}$$

If it is assumed that all resonances except (at most) one, denoted by E_Λ, are at energies very large compared to the moderate values of p_0 under consideration, Eq. (3) may be used to rewrite Eq. (4) as

$$\delta_{lJI}'>\tfrac{1}{2}\sin 2\delta_{lJI}[2g_\Lambda'/g_\Lambda+(2l+1+2p^2)/pp_0^2]. \tag{5}$$

The justification of this one-level approximation lies not only in its simplicity but also in the fact that it seems to be in accord with the data as is demonstrated in Sec. III.

From Eq. (2), we find

$$\eta_l'=\tfrac{1}{2}\sin 2\eta_l[-q_l'/q_l+(2l+1+2p^2)/pp_0^2]. \tag{6}$$

Hence the condition on α_{lJI}', where α_{lJI} is given by Eq. (1), reads

$$\alpha_{lJI}'>\tfrac{1}{2}\sin 2\delta_{lJI}\frac{d\ln g_\Lambda^2}{dp}-\tfrac{1}{2}\sin 2\eta_l\frac{d\ln q_l}{dp}+\frac{2l+1+2p^2}{2pp_0^2}(\sin 2\delta_{lJI}+\sin 2\eta_l). \tag{7}$$

We shall be concerned with the case in which α_{lJI} is positive. Since δ_{lJI} is everywhere positive and η_l everywhere negative, we then have, from Eq. (1),

$$|\delta_{lJI}|>|\eta_l|. \tag{8}$$

Therefore, as long as

$$\delta_{lJI}<\tfrac{1}{4}\pi, \tag{9}$$

we are assured that $(\sin 2\delta_{lJI}+\sin 2\eta_l)$ is positive. Consequently, the inequality (7) leads to the condition

$$\alpha_{lJI}'>\tfrac{1}{2}\sin 2\delta_{lJI}\frac{d\ln g_\Lambda^2}{dp}-\tfrac{1}{2}\sin 2\eta_l\frac{d\ln q_l}{dp}. \tag{10}$$

This condition is very similar in form to that given by Wigner. His criterion involves a "radius of the scatterer," denoted by a, and it is easy to see that Eq. (10) involves a quantity of the dimension of a length which in this case is closely related to the size of the nucleon proper field, namely,

$$a(p)=-d\ln q_l/dp. \tag{11}$$

Since the rate at which $q_l(p)$ decreases is governed by the condition of orthogonality to the bound pion field, it will be determined by the mean radius of the bound pion wave functions. Thus, $a(p)$ is of the order of magnitude of this radius, which should lie somewhere between the pion Compton wavelength (1 in our units) and the radius of the nucleon source function. In fact, for reasonable assumptions concerning the structure of the pion cloud, a will have a value rather close to the size of the source function; hence $a<1$.

The term involving the derivative of $\ln g_\Lambda^2$ is expected to be quite small for moderate energies since the principal change in g_Λ occurs near the cutoff. Thus an adequate substitute for Eq. (10) is

$$\alpha_{lJI}'>-\tfrac{1}{2}a. \tag{12}$$

This condition is considerably stronger than Wigner's, both because of the factor $\frac{1}{2}$ (not to speak of the factor $|\sin\eta_l|$ which has been dropped) and the fact that we are assured that $a<1$.

Unfortunately, the validity of the inequality (12) is restricted by the condition Eq. (9), which is important because the quantity $(2l+1+2p^2)/2pp_0^2$ appearing in Eq. (7) is larger than a for the values of p that are of interest. In qualitative terms, Eq. (9) states that it is necessary for the first resonance to dominate the behavior of the phase shift in order that the slope of the phase shift remain positive. If the radius of the pion cloud is too large, the orthogonality condition may take over from the resonance and cause a rapid decrease in the slope of α_{lJI} as a function of p. Therefore, in order to make a convincing application of Eq. (12), we must have further knowledge concerning the resonance, that is, we must examine the entire behavior of the phase shift as a function of energy, not just its local behavior.

III. INTERPRETATION OF THE PHASE SHIFTS

Although a rather complicated behavior of the phase shifts is admitted by Eqs. (1), (2), and (3), it seems reasonable to attempt the simplest possible interpretation. Therefore we try a one-level approximation, i.e., assume that no more than one term need be included in the sum over resonances of Eq. (3). The two constants in this term will be fixed then by fitting the data.

As usual, attention is limited to the S- and P-waves. A phenomenological method is used to determine the corresponding two orthogonality phase shifts η_0 and η_1. Taking the S-waves first, we note that the Bethe-de

Hoffmann solutions for the $I=\frac{3}{2}$ phase shift, α_3, is negative and that α_3 decreases smoothly with energy. This is very suggestive of the behavior of an orthogonality phase shift, hence we assume that *there is no resonance* contributing to the $I=\frac{3}{2}$, S-wave and that

$$\eta_0 \approx \alpha_3. \tag{13}$$

The $I=\frac{1}{2}$ phase shift, α_1, on the other hand, is positive so that a resonance is required to account for its behavior. Up to an energy of 144 Mev, α_1 increases smoothly with energy. This suggests a remote resonance so that it should be possible to replace Eq. (3) by

$$\tan\delta_1 = \text{const}\times\pi p p_0. \tag{14}$$

Now one consequence of Eq. (13) is that the experimental values of α_3 and α_1 may be used to obtain δ_1. From Eq. (1), we find

$$\delta_1 \approx \alpha_1 - \alpha_3. \tag{15}$$

The Bethe-de Hoffmann values of α_1 and α_3 are given in Table I along with the values of δ_1 obtained from Eq. (15). In the last column of the table are found the values of tan $\delta_1/\pi p p_0$, which should be constant according to Eq. (14). Up to an energy of 144 Mev, they are constant to a sufficient degree, but beyond that energy the behavior of $\tan\delta_1/\pi p p_0$ is erratic. Just these points are the ones re-evaluated by Orear on the grounds that they were not consistent with the Wigner condition. Since $\delta_1 < \frac{1}{4}\pi$, Eq. (9) is satisfied, and we see that Orear's use of the condition Eq. (12) is certainly justified. His re-evaluated phase shifts at 217 Mev are included in Table I, and the corresponding value of $\tan\delta_1/\pi p p_0$ is in good agreement with the values at lower energy. The results obtained here are taken directly from Orear's Fig. 2, since his straight line extrapolation of $\tan\alpha_1$ and $\tan\alpha_3$ does not lead to as good an agreement with the data. In fact, it is to be expected on the basis of Eqs. (14) and (15) that at least one of the two functions curves away from the energy axis, the curvature being quite small at low energy.

We also take this opportunity to give a similar simple interpretation of the P-wave phase shifts. The usual values[4] of α_{11}, α_{31}, and α_{13} are quite small, a result which would suggest both that there is no resonance in any of these states and that η_1 is correspondingly small. Hence we assume

$$\eta_1 \approx 0. \tag{16}$$

Then, for the important $I=\frac{3}{2}$, $J=\frac{3}{2}$ state we have

$$\alpha_{33} \approx \delta_{33}, \tag{17}$$

and in the one-level approximation to Eq. (3) we find

$$\tan\alpha_{33} = \pi p p_0 (p/p_0)^2 g_{33}{}^2/(E_{33}-p_0), \tag{18}$$

where g_{33} should be nearly constant for the values of p under consideration. We may write this in a form

TABLE I. Analysis of S-wave phase shifts obtained by Bethe and de Hoffmann[a] and by Orear.[b] Values in parentheses are excluded by the arguments presented in the text.

Energy Mev	Reference	α_1	α_3	$\alpha_1-\alpha_3$	$(\pi p p_0)^{-1}$ $\times\tan(\alpha_1-\alpha_3)$
26	4	4.5°	−4.0°	8.5°	0.078
40	4	5.5	−5.0	10.5	0.072
61.5	4	9.5	−5.5	15.0	0.077
120	4	8	−12	20	0.058
144	4	14	−13	27	0.068
169	4	(7)	(−4)	(11)	(0.022)
194	4	(−14)	(−13)	(−1)	(−0.002)
217	4	(−4)	(−20)	(16)	(0.026)
217	3	22	−14	36	0.065

[a] See reference 4.
[b] See reference 3.

suggested by Chew and Low,[6] namely,

$$p_0^{-1}p^3 \cot\alpha_{33} = (\pi g_{33}{}^2)^{-1}(E_{33}-p_0). \tag{19}$$

Therefore a plot of the experimental values of the expression on the left-hand side of the equation against p_0 should take the form of a straight line, as Chew and Low have shown to be the case. It is important to notice here that the straight line results from a simple resonance phenomenon.

Orear gives[3] for the values of the constants which best fit the Chew-Low curve[7]:

$$(\pi g_{33}{}^2)^{-1} = 3.8,$$
$$(\pi g_{33}{}^2)^{-1}E_{33} = 8.0.$$

The corresponding resonance energy (including the rest energy of the pion), is

$$E_{33} = 2.1 = 295 \text{ Mev}, \tag{20}$$

and the coupling constant is

$$g_{33} = 0.29. \tag{21}$$

There is some difficulty in the understanding of the result expressed by Eq. (16), namely that η_1 is small for values of the energy considered here; i.e., for $p \lesssim 2$. The function η_l may be calculated easily in the Tomonaga approximation,[8] wherein it is assumed that just one radial function is required to describe the bound pion states. Then[5]

$$\tan\eta_l = -\pi p p_0 f_l{}^2(p)/\mathrm{P}\int_0^\infty k^2 dk f_l{}^2(k)(k_0-p_0)^{-1}, \tag{22}$$

where P denotes principal value of the integral and $f_l(k)$ is the Fourier-Bessel transform of the radial

[6] G. Chew and F. Low, *Proceedings of the Fifth Annual Rochester Conference on High Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1955).

[7] In order to obtain the best fit, use was made of the suggestion of Chew and Low that the quantity p_0 appearing in Eq. (19) be replaced by $\omega^* = p_0 + p^2/2M$, in order to take some account of the nuclear recoil.

[8] Maki, Sato, and Tomonaga, Progr. Theoret. Phys. (Japan) **9**, 607 (1953).

function. For $l=1$, it is usually assumed that

$$f_1(k)=k/k_0^{\frac{3}{2}},\ k<K \\ =0 \quad ,\ k>K, \tag{23}$$

with K an appropriately chosen cutoff. In order that η_1 be very small for moderate energy, the cutoff must be large compared to p. In that case

$$\tan\eta_1\approx-\pi(p/p_0)^3(p_0/K). \tag{24}$$

Equation (16) certainly implies that $\tan\eta_1<0.2$ for $p\approx2$, but then we must have

$$K\gtrsim 9\pi, \tag{25}$$

which would seem to be an unreasonably large value for K.

The difficulty may result from our use of the Tomonaga approximation. It has been suggested[9] that there is an important correlation between the pions in the bound P-states of the nucleon, in which case the use of a single radial function is inadequate, and the form of Eq. (22) must be revised.[5] No adequate calculation of η_1 has been carried out for the case of correlation, largely because an appropriate substitute for Eq. (23) has not been suggested. It is not clear that such a calculation would lead to small η_1, but the possibility remains until the contrary is demonstrated.

[9] W. G. Holladay and R. G. Sachs, Phys. Rev. **98**, 1155 (1955) and W. G. Holladay, Phys. Rev. **101**, 1198, 1202 (1956).

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Cosmic-Ray Observations at Very High Altitudes During Periods of Intense Solar Activity*

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The records of a number of balloon flights which coincided with outstanding solar disturbances during the period 1947–1952 have been subjected to detailed statistical analysis in a search for associated fluctuations in the primary cosmic-ray flux. The intensity at very high altitudes was enhanced in four cases, and normal on five occasions when rf radio disturbances and/or visually-observed solar flares occurred. In some instances, outstanding chromospheric eruptions are not accompanied by any detectable increase in the flux of primary cosmic-ray particles even down to the very low energy required for just penetrating 10 g/cm^2 of matter. On the other hand, additional low-energy particles apparently associated with solar disturbances sometimes are detected at very high altitudes, even when the flares are small and the observing station is outside the morning impact zone.

I. INTRODUCTION

ONLY four unusual increases in the cosmic-ray intensity coincident with the occurrence of outstanding solar flares have been observed with instruments operating at low altitudes in almost two decades.[1] Thus, it is self-evident that certain special conditions must prevail for events of this type to be detected by the existing stations.

One important factor relating to this matter concerns the accessibility of the various allowed regions on the earth for charged particles originating at the sun. A detailed examination of the form of the individual trajectories is required in this case. This problem was first attacked by Schlüter[2] whose results regarding the intensity distribution at the earth, based upon the integration of twenty Störmer orbits, indicated that a maximum should occur at a particular local time, which for positive particles of momentum several Bev$\times Z/c$ originating at the sun in the equatorial plane is 0900 hr. Firor[3] has derived the distribution of impact zones on the earth for particles of magnetic rigidities 1–10 Bv which originally approach from the sun, and has shown that at intermediate latitudes the relative intensities at 0900 hr, 0400 hr, and at all other times are in the ratio 7:3:1. Except at very high latitudes where perturbations may be introduced by nondipole terms in the terrestrial magnetic field, the observed world-wide increases agree with theoretical predictions relating the expected magnitude of the increase to location of the station at the time of the disturbance. Furthermore, data obtained with the Climax neutron monitor[4] appeared to support the supposition that additional particles approach the earth from the direction of the sun at the time of perhaps even all flares, at least when the detector is in a morning impact zone.

* Assisted by the Office of Naval Research and by the U. S. Atomic Energy Commission.

[1] Forbush, Stinchcomb, and Schein, Phys. Rev. **79**, 501 (1950). *Note added in proof.*—The fifth unusual increase, world-wide in character and of much greater magnitude than any observed previously, commenced at 350 UT, February 23, 1956. This event has been reported by numerous stations.

[2] A. Schlüter, Z. Naturforsch. **6a**, 613 (1951).

[3] J. Firor, Phys. Rev. **94**, 1017 (1954).

[4] Simpson, Fonger, and Trieman, Phys. Rev. **90**, 934 (1953).