Nucleon Scattering and the Feshbach-Lomon Model*†

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A method, based upon the results of Feshbach and Lomon, is established by which a given potential may be simply tested for its capability of providing an adequate charge-independent description of nucleon-nucleon scattering to 300 Mev. The view is taken that the scattering is better represented by replacing the (in general) strongly energy-dependent scattering phase shifts with a set of suitably defined logarithmic derivatives Γ_{JL}. These, when evaluated at a "characteristic interaction distance" $\bar{r}$, will depend at most weakly upon energy. The success of the Feshbach-Lomon (FL) model in the interval 0 to 300 Mev is here regarded more generally as establishing this view in the above energy range. These ideas are given their mathematical formulation through a generalization of effective-range methods familiar in low-energy scattering. Here $\bar{r}$, which plays the role of an effective range, is shown by means of stationary expressions for Γ_{JL} to attain a state-independent value equal to the FL "core" radius, provided that Γ_{JL} has the desired weak energy dependence. It is then only necessary to find the behavior with energy of the parameters Γ_{JL}, as determined by a given potential, in order to test whether or not the latter is capable of giving a charge-independent description of the scattering. The method is illustrated in singlet S-states with two-parameter monotonic static potentials. It is found that Gaussian, exponential, and Yukawa potentials are not consistent with the analysis unless these potentials also contain a repulsive core whose radius is suitably restricted. A rectangular potential is barely possible because of the uncertainties in the FL fit arising from insufficient experimental data. An Appendix is devoted to a discussion of the variational principle used in the text.

I. INTRODUCTION

In the preceding paper,[1] an explicitly charge-independent fit of the nucleon-nucleon bound state, and scattering data has been obtained. A model was used which confines the nuclear forces within a radius r_0 which depends upon the state and which, for all but one state, the $^1S_0$, was taken to be energy independent. In this region the forces are assumed to be sufficiently strong to be essentially independent of the relative energy of the colliding particles, and may thus be represented by an energy-independent boundary condition applied to the set of logarithmic derivatives $f_{JL}(r_0)$ of the interaction wave function. We wish to point out that the success of this specific model may be more generally regarded as establishing, within the above energy range, an alternative and highly compact representation of experimental results. This representation embodies an extension to higher energies of the effective-range methods[2] familiar in the analysis of low-energy nucleon scattering. It will be shown in this paper that such a viewpoint results in a simple procedure for determining whether a given potential is capable of giving a charge-independent description of the scattering.

The success of FL in fitting data in the range 0 to 300 Mev suggests that, from a formal standpoint, the scattering may be more appropriately represented in each state by two parameters, $\Gamma_{JL}(\bar{r})$ and $\bar{r}$, which depend only weakly upon the energy, than by the phase shift δ_{JL}, which is, in general, strongly energy dependent.[3] ($\Gamma_{JL}=f_{JL}+1$). In II, we shall make use of a procedure due to Schwinger[4] to define an appropriate set of generalized logarithmic derivatives Γ on the surface of a sphere of radius $\bar{r}$, which we then show to be Hermitian. A stationary expression for Γ is then invoked, by means of which $\bar{r}$ is so chosen that Γ has the desired weak energy dependence. The resulting condition for $\bar{r}$ is manifestly a generalization of that used to define the effective range in low-energy scattering theory. This connection is made explicit by consideration of the $^1S_0$ and $^3S_1+^3D_1$ states of the two-nucleon system. The "effective interaction distances" $\bar{r}$ in these states are shown to be equal to each other within experimental error and to have the value r_0 found by FL to yield the best over-all agreement with the scattering data. Thus, only a single range parameter is needed to describe the low-energy scattering in singlet and triplet states. In III, we investigate the energy dependence of the logarithmic derivative in the $^1S_0$ state and compare with the FL fit. It is demonstrated that with sufficiently complete and accurate data for energies less than 50 Mev, it would be possible to determine the character of the interaction potential. For example, if it is assumed that Γ does not vary more than 20 percent in energy from 0 to 50 Mev, then the rectangular, Gaussian, exponential, and Yukawa wells would fail to fit the data. However, a potential consisting of a repulsive core together with an exterior well of rectangular or Yukawa shape could satisfy the above criterion. In the former case a repulsive core radius r_c less than 0.4×10^{-13} cm, and in the latter r_c less than 0.7×10^{-13} cm would be required. A brief

* A portion of this material may be found in the author's thesis, Harvard, 1954 (unpublished).

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[1] H. Feshbach and E. Lomon, preceding paper [Phys. Rev. **102**, 891 (1956)], referred to as FL.

[2] J. Schwinger, lectures on nuclear physics, Harvard, 1947 (unpublished); Phys. Rev. **72**, 724 (1947). J. M. Blatt, Phys. Rev. **74**, 92 (1948). J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

[3] G. Breit and W. G. Bouricius, Phys. Rev. **79**, 1029 (1949) have discussed this point of view most thoroughly for low-energy S-wave scattering.

[4] J. Schwinger, lectures on nuclear physics, Harvard, 1955 (unpublished).

discussion of the variational principle used in the text, together with its explicit realization for higher angular momenta and tensor forces, is included in the appendix.

II. THE EXTENDED EFFECTIVE-RANGE DESCRIPTION

Consider the Schrödinger equation for the interaction of two nucleons,

$$(k^2-H)\psi=0;\quad r\psi=0 \quad \text{when} \quad r=0, \tag{1a}$$

and its adjoint,

$$\psi^\dagger(k^2-H)^\dagger=0, \tag{1b}$$

where $k^2=ME/\hbar^2$, $H=-\nabla^2+H'$, and H' is the interaction Hamiltonian, which we shall assume to be energy independent and of short range. Let us also introduce a comparison wave function Ψ, satisfying

$$(k^2+\nabla^2)\Psi=0. \tag{2}$$

Inside the interaction volume, Ψ is then an extrapolation of ψ. By elementary manipulation one obtains the relations

$$\nabla\cdot[\psi^\dagger\nabla\psi-(\nabla\psi)^\dagger\psi]=0, \tag{3}$$

$$\nabla\cdot[\Psi^\dagger\nabla\Psi-(\nabla\Psi)^\dagger\Psi]=0. \tag{4}$$

Let us integrate Eq. (3) over all space, but Eq. (4) only outside a sphere of radius $r=\bar{r}$. Then in view of the identity of ψ, Ψ at large distances, we must have

$$\int d\mathbf{S}\cdot\left[\Psi^\dagger\nabla\Psi-(\nabla\Psi)^\dagger\Psi\right]_{r=\bar{r}}=0. \tag{5}$$

Observing that only the radial part of the gradient operator contributes to this integral, we define a generalized logarithmic derivative Γ on the sphere $r=\bar{r}$ by the expressions

$$\begin{aligned}\left[\frac{\partial}{\partial r}(r\Psi)\right]_{r=\bar{r}}&=\Gamma\bar{r}\Psi(\bar{r}),\\ \left[\frac{\partial}{\partial r}(r\Psi)^\dagger\right]_{r=\bar{r}}&=\bar{r}\Psi^\dagger(\bar{r})\Gamma^\dagger.\end{aligned} \tag{6}$$

Equation (5) thus expresses the hermiticity of Γ:

$$\int d\omega\left[r\Psi^\dagger(\Gamma-\Gamma^\dagger)r\Psi\right]_{r=\bar{r}}=0. \tag{7}$$

It is proven in the appendix that

$$\begin{aligned}\int_{r=\bar{r}} d\omega r\Psi^\dagger\Gamma r\Psi &=\int_{\text{all space}}(dr)[r^{-2}(\nabla r\psi)^\dagger\cdot\nabla(r\psi)-\psi^\dagger(k^2-H')\psi]\\ &\quad-\int_{r\geqslant\bar{r}}(dr)[r^{-2}\nabla(r\Psi)^\dagger\cdot\nabla(r\Psi)-\Psi^\dagger k^2\Psi]\end{aligned} \tag{8}$$

is a stationary expression for Γ under independent variation of ψ, Ψ or their adjoints. It is convenient to use Eq. (8) as a means of specifying the energy dependence of Γ. Under variation with respect to the energy, the wave functions in their dependence on energy do not contribute by virtue of the stationary property, and one obtains the rigorous expression

$$\int_{r=\bar{r}} d\omega r\Psi^\dagger(\partial\Gamma/\partial k^2)r\Psi=\int_{r\geqslant\bar{r}}(dr)\Psi^\dagger\Psi-\int_{\text{all space}}(dr)\psi^\dagger\psi. \tag{9}$$

Let us now view the wave functions ψ, Ψ as being decomposed into mutually orthogonal parts, each part corresponding to a constant of the motion. Equation (9) is then to be regarded as a set of independent equations corresponding, for example, to possible values of the total angular momentum J, spin S, and charge state τ. Thus with central forces Eq. (9) assumes the form

$$U_L^2(\bar{r}_L)d\Gamma_{LL}/dk^2=\int_{\bar{r}_L}^\infty drU_L^2-\int_0^\infty dru_L^2, \tag{10}$$

as shown in the appendix. Here u_L, U_L are the radial parts of ψ, Ψ corresponding to angular momentum L. The orthogonality of these partial waves requires Γ to be a diagonal matrix, whose elements are Γ_{LL}. For the coupled waves present in triplet scattering with tensor forces, we find in the appendix that

$$\begin{aligned}\left[U_J^2(r)\frac{\partial\Gamma_{UJ}}{\partial k^2}+W_J^2(r)\frac{\partial\Gamma_{WJ}}{\partial k^2}\right]_{r=\bar{r}_J}&=\int_{\bar{r}_J}^\infty dr[U_J^2(r)+W_J^2(r)]\\ &\quad-\int_0^\infty dr[u_J^2(r)+w_J^2(r)].\end{aligned} \tag{11}$$

Here u_J, w_J are the radial wave functions for $L=J-1$, $J+1$, respectively, and U_J, W_J are the corresponding comparison functions. These wave functions, together with $\bar{r}_J$ and the logarithmic derivatives, may refer to either of the mutually orthogonal eigenwave mixtures which, in virtue of the tensor coupling, remain unchanged during the scattering process.

In each scattering state ξ, we now require $\bar{r}_\xi$ to be energy independent, and of value such that the left-hand side of Eq. (9) vanishes when evaluated at some specified energy $\bar{E}$,

$$\left[\frac{\partial\Gamma_\xi}{\partial k^2}\right]_{r=\bar{r}_\xi}=0,\quad E=\bar{E}. \tag{12}$$

The logarithmic derivatives Γ_ξ, which in the vicinity of $\bar{E}$ depend at most weakly upon energy, together with

the (by choice) energy-independent characteristic distances $\bar{r}_\xi$, completely specify the scattering. Indeed the Γ's are *formally* identical with the logarithmic derivatives $f+1$, assumed energy independent by FL in their fit of the scattering data. However, while the latter incorporate the effect of a very strong nuclear force confined within a region $r \leqslant r_0$, the former embody no assumptions whatever about the force except its static nature and short range. The increased generality of description is reflected in the correspondingly weaker statement that Γ_ξ is no more than locally energy-independent.

The form of Eq. (9) suggests that the foregoing description is closely analogous to the effective-range representation of low-energy scattering. In the $^1S_0$ state, Eq. (8) reduces to

$$U^2(\bar{r}_0)\Gamma_0(\bar{r}_0)=\int_0^\infty dr\{(du/dr)^2-[k^2-\lambda f(r)]u^2\} -\int_{r_0}^\infty dr[(dU/dr)^2-k^2U^2], \quad (13)$$

where u, U satisfy the equations

$$[(d^2/dr^2)+k^2-\lambda f(r)]u=0, \quad (14)$$

$$[(d^2/dr^2)+k^2]U=0, \quad (15)$$

in which $\lambda=MV_0/\hbar^2$ and $V_0f(r)$ is the $^1S_0$ potential. Let us set $\bar{r}_0=0$ momentarily, and follow Schwinger[5] in introducing energy expansions:

$$\begin{aligned} u(r)&=u_0(r)+k^2u_1(r)+\cdots, & u_1(0)&=0,\\ U(r)&=U_0(r)+k^2U_1(r)+\cdots, & U_1(0)&=0, \end{aligned} \quad (16)$$

and

$$\Gamma_0(0)=k\cot\delta_0=a_0^{-1}+\tfrac{1}{2}r_{e0}k^2-Pr_{e0}^3k^4+\cdots. \quad (17)$$

Equation (13) then yields the customary definitions of the singlet scattering length a_0, effective range r_{e0}, and shape-dependent parameter P:

$$a_0^{-1}=U_0^{-2}(0)\left\{\int_0^\infty[(du_0/dr)^2-\lambda f(r)u_0^2]dr -\int_0^\infty dr(dU_0/dr)^2\right\}, \quad (18)$$

$$\tfrac{1}{2}r_{e0}=U_0^{-2}(0)\int_0^\infty (U_0^2-u_0^2)dr, \quad (19)$$

$$Pr_{e0}^3=U_0^{-2}(0)\int_0^\infty (U_0U_1-u_0u_1)dr. \quad (20)$$

It is thus appropriate to apply Eq. (12) at $\bar{E}=0$, which then gives the definnig equation for $\bar{r}_0$:

$$\int_{\bar{r}_0}^\infty drU_0^2-\int_0^\infty u_0^2dr=0. \quad (21)$$

[5] J. Schwinger, Phys. Rev. 78, 135 (1950).

Using Eq. (19) and the zero-energy solution of Eq. (15),

$$U_0(r)=1+a_0^{-1}r, \quad (22)$$

we obtain an expression for r_0 independent of the details of the force,

$$\tfrac{1}{2}r_{e0}=\bar{r}_0+a_0^{-1}\bar{r}_0^2+\tfrac{1}{3}a_0^{-2}\bar{r}_0^3. \quad (23)$$

From the experimental values[6]

$$r_{e0}=(2.52\pm0.23)\times10^{-13}\text{ cm},$$

$$a_0^{-1}=(0.422\pm0.009)\times10^{12}\text{ cm}^{-1},$$

we have

$$\bar{r}_0=(1.20\pm0.11)\times10^{-13}\text{ cm}. \quad (24)$$

This value is in good agreement with the "core" radius $r_0\sim1.32\times10^{-13}$ cm found empirically by FL to yield the best over-all fit of nucleon-nucleon scattering data; and of course also agrees with the boundary radius of 0.47 e^2/mc^2 found by Breit,[3] using the same criterion (i.e., weak energy dependence) but rather different methods.

Turning now to the coupled $^3S_1+^3D_1$ mixture, we apply condition (12) to Eq. (11) at $E=-\epsilon$, the binding energy of the deuteron. By virtue of the normalization

$$\int_0^\infty dr(u_1^2+w_1^2)=1, \quad (25)$$

we have

$$\int_{r_1}^\infty dr[U_1^2+W_1^2]=1, \quad \bar{E}=-\epsilon, \quad (26)$$

in which the S, D comparison functions U_1, W_1 refer to the S-dominant eigenwave mixture. Equation (26) is now to be compared with the "bound state" definition of the effective range,

$$\bar{U}_1^2(0)(\tfrac{1}{2}r_{e1})\equiv\int_0^\infty dr\bar{U}_1^2-1 \quad (27)$$

where $\bar{U}_1(r)$ is the solution of (15) at $E=-\epsilon$,

$$\bar{U}_1(r)=\bar{U}_1(0)\exp(-\eta r), \quad \eta=(M\epsilon/\hbar^2)^{\frac{1}{2}}, \quad (28)$$

and thus corresponds to neglect of the small D-state admixture in $U_1(r)$. The latter and $W_1(r)$ are the solutions of the radial equations, uncoupled in the absence of interaction,

$$[(d^2/dr^2)-\eta^2]U_1(r)=0, \quad (29)$$

$$[(d^2/dr^2)-(6/r^2)-\eta^2]W_1(r)=0, \quad (30)$$

which we write as

$$U_1(r)=\bar{U}_1(0)(1-\zeta^2)^{\frac{1}{2}}\exp(-\eta r), \quad (31)$$

$$W_1(r)=\bar{U}_1(0)\zeta[1+3(\eta r)^{-1}+3(\eta r)^{-2}]\exp(-\eta r). \quad (32)$$

In this way $\bar{U}_1(0)$ measures the total amplitude at large

[6] Recent data summarized by H. S. W. Massey, *Proceedings of the International Conference on Nuclear and Meson Physics, Glasgow, 1954* (Pergamon Press, London, 1955).

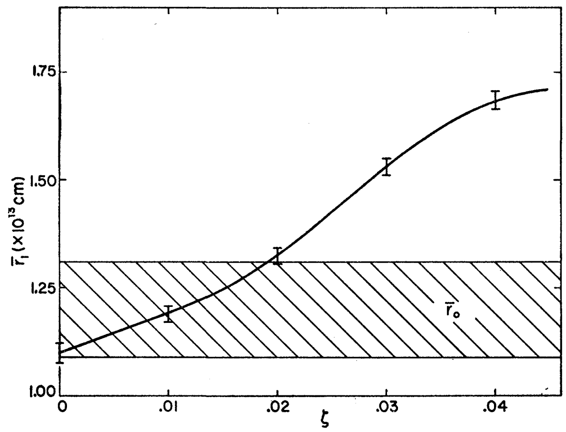

FIG. 1. The variation of the triplet characteristic interaction distance $\bar{r}$ with the admixture parameter ζ [Eq. (37)]. The vertical lines indicate the error associated with the triplet effective range r_{e1}. The shaded region indicates possible values of the corresponding singlet quantity $\bar{r}_0$. The distances $\bar{r}_1$, $\bar{r}_0$ may, consistently with experiment, be taken as equal.

distances,

$$\lim_{r\to\infty}[U_1{}^2+W_1{}^2]^{\frac{1}{2}}\sim\bar{U}_1(0)\exp(-\eta r), \tag{33}$$

while ζ similarly measures the relative amplitude of D state,

$$\lim_{r\to\infty} W_1/(U_1{}^2+W_1{}^2)^{\frac{1}{2}}\sim\zeta. \tag{34}$$

The quantity $\bar{U}_1(0)$ is determined from Eq. (27) to be

$$\bar{U}_1(0)=2\eta/(1-\eta r_{e1})^{\frac{1}{2}}, \tag{35}$$

while ζ is fixed by reference to the quadrupole moment,

$$Q=\frac{\sqrt{2}}{10}\int_0^\infty drr^2(u_1w_1-2^{-\frac{3}{2}}w_1{}^2)\simeq2.77\times10^{-27}\text{ cm}^2. \tag{36}$$

The error incurred in Eq. (36) by replacing the interaction wave functions by their corresponding comparison functions is small, by virtue of the insensitivity of Q to changes in the former at small distances. The quantity ζ thus determined is $\sim$0.02, although this value could fluctuate by perhaps 50% without destroying agreement with the observed value of Q and the asymmetry of the low-energy n-p angular distribution.

The condition (26) then yields the following expression for $\bar{r}_1$:

$$1-\eta r_{e1}=[1+6\zeta^2(\eta\bar{r}_1)^{-3}(1+\eta\bar{r}_1)^2]\exp(-2\eta\bar{r}_1). \tag{37}$$

In Fig. 1 we have plotted $\bar{r}_1$ in its dependence upon the admixture ζ, using the experimental values[7]

$$\epsilon=(2.226\pm0.003)\text{ Mev},$$
$$r_{e1}=(1.720\pm0.035)10^{-13}\text{ cm},$$

and have also indicated the permitted range of values of $\bar{r}_0$. We may evidently consider the characteristic distances r_1, r_0 to be the same within experimental uncertainties. Thus even at low energies the compactness of this "extended effective range" description is manifest, in that only one characteristic distance need be used to describe the ground states of the two-nucleon system, as opposed to the two range corrections r_{e0}, r_{e1} in the usual treatment.

It is clear from the above development that $\bar{r}$ has no more significance than the effective range itself. In the $^1S_0$ state, it is closely equal to half the effective range, and therefore may be thought of as a mean interaction distance. In general, however, it is simply to be viewed as one of two parameters that have been chosen to represent a single strongly energy-dependent quantity, the phase shift, in such a way as to make it an explicit function of energy, rather than an implicit one, over a certain energy range. To the extent that an energy-independent core region is actually a true representation of the nuclear force, these parameters have physical significance.

III. IMPLICATIONS FOR THE NUCLEAR FORCE

Since the value of $\bar{r}$ agrees well with the core radius of the preceding paper, we may apply the results of FL directly to our own work. Regarding the logarithmic derivative Γ as energy independent results in a quantitative fit of all nucleon scattering data to 200 Mev. In virtue of our more general description, this fit is no longer to be regarded solely as a consequence of a specific model of nuclear forces, but rather as constituting a strong condition which any proposed nuclear potential must satisfy if it is to yield a charge-independent representation of the data. More specifically, the energy-independent character of Γ in S-states is maintained only within an energy region corresponding to the shape-independent approximation, where all short-range potentials are equivalent; i.e., 0 to 15 Mev. The further energy independence required by the FL fit must therefore be determined by the details of the potential. Thus what has been established in the above simple fashion is a means of testing potentials for their agreement with experiment without recourse to a phase-shift analysis. One need only determine the behavior with energy of the Γ_ξ corresponding to the potential in question.

In order to illustrate this procedure, we turn our attention to $^1S_0$ scattering from a number of potentials in the limit of zero binding (infinite scattering length). Let us first consider the rectangular well, for which an exact expression for Γ can be obtained.[8] The functions u, U are given by

$$u\sim\sin\alpha x,\quad x\leqslant1,$$
$$u=U\sim\sin kx\cot\delta+\cos kx,\quad x\geqslant1, \tag{38}$$

[7] For r_{e1}, see Hughes, Burgy, and Ringo, Phys. Rev. 79, 227 (1950); for ϵ, see E. Melkonian, Phys. Rev. 76, 1744 (1949).

[8] Henceforth, all quantities refer to the $^1S_0$ state.

where we now use the dimensionless notation $x=r/R$, $\kappa=kR$, $\alpha^2=\kappa^2+\pi^2/4$, and R is the range of the force. Matching logarithmic derivatives at $x=1$ yields the desired expression for $\Gamma(\kappa)$,

$$\Gamma(\kappa)=\kappa\cot(\kappa\bar{x}+\delta)=\kappa\frac{(\alpha\cot\alpha)\cot\kappa(\bar{x}-1)-\kappa}{\alpha\cot\alpha+\kappa\cot\kappa(\bar{x}-1)},\quad (39)$$

where according to Eq. (23) $\bar{x}=\bar{r}/R=\frac{1}{2}$.

An estimate of the energy dependence of $\Gamma(\kappa)$ is to be made by comparing its value at a given energy with its correct zero-energy value,[9]

$$\Gamma(0)=a^{-1}R/(1+a^{-1}\bar{r})=0.0965\pm0.0077(\text{rect.}).\quad (40)$$

The expression

$$\epsilon(\kappa)=(\Gamma(\kappa)-\Gamma(0))/\Gamma(0)\quad (41)$$

is shown in Fig. 2.

It is nevertheless of interest to use the rectangular well as a means of testing the accuracy of results obtained through the use of variational principle, Eq. (13). Consider the expression

$$\Gamma'(\kappa)=\Gamma_0+\Gamma_1\kappa^2+\Gamma_2\kappa^4.\quad (42)$$

The coefficient Γ_1 is required to vanish by our choice of $\bar{r}$; then Γ_2 is primarily responsible for the energy dependence of Γ over an interval whose extent we now determine. By differentiating Eq. (13) twice with respect to energy, making use of its stationary character in the manner already described, and evaluating the result at $E=0$, we obtain

$$\Gamma_2=U_0^{-2}(\bar{x})\left[\int_{\bar{x}}^{\infty}dxU_0U_1-\int_0^{\infty}dxu_0u_1\right],\quad (43)$$

which may be re-expressed in terms of the shape-dependent parameter P, using Eq. (20):

$$\Gamma_2=-\bar{x}^3(8P+\tfrac{1}{3}).\quad (44)$$

A simple calculation yields the value $\Gamma_2(\text{rect.})=-0.0096$, consistent with $P(\text{rect.})=-0.032$ as given in Blatt-Jackson.[2] The expression

$$\epsilon'(\kappa)=\Gamma_2\Gamma_0^{-1}\kappa^4,\quad (45)$$

where Γ_0 is given by Eq. (40), is seen from Fig. 2 to be accurate within 10% for energies less than 50 Mev.

We assume that Eq. (42) gives an equally adequate representation of the Gaussian, exponential, and Yukawa potentials. The corresponding $\epsilon'(\kappa)$ are also plotted in Fig. 2. We now compare this energy dependence with that determined by the FL fit for the $^1S_0$ state, restricting our discussion to energies below 50 Mev. In this energy region p-p scattering is determined mostly by the $^1S_0$ state, although at the higher energies there is a small contribution coming from the $^3P_0$ state. The uncertainty in the change in Γ over this range, $\Delta\Gamma$, arises mostly from the incomplete determination of the contribution of the latter state. From the FL analysis, it is reasonable to assume that $\Delta\Gamma$ varies between zero and the change which FL obtain in their fit B. (Fit A seems to be eliminated by recent experimental data.) The corresponding condition on P is

$$-0.0416<P<-0.0367.\quad (46)$$

This condition clearly eliminates the Yukawa, exponential, and Gaussian potentials.[10] The square-well potential is still possible but only barely so. It is clear that it would be worthwhile to make a more thorough experimental investigation of this energy region.

To demonstrate what information such research might yield, suppose that Γ only changes by 20% in going from 0 to 50 Mev. Then

$$-0.0416<P<-0.0402.\quad (47)$$

We shall now show that such a limitation on the permitted values of P would require the use of core-type potentials, as is already indicated by intermediate energy scattering experiments. We shall also see that the FL fit as given by Eq. (46) if applied rigorously would also require a core. As a first orientation, it may be noted that as P changes from positive to negative

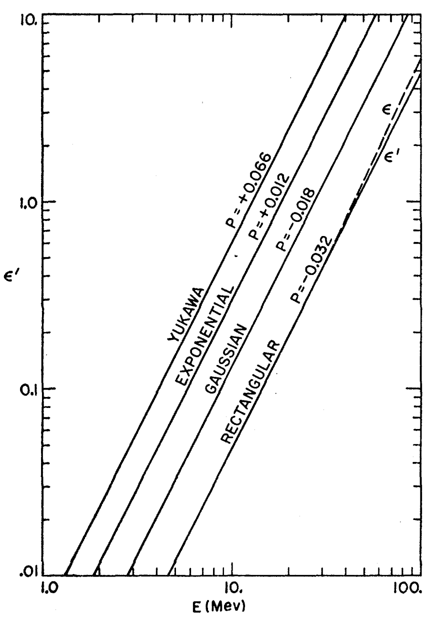

FIG. 2. The dependence of the $^1S_0$ logarithmic derivative on energy, as measured approximately by the parameter ϵ' [Eq. (45)], for four well shapes. The accuracy of this measure is indicated for the rectangular well by plotting the exact dependence ϵ [Eq. (41)] using Eq. (39).

[9] FL obtain 0.08 for this quantity by analysis of low-energy n-p data. A similar analysis by G. Breit and W. G. Bouricius (reference 3) gives the same value.

[10] The restriction (46) on P is of course much more severe than that imposed on the basis of low-energy (<10 Mev) measurements alone, since the low-energy cross sections depend only weakly on this quantity. See J. M. Blatt and J. D. Jackson, Revs. Modern Phys. 22, 77 (1950).

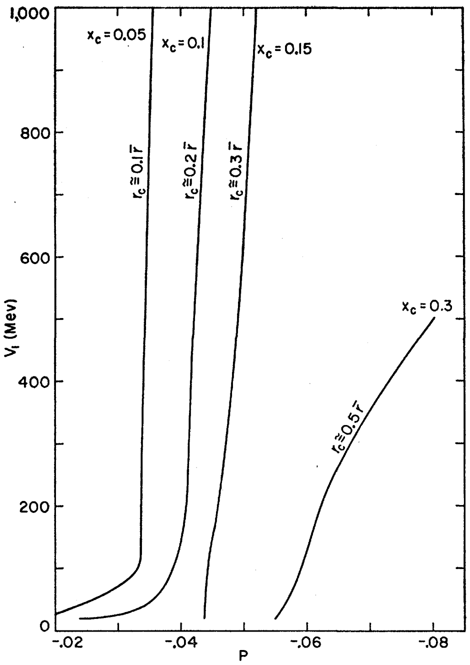

FIG. 3. The dependence of the core height V_1 on the shape dependent parameter P for various core radii r_c, using an exterior rectangular well [Eq. (47)] to achieve proper (zero) binding. Permissible core parameters must yield a value of P in the neighborhood of $P=-0.04$ (constant logarithmic derivative).

values, the potential becomes progressively shorter tailed and less attractive at the origin. Since P for a rectangular well is -0.032, the requirement (47) as well as (46) of the FL fit suggests a core without adducing further evidence.

Consider then the potential

$$V(x)=\begin{cases} V_1, & x<x_c \\ -V_0, & x_c<x<1 \\ 0, & x>1, \end{cases} \tag{48}$$

for which exact solutions may be obtained. The calculation, which is perfectly straightforward, will not be discussed except to observe that Γ in its dependence on energy is most easily found by using an obvious modification of Eq. (13):

$$U^2(\bar{x})\Gamma(\bar{x})-u^2(x_c)\gamma_c = \int_{x_c}^1 dx\left[\left(\frac{du}{dx}\right)^2-\frac{MR^2}{\hbar^2}(E+V_0)u^2\right] - \int_1^{\bar{x}} dx\left[\left(\frac{dU}{dx}\right)^2-\frac{MR^2}{\hbar^2}U^2\right],$$

where $\gamma_c=[u^{-1}du/dx]_{x=x_c}$, and the other symbols have their usual significance. In the limit of zero binding, V_1 is shown in Fig. 3 as a function of P for several values of x_c. The value of the core radius r_c changes by no more than $\pm 0.03\,\bar{r}$ for a given value of $x_c=r_c/R$, so that each of the curves in Fig. 3 also represents a characteristic core size. The matching conditions at $x=x_c$ cannot be maintained for $V_1<\sim 25$ Mev; for large V_1 the curves approach asymptotic values of P. In the limit of small r_c and large V_1, this asymptotic value approaches $P=-0.032$, i.e., a pure rectangular well. The effect upon the scattering of an increase in core size is evidently just the reverse of that produced by going to longer-tailed attractive wells, which implies the existence of many tail-core combinations for a given value of P. We note that, in order for Eq. (47) to hold r_c is confined to the approximate region

$$0.1\times 10^{-13}\text{ cm}<r_c<0.4\times 10^{-13}\text{ cm}, \quad \textit{rect.} \tag{49}$$

for reasonably hard cores.[11] This is to be compared with $r_c=0.5-0.6\times 10^{-13}$ cm, the value used by Jastrow[12] in his phenomenological fit of p-p scattering, as well as by Lévy[13] in conjunction with a meson-theoretical potential. The small discrepancy is due to their use of a fairly long-tailed attractive well, which in our calculations has the effect of increasing somewhat the permissible core size. A numerical analysis making use of an exterior Yukawa well yields the approximate limiting value

$$r_c\leqslant 0.7\times 10^{-13}\text{ cm}, \quad \textit{Yukawa}. \tag{50}$$

In summary, we have seen that the local energy independence of Γ in S-states is valid over an energy range corresponding to the shape-independent approximation. The additional energy independence required by the FL analysis prohibits the use of simple two-parameter potentials except possibly the rectangular well, but does permit core-type configurations. Such potentials will yield the correct $^1S_0$ phase shift up to energies of at least 50 to 60 Mev. It has been noted that there is still considerable freedom in the choice of potential. $^1S_0$ scattering to 50 Mev implies no restriction on the form of the attractive well, although once a well shape is chosen the core size is limited.[14] A further (weak) restriction may be obtained by invoking charge independence in S-states,[5] which tends to exclude cores greater than $\sim 0.3\times 10^{-13}$ cm.[15] Such small cores are not necessarily in conflict with meson theory.[16]

While all our examples have been concerned with static potentials, the methods may be easily extended to treat velocity-dependent forces. These can be reconciled with the FL analysis provided they are at most weakly energy dependent. Calculations, to be reported

[11] A pathological case not included in this range consists of an infinitely strong repulsive core surrounded by a δ-function attractive shell.

[12] R. Jastrow, Phys. Rev. **81**, 165 (1951).

[13] M. Lévy, Phys. Rev. **88**, 725 (1952).

[14] It proves possible to establish the necessity of a core-type potential in S-states under some fairly general assumptions. The force characteristics thus deduced accurately confirm those obtained here. For a preliminary report, see R. B. Raphael, Phys. Rev. **99**, 619(A) (1955).

[15] E. Salpeter, Phys. Rev. **91**, 994 (1953).

[16] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).

more fully at a later date, reveal that such forces are capable of reducing ϵ [Eq. (41)] in S-states considerably below its corresponding static force value. We point out that the virtual excitation, in close collisions, of nucleonic isobaric states produces an S-state force having this behavior in the energy range considered above.[17]

The fit of the two-nucleon scattering data obtained from the constancy of Γ in all states is quantitative to energies as high as 200 Mev, so that the testing procedures may be extended. Uncertainties in the FL analysis due to causes already mentioned will of course increase, so that limitations on the energy variation of Γ will become less severe.

The methods we have presented are of course not restricted to S-states. The forces obtaining in higher angular momentum states with tensor interactions may be analyzed with the aid of the development in the Appendix. It is hoped that these methods may be of use in treating the consequences of specific models of the two-body force without the necessity of direct comparison with experiment at a variety of energies.

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APPENDIX

We shall verify that Eq. (8) in the text is a stationary expression for the generalized logarithmic derivative $\Gamma(\bar{r})$ with respect to independent variations of ψ, Ψ or their adjoints, subject only to the restriction

$$(r\psi)(0)=0;\quad \psi(r)\rightarrow\Psi(r),\quad r\rightarrow\infty. \tag{A1}$$

The stationary value is attained for functions ψ, Ψ satisfying Eqs. (1) and (2) in the text. Variation with respect to $\psi^\dagger$, $\Psi^\dagger$ for example gives

$$\lim_{R\to\infty}\left\{\int d\omega\left[r\delta\psi^\dagger\frac{\partial}{\partial r}(r\psi)\right]_{r=0}^{r=R}-\int d\omega\left[r\delta\Psi^\dagger\frac{\partial}{\partial r}(r\Psi)\right]_{r=\bar{r}}^{r=R}\right\}$$

$$+\int_{r\geqslant\bar{r}}(dr)\delta\Psi^\dagger(k^2+\nabla^2)\Psi-\int_{\text{all space}}(dr)\delta\psi^\dagger(k^2-H)\psi$$

$$=\int d\omega\left[r\delta\Psi^\dagger\Gamma r\Psi+r\Psi^\dagger\delta\Gamma r\Psi\right]_{r=\bar{r}}. \tag{A2}$$

By virtue of the condition in Eq. (A1), the surface integrals on the left-hand side become

$$\int d\omega\left[r\delta\Psi^\dagger\frac{\partial(r\Psi)}{\partial r}\right]_{r=\bar{r}};$$

hence the condition that Γ be stationary under this type of variation is expressed by the differential Eqs. (1) and (2) of the text and the requirement of Eq. (6).

By insertion of the appropriate partial-wave expansions into the stationary expression Eq. (8), we may easily obtain the variational principles appropriate to the different scattering states, together with the differential equations obeyed by the radial functions in these states. For example, let us assume

$$H'=-\lambda[F(r)+\gamma G(r)S_{12}], \tag{A3}$$

where S_{12} is the tensor operator

$$S_{12}=(3/r^2)(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \tag{A4}$$

$\lambda=MV_0/\hbar^2$, and γ is the ratio of the tensor strength V_1 to the central strength V_0. We make use of

$$r\Psi_{\text{singlet}}=\textstyle\sum_L c_L p_L(\cos\theta)U_L(r). \tag{A5}$$

with a similar expression for ψ_{singlet}; and, after performing the angular integrations, obtain the form of Eq. (8) appropriate to singlet states of scattering:

$$U_L^2(\bar{r})\Gamma_{LL}(\bar{r})$$

$$=\int_0^\infty dr\left\{(du_L/dr)^2-\left[k^2+\lambda F(r)-\frac{L(L+1)}{r^2}\right]u_L^2\right\}$$

$$-\int_{r_L}^\infty dr\left\{(dU_L/dr)^2-\left[k^2-\frac{L(L+1)}{r^2}\right]U_L^2\right\}. \tag{A6}$$

Equation (A6) is a stationary expression for $\Gamma_{LL}(\bar{r})$ when the radial functions u_L, U_L satisfy

$$\begin{aligned}\{(d^2/dr^2)+[k^2+\lambda F(r)-L(L+1)/r^2]\}u_L(r)=0,\\ \{(d^2/dr^2)+[k^2-L(L+1)/r^2]\}U_L(r)=0,\end{aligned} \tag{A7}$$

and provided we make the identification

$$\Gamma_{LL}(\bar{r}_L)=U_L^{-1}(\bar{r}_L)[dU_L/dr]_{r=\bar{r}_L}. \tag{A8}$$

Thus, in singlet states, Γ is a diagonal matrix with elements Γ_{LL}. In virtue of the stationary property of Eq. (A6), expansion of the radial functions u_L, U_L in the manner of Eq. (16) of the text gives rise to the rigorous expression Eq. (10).

Turning now to triplet states, we note that Γ_{triplet} is in general not diagonal, owing to mixing of states induced by the tensor coupling. We may, however, go to a representation in which Γ_{triplet} is diagonal by taking linear combinations of states with the same parity. In this "eigenstate representation" the scattering matrix is diagonal. We follow Rohrlich and Eisenstein[18] in

[17] R. B. Raphael and J. Schwinger, Phys. Rev. **90**, 373(A) (1953).

[18] F. Rohrlich and J. Eisenstein, Phys. Rev. **75**, 705 (1949), referred to as RE.

introducing the partial wave expansion

$$\Psi_{\text{triplet}}=\sum_{J,m} c_J{}^m[c_\alpha\Psi_\alpha{}^{J,m}+c_\beta\Psi_\beta{}^{J,m}+c_\gamma\Psi_\gamma{}^{J,m}], \tag{A9}$$

where

$$\begin{aligned} r\Psi_\alpha{}^{J,m}&=\Phi_{UJ}U_J{}^{(\alpha)}-\Phi_{WJ}W_J{}^{(\alpha)},\\ r\Psi_\beta{}^{J,m}&=\Phi_{VJ}V_J{}^{(\beta)},\\ r\Psi_\gamma{}^{J,m}&=\Phi_{UJ}U_J{}^{(\gamma)}+\Phi_{WJ}W_J{}^{(\gamma)}, \end{aligned} \tag{A10}$$

are a set of mutually orthogonal "parity" wave functions, the eigenfunctions of the tensor Hamiltonian. Here the Φ's are the orthonormal spin-angle vectors introduced by Corben and Schwinger,[19] while U_J, V_J, W_J represent the comparison radial functions for the states $L=J-1$, J, $J+1$, respectively. An analogous expansion is introduced for the interaction wave function ψ_{triplet}. The effect of S_{12} operating on Φ, as given in RE, is

$$\begin{aligned} S_{12}\Phi_{UJ}&=-2\frac{J-1}{2J+1}\Phi_{UJ}+6\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\Phi_{WJ},\\ S_{12}\Phi_{WJ}&=6\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\Phi_{UJ}-2\frac{J+2}{2J+1}\Phi_{WJ},\\ S_{12}\Phi_{VJ}&=2\Phi_{VJ}. \end{aligned} \tag{A11}$$

We are now in a position to carry out the angular integrations in Eq. (8). The orthogonality of the wave functions, Eq. (A10), results in three separate expressions corresponding to the eigenstates[20] (α, β, γ):

$$\begin{aligned} &[U_J{}^2(\bar r_J)\Gamma_{UJ}(\bar r_J)]^{(\alpha,\gamma)}+[W_J{}^2(\bar r_J)\Gamma_{WJ}(\bar r_J)]^{(\alpha,\gamma)}\\ &=\int_0^\infty dr\bigg\{\bigg[(du_J/dr)^2-\bigg(k^2+\lambda f_J(r)-\frac{J(J-1)}{r^2}\bigg)u_J{}^2\bigg]\\ &\quad+\bigg[(dw_J/dr)^2-\bigg(k^2+\lambda h_J(r)-\frac{(J+1)(J+2)}{r^2}\bigg)w_J{}^2\bigg]\\ &\quad-2\lambda g_J(r)u_Jw_J\bigg\}^{(\alpha,\gamma)}-\int_{\bar r_J{}^{(\alpha,\gamma)}}^\infty dr\bigg\{\bigg[(dU_J/dr)^2\\ &\quad-\bigg(k^2-\frac{J(J-1)}{r^2}\bigg)U_J{}^2\bigg]+\bigg[(dW_J/dr)^2\\ &\quad-\bigg(k^2-\frac{(J+1)(J+2)}{r^2}\bigg)W_J{}^2\bigg]\bigg\}^{(\alpha,\gamma)}, \end{aligned} \tag{A12}$$

$$\begin{aligned} &[V_J{}^2(\bar r_J)\Gamma_{VJ}(\bar r_J)]^{(\beta)}\\ &+\int_0^\infty dr\bigg[(dv_J/dr)^2-\bigg(k^2-\frac{J(J+1)}{r^2}+\lambda l_J(r)\bigg)v_J{}^2\bigg]^{(\beta)}\\ &-\int_{\bar r_J{}^{(\beta)}}^\infty dr\bigg[(dV_J/dr)^2-\bigg(k^2-\frac{J(J+1)}{r^2}\bigg)V_J{}^2\bigg]^{(\beta)}. \end{aligned} \tag{A13}$$

Here we use the notation of RE,

$$\begin{aligned} f_J(r)&=F(r)-2\frac{J-1}{2J+1}G(r),\\ g_J(r)&=6\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\gamma G(r),\\ h_J(r)&=F(r)-2\frac{J+2}{2J+1}\gamma G(r),\\ l_J(r)&=F(r)+2\gamma G(r). \end{aligned} \tag{A14}$$

Again we obtain the differential equations satisfied by the various radial functions in the form of conditions such that Eqs. (A12) and (A13) are stationary expressions for the Γ's. They are

$$\begin{aligned} &\bigg(\frac{d^2}{dr^2}+k^2-\frac{J(J-1)}{r^2}\bigg)u_J(r)\\ &\qquad=\lambda[f_J(r)u_J(r)+g_J(r)w_J(r)],\\ &\bigg(\frac{d^2}{dr^2}+k^2-\frac{(J+1)(J+2)}{r^2}\bigg)w_J(r)\\ &\qquad=\lambda[g_J(r)u_J(r)+h_J(r)w_J(r)],\\ &\bigg(\frac{d^2}{dr^2}+k^2-\frac{J(J+1)}{r^2}\bigg)v_J(r)=\lambda l(r)v_J(r). \end{aligned} \tag{A15}$$

The corresponding comparison functions satisfy the set of Eqs. (A15) with $\lambda=0$. Then by virtue of Eq. (A1), a variation of this type applied to Eq. (A12) results in

$$\begin{aligned} &U_J{}^2(\bar r_J)\delta\Gamma_{UJ}(\bar r_J)+W_J{}^2(\bar r_J)\delta\Gamma_{WJ}(\bar r_J)\\ &\quad=2[(dU_J/dr)-U_J\Gamma_{UJ}]_{\bar r_J}\delta U_J(\bar r_J)\\ &\qquad+2[(dW_J/dr)-W_J\Gamma_{WJ}]_{\bar r_J}\delta W_J(\bar r_J) \end{aligned} \tag{A16}$$

for either of the mixtures (α,γ). The variations of U_J, W_J at $r=\bar r_J$ are unrelated, and hence Eq. (A12) is indeed stationary provided we define

$$\begin{aligned} \Gamma_{UJ}(\bar r_J)&=U_J{}^{-1}(\bar r_J)\bigg[\bigg(\frac{d}{dr}\bigg)U_J(r)\bigg]_{\bar r_J};\\ \Gamma_{WJ}(\bar r_J)&=W_J{}^{-1}(\bar r_J)\bigg[\bigg(\frac{d}{dr}\bigg)W_J(r)\bigg]_{\bar r_J}. \end{aligned} \tag{A17}$$

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[19] H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1940).

[20] In these expressions the superscripts designating the parity eigenstates are meant to apply to all the wave functions, logarithmic derivatives, and characteristic distances $\bar r$ occurring within the brackets.

The stationary nature of Eq. (A13) follows analogously.

It is a consequence of describing the scattering in terms of the eigenstates of the tensor Hamiltonian that the variations δU_J, δW_J are asymptotically related. Thus asymptotically, U_J, W_J are given by

$$\begin{aligned} U_J{}^{(\alpha,\gamma)} &\sim A_J{}^{(\alpha,\gamma)}\{\sin[kr-(\pi/2)(J-1)\\ &\qquad +\delta_{UJ}{}^{(\alpha,\gamma)}]/\cos\delta_{UJ}{}^{(\alpha,\gamma)}\},\\ W_J{}^{(\alpha,\gamma)} &\sim C_J{}^{(\alpha,\gamma)}\{\sin[kr-(\pi/2)(J+1)\\ &\qquad +\delta_{WJ}{}^{(\alpha,\gamma)}]/\cos\delta_{WJ}{}^{(\alpha,\gamma)}\}. \end{aligned} \tag{A18}$$

We must choose $\delta_{UJ}{}^{(\alpha,\gamma)}=\delta_{WJ}{}^{(\alpha,\gamma)}$ in order to uniquely fix the pairs of linearly independent radial functions $(u_J,w_J)^\alpha$, $(u_J,\omega_J)^\gamma$ so that the coupled wave mixture remains unchanged before and after the scattering. We may then define admixture parameters $\eta_J{}^{(\alpha,\gamma)}=-(C_J/A_J)^{(\alpha,\gamma)}$ giving the asymptotic mixing of the $L=J\pm1$ waves. Thus we have at large distances the relation

$$\delta W_J{}^{(\alpha,\gamma)}(r)=\eta_J{}^{(\alpha,\gamma)}\delta U_J{}^{(\alpha,\gamma)}(r),\quad r\rightarrow\infty. \tag{A19}$$

The reciprocal relationship between $\eta_J{}^\alpha$, $\eta_J{}^\gamma$, which follows from the unitarity of the S-matrix and the reciprocity of scattering, may also be obtained from the variational principle of Eq. (A12) in a particularly simple fashion. We introduce the scale transformation

$$\begin{aligned} w_J{}^{(\alpha,\gamma)} &\rightarrow \eta_J{}^{(\alpha,\gamma)}w_J{}^{(\alpha,\gamma)},\\ W_J{}^{(\alpha,\gamma)} &\rightarrow \eta_J{}^{(\alpha,\gamma)}W_J{}^{(\alpha,\gamma)}. \end{aligned} \tag{A20}$$

Equation (A12) may then be written as

$$[U_J{}^2\Gamma_{UJ}+\eta_J{}^2W_J{}^2\Gamma_{WJ}]_{r=\bar{r}_J}=a_J-2b_J\eta_J+c_J\eta_J{}^2. \tag{A21}$$

Variation with respect to η_J yields, by virtue of the stationary property,

$$\eta_J=b_J/[c_J-W_J{}^2(r_J)\Gamma_{WJ}(r_J)]. \tag{A22}$$

Introducing Eq. (A22) into Eq. (A21), one obtains a quadratic equation in $W_J{}^2\Gamma_{WJ}$, whose solutions are the eigensolutions of the scattering problem. The well-known relation

$$\eta_J{}^\alpha\eta_J{}^\gamma=-1 \tag{A23}$$

then follows immediately from Eq. (A22).

Finally, it may be noted that the D-wave admixture for the deuteron ground state may be written

$$\begin{aligned} \eta_D{}^2 &= \zeta^2/(1+\zeta^2)\\ &= -\left[\frac{U_1{}^2(\partial/\partial E)\Gamma_{U1}}{W_1{}^2(\partial/\partial E)\Gamma_{W1}}\right]_{r=\bar{r}},\quad E=-\epsilon, \end{aligned} \tag{A24}$$

as follows from the stationary character of Eq. (A21). This then implies the more explicit expression (37) of the text.

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Derivation of Low Scattering Formalism

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The Yang-Feldman formalism is applied to the derivation of the transition amplitude for boson-fermion scattering in a form which makes evident its connection with a method of derivation suggested by Wick.

LOW[1] has proposed a form of scattering theory which has proved extremely useful in the discussion of the consequences of causality for scattering.[2] Alternative derivations have been given by Goldberger,[3] Nambu,[4] Schweber,[5] and Wick,[6] the last of these only for the case of a fixed source. In this note we propose a concise derivation based upon the Yang-Feldman formalism[7] which makes evident the relationship to the method of Wick and shows that the latter is applicable to the relativistic case.

We consider a system described by a Hamiltonian

$$H=H_0+H_1, \tag{1}$$

where H_0 is a sum of Hamiltonians for noninteracting fields describing particles with their experimental masses and H_1 is their mutual interaction. We merely assume that H is expressed in terms of sets of canonical variables, which in the absence of interaction are the usual particle creation and annihilation operators,

$$H=H[a^\dagger(t),a(t);\,b^\dagger(t),b(t);\,\cdots]. \tag{2}$$

Here, the set $a^\dagger(t)$, $a(t)$ refer to the mesons, $b^\dagger(t)$, $b(t)$, to the nucleons, etc., and we have suppressed the dependence on spatial or three-momentum coordinates. For simplicity of notation we work with neutral mesons.

The field equations for the meson operators are

$$\begin{aligned} i\dot{a}(t) &= [a(t),H]=\delta H/\delta a^\dagger(t)\\ &= \omega a(t)+\delta H_1/\delta a^\dagger(t), \end{aligned} \tag{3}$$

[1] F. E. Low, Phys. Rev. **97**, 1392 (1955).
[2] See, for example, R. Oehme, Phys. Rev. **100**, 1503 (1955), and M. L. Goldberger, Phys. Rev. **99**, 979 (1955), from which the previous literature may be traced.
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