is no marked tendency for the contributions of the higher l values to decrease.

V. ACKNOWLEDGMENTS

We are indebted to Dr. Eugene Baumgartner, Dr. J. B. French, Dr. H. W. Fulbright, and Dr. B. J. Raz for several interesting discussions; to Mr. Yoshio Hashimoto for his assistance throughout the experiment and in particular for the square-well calculations; and especially to Dr. David Saxon for his cooperation in supplying us with his preliminary calculations on these data.

PHYSICAL REVIEW VOLUME 102, NUMBER 6 JUNE 15, 1956

Calculation of Electric Dipole Coulomb Excitation and Dipole Bremsstrahlung*

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(Received January 16, 1956)

Numerical results are obtained for electric dipole Coulomb excitation. Tables of the dimensionless excitation function a_0 and the directional correlation parameter a_2 are presented over a very wide range of their arguments. It is shown that the dipole bremsstrahlung cross section can be written in terms of the a_μ, so that the tabulation also serves for the calculation of the bremsstrahlung differential cross section.

I. INTRODUCTION

THE formalism and mathematical techniques necessary to an adequate quantum mechanical calculation for the general multipole in Coulomb excitation have been presented earlier.[1] These techniques have already been applied to the case of electric quadrupole excitation.[2] In the present paper numerical results for electric dipole excitation are given. Although at present electric dipole excitation is not as important experimentally as electric quadrupole excitation,[3] a few low-lying (1−) states have been reported[4] and there is reason to believe that low-lying states of odd parity may exist in the fissionable elements.

Results are given for the dimensionless excitation function a_0 and the correlation parameter a_2, for values of the arguments in the range $0.001 \leq \eta \leq 40$, $1 \leq \rho \leq 1.8$, where $\eta \equiv Z_1 Z_2 e^2/\hbar v_{\text{initial}}$ and $\rho \equiv k_{\text{initial}}/k_{\text{final}}$. This range includes all energies of experimental interest and covers energy losses of up to 70%. Numerical values for the dimensionless excitation function a_0 and the particle parameter a_2 are presented in Tables I and II, respectively. These values are plotted in two different ways in order to exhibit the general behavior of these functions. The qualitative features of the present calculation are very similar to those discussed in II.

The limitations of this work have been discussed in I. In particular, center-of-mass corrections and retardation effects have been neglected in the calculation presented below. For the excitation of medium and heavy nuclei, the error due to the assumption that the position of the target nucleus defines the center of mass should be small. Moreover, since it is expected that the present results will be applied to the excitation of very heavy nuclei, such corrections should be especially small. The neglect of retardation is a more serious source of error for electric dipole than for electric quadrupole excitation. In the electric dipole case the retardation expansion enters in lower order in $(k_{\text{rad}} r_{tp})^2$ than for quadrupole excitation. In addition the effect of the higher angular momenta is more pronounced in the dipole case. Nevertheless, the effect of retardation is expected to be less than one percent over most of the experimental region. Calculations to treat the exact dipole operator are now in progress and a quantitative discussion of retardation will appear later.

The relation between the calculations for electric dipole excitation and dipole bremsstrahlung will be discussed in Sec. III. It will be seen that the functions $a_{0,2}$ are likewise required in the calculation of dipole bremsstrahlung.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] Biedenharn, McHale, and Thaler, Phys. Rev. **100**, 376 (1955). This reference is hereinafter designated in the text as I.

[2] Goldstein, McHale, Thaler, and Biedenharn, Phys. Rev. **100**, 436 (1955); and Biedenharn, Goldstein, McHale, and Thaler, Phys. Rev. **101**, 662 (1956). The latter reference is designated in the text as II.

[3] The absence of low-lying odd-parity states is accounted for by the fact that for nuclei whose intrinsic shapes are symmetric with respect to reflection, no states of parity opposite to that of the ground state can appear in the rotational spectrum. This question is discussed in detail by A. Bohr and his collaborators. For a survey of this work and a complete set of references, see, e.g., A. Bohr, *Rotational States of Atomic Nuclei* (Einar Munksgaard, Copenhagen, 1954). However, the theoretical interpretation of the anisotropies in the fragment distribution for photofission and fission produced by fast particles requires the abandonment of the assumption of reflective symmetry, so that, for $A \gtrsim 200$, low-lying (1−) states (capable of excitation by the Coulomb excitation process) should appear. For a discussion of this question, see A. Bohr, *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (to be published), Paper U.N. 911.

[4] Stephens, Asaro, and Perlman, Phys. Rev. **96**, 1568 (1954), and **100**, 1543 (1955).

TABLE I. The dimensionless excitation function a_0 vs ρ, η. The coefficient a_0 is tabulated for representative values of its arguments. The entries are presented as four-digit numbers followed in parentheses by the powers of ten which multiply them. The last significant figure is uncertain.

ρ \ η	1.01	1.025	1.05	1.1	1.15	1.2	1.25	1.3	1.35	1.4	1.6	1.8
0.001	2.643 (0)	2.249 (0)	1.948 (0)	1.674 (0)	1.530 (0)	1.438 (0)	1.372 (0)	1.323 (0)	1.284 (0)	1.253 (0)	1.171 (0)	1.125 (0)
0.01	2.661 (0)	2.246 (0)	1.945 (0)	1.669 (0)	1.523 (0)	1.429 (0)	1.362 (0)	1.311 (0)	1.271 (0)	1.238 (0)	1.151 (0)	1.099 (0)
0.1	2.651 (0)	2.224 (0)	1.912 (0)	1.614 (0)	1.450 (0)	1.339 (0)	1.256 (0)	1.190 (0)	1.135 (0)	1.088 (0)	9.467 (−1)	8.454 (−1)
0.25	2.608 (0)	2.166 (0)	1.833 (0)	1.500 (0)	1.305 (0)	1.168 (0)	1.060 (0)	9.714 (−1)	8.960 (−1)	8.302 (−1)	6.283 (−1)	4.853 (−1)
0.5	2.499 (0)	2.032 (0)	1.665 (0)	1.279 (0)	1.044 (0)	8.749 (−1)	7.434 (−1)	6.371 (−1)	5.489 (−1)	4.748 (−1)	2.710 (−1)	1.565 (−1)
0.75	2.372 (0)	1.884 (0)	1.488 (0)	1.064 (0)	8.093 (−1)	6.309 (−1)	4.983 (−1)	3.966 (−1)	3.172 (−1)	2.545 (−1)	1.072 (−1)	4.548 (−2)
1.0	2.250 (0)	1.743 (0)	1.326 (0)	8.805 (−1)	6.217 (−1)	4.497 (−1)	3.294 (−1)	2.430 (−1)	1.801 (−1)	1.338 (−1)	4.139 (−2)	1.287 (−2)
2.5	1.747 (0)	1.172 (0)	7.073 (−1)	2.986 (−1)	1.336 (−1)	6.103 (−2)	2.817 (−2)	1.308 (−2)	6.091 (−3)	2.842 (−3)	1.356 (−4)	6.466 (−6)
5.0	1.308 (0)	6.928 (−1)	2.861 (−1)	5.615 (−2)	1.160 (−2)	2.440 (−3)	5.173 (−4)	1.101 (−4)	2.349 (−5)	5.016 (−6)	1.045 (−8)	2.166 (−11)
7.5	1.033 (0)	4.345 (−1)	1.225 (−1)	1.110 (−2)	1.054 (−3)	1.016 (−4)	9.859 (−6)	9.596 (−7)	9.354 (−8)	9.125 (−9)	8.255 (−13)	...
10.0	8.339 (−1)	2.793 (−1)	5.363 (−2)	2.237 (−3)	9.737 (−5)	4.294 (−6)	1.904 (−7)	8.465 (−9)	3.768 (−10)	1.678 (−11)	6.577 (−17)	...
12.5	6.825 (−1)	1.819 (−1)	2.376 (−2)	4.552 (−4)	9.066 (−6)	1.827 (−7)	3.701 (−9)	7.511 (−11)	1.526 (−12)	3.100 (−14)	...	...
15.0	5.635 (−1)	1.195 (−1)	1.060 (−2)	9.311 (−5)	8.481 (−7)	7.808 (−9)	7.219 (−11)	6.687 (−13)	6.197 (−15)	5.744 (−17)	...	...
20.0	3.903 (−1)	5.231 (−2)	2.135 (−3)	3.934 (−6)	7.480 (−9)	1.435 (−11)	2.764 (−14)	5.329 (−17)	1.028 (−19)	1.982 (−22)	...	...
25.0	2.745 (−1)	2.317 (−2)	4.343 (−4)	1.674 (−7)	6.637 (−11)	2.652 (−14)	1.063 (−17)	4.265 (−21)	1.711 (−24)	...	...	...
30.0	1.946 (−1)	1.033 (−2)	8.883 (−5)	7.152 (−9)	5.909 (−13)	4.916 (−17)	4.099 (−21)	3.421 (−25)	...	...	...	...
35.0	1.389 (−1)	4.633 (−3)	1.824 (−5)	3.064 (−10)	5.271 (−15)	9.126 (−20)	1.583 (−24)	...	...	...	...	...
40.0	9.940 (−2)	2.082 (−3)	3.752 (−6)	1.315 (−11)	4.709 (−17)	1.696 (−22)	6.120 (−28)	...	...	...	...	...

II. SUMMARY OF FORMULAS

The electric dipole excitation function in the long-wavelength approximation is given by the expression

$$b_0=\sum_{l=0}^{\infty}\left[l(1,2;l-1)^2+(l+1)(-1,2;l+1)^2\right],\quad (1)$$

where

$$(m,n;l)\equiv\int_0^\infty dr\, r^{-n}F_l(\eta_1,k_1r)F_{l+m}(\eta_2,k_2r)$$
$$=k_2^{n-1}\int_0^\infty dr'(r')^{-n}F_l(\eta,\rho r')F_{l+m}(\rho\eta,r'),\quad (2)$$

and $\eta\equiv\eta_1$, $\rho\equiv(k_1/k_2)=(\eta_2/\eta_1)$.

The total cross section in this approximation[5] may be written as

$$\sigma_1^{(e)}=\left(\frac{k_2}{k_1}\right)\left(\frac{2J_f+1}{2J_i+1}\right)(f\|r\|i)^2\left(\frac{8\pi Z_1Z_2Me^2}{3k_1k_2\hbar^2}\right)^2 b_0.\quad (3)$$

The reduced radial matrix element is similarly defined by

$$\langle J_f\mu'|rY_1{}^m|J_i\mu\rangle\equiv(f\|r\|i)C_{\mu m\mu'}{}^{J_i1J_f}.\quad (4)$$

To facilitate comparison of these and previous formulas with the work of the Copenhagen group, it should be noted that

$$B_e(\lambda)=(Z_2e)^2\left(\frac{2J_f+1}{2J_i+1}\right)(f\|r^\lambda\|i)^2,$$

and that

$$f_{e\lambda}=\frac{64\pi^2}{(2\lambda+1)^2}\frac{a_0(e\lambda)}{\rho}\eta^{2\lambda-2},$$

where $a_0(e\lambda)=k_2^{-2\lambda}b_0(e\lambda)$.

The correlation of the emitted gamma ray with respect to the direction of the incident beam is calculated from the formula

$$W(\theta)=A_0+a_2A_2P_2(\cos\theta),\quad (5)$$

where a_2 is the particle parameter defined by Eqs. (6)–(7) and the A_μ are the $\gamma-\gamma$ correlation coefficients[6] such that the corresponding $\gamma-\gamma$ correlation is given by: $W(\theta)=\sum_\mu A_\mu P_\mu(\cos\theta)$.

The particle parameter a_2 is defined as

$$a_2\equiv(b_2/b_0),\quad (6)$$

where

$$b_2=-\sum_{l=0}^{\infty}\left[\frac{l(l-1)}{2l+1}(1,2;l-1)^2+\frac{(l+1)(l+2)}{2l+1}(-1,2;l+1)^2-6\cos(\sigma_{l+1}-\sigma_{l-1})\frac{l(l+1)}{2l+1}\times(1,2;l-1)(-1,2;l+1)\right],\quad (7)$$

[5] In previous formulas for the total cross section [I, Eq. (13) and II, Eq. (3)] the factor 4π should be deleted.

[6] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).

TABLE II. The particle parameter a_2 vs ρ, η. The parameter a_2 is tabulated for representative values of its arguments. The last significant figure is uncertain.

$\eta \backslash \rho$	1.01	1.025	1.05	1.1	1.15	1.2	1.25	1.3	1.35	1.4	1.6	1.8
0.001	0.4105	0.2601	0.09045	−0.1568	−0.3458	−0.5013	−0.6324	−0.7455	−0.8441	−0.9308	−1.193	−1.368
0.01	0.4142	0.2627	0.09134	−0.1561	−0.3456	−0.5010	−0.6323	−0.7454	−0.8440	−0.9306	−1.193	−1.367
0.1	0.4124	0.2614	0.08992	−0.1568	−0.3454	−0.4998	−0.6301	−0.7422	−0.8398	−0.9257	−1.185	−1.357
0.25	0.4066	0.2543	0.08284	−0.1610	−0.3451	−0.4947	−0.6201	−0.7275	−0.8206	−0.9022	−1.147	−1.309
0.5	0.3884	0.2316	0.05950	−0.1772	−0.3500	−0.4869	−0.5996	−0.6947	−0.7761	−0.8467	−1.055	−1.190
0.75	0.3642	0.2010	0.02661	−0.2043	−0.3664	−0.4912	−0.5917	−0.6750	−0.7453	−0.8055	−0.9794	−1.089
1.0	0.3389	0.1682	−0.009740	−0.2374	−0.3916	−0.5073	−0.5987	−0.6732	−0.7353	−0.7879	−0.9369	−1.029
2.5	0.2147	0.007991	−0.1875	−0.4074	−0.5386	−0.6288	−0.6955	−0.7473	−0.7888	−0.8228	−0.9141	−0.9670
5.0	0.08335	−0.1507	−0.3522	−0.5536	−0.6622	−0.7326	−0.7826	−0.8202	−0.8497	−0.8735	−0.9355	−0.9704
7.5	−0.006246	−0.2521	−0.4493	−0.6326	−0.7262	−0.7851	−0.8261	−0.8566	−0.8802	−0.8992	−0.9479	...
10.0	−0.07426	−0.3253	−0.5155	−0.6836	−0.7665	−0.8176	−0.8528	−0.8788	−0.8989	−0.9148	−0.9558	...
12.5	−0.1296	−0.3818	−0.5643	−0.7198	−0.7945	−0.8400	−0.8712	−0.8941	−0.9116	−0.9256	...	...
15.0	−0.1762	−0.4272	−0.6022	−0.7470	−0.8154	−0.8566	−0.8847	−0.9052	−0.9210	−0.9334	...	...
20.0	−0.2511	−0.4963	−0.6577	−0.7858	−0.8447	−0.8797	−0.9035	−0.9207	−0.9339	−0.9444	...	...
25.0	−0.3093	−0.5474	−0.6971	−0.8126	−0.8645	−0.8954	−0.9162	−0.9312	−0.9426	...	...	...
30.0	−0.3569	−0.5867	−0.7266	−0.8320	−0.8791	−0.9068	−0.9253	−0.9387	...	...	...	...
35.0	−0.3960	−0.6180	−0.7499	−0.8473	−0.8903	−0.9154	−0.9323	...	...	...	...	...
40.0	−0.4300	−0.6445	−0.7689	−0.8594	−0.8993	−0.9224	−0.9380	...	...	...	...	...

and the $\sigma_l \equiv \arg\Gamma(l+1+i\eta)$ are the Coulomb phase shifts.

Since the excitation function b_0 has the dimensions of inverse length squared, it is convenient to tabulate the dimensionless excitation function

$$a_0 \equiv (k_2)^{-2} b_0. \tag{8}$$

The derivation of these formulas is discussed in general in I.

III. DIPOLE BREMSSTRAHLUNG

A completely quantum mechanical calculation of the differential cross section for the dipole emission of radiation of frequency ω in bremsstrahlung has been carried out by Sommerfeld.[7] Drell and Huang[8] have performed an expansion in ρ (about $\rho=1$) of the Sommerfeld formula, which enabled them to obtain good agreement with the experimental bremsstrahlung cross section[9] for 2-Mev protons on tin ($\eta=5.64$, $\rho=1.04$). The present calculations represent a somewhat different approach and have the advantage that they need not be restricted to values of ρ near $\rho=1$.

The result of Sommerfeld[7] was presented in terms of the summed Coulomb field. In terms of the angular momentum expansion, this result may equivalently be written as

$$\frac{d\sigma}{d\Omega}=\frac{1}{6\pi}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{Mc}\right)^2 \rho^{-3}(\rho^2-1)^2 \times\left(\frac{Z_1A_2-Z_2A_1}{A_1+A_2}\right)^2\frac{d\omega}{\omega}\left[\beta_0+\tfrac{1}{2}\beta_2 P_2(\cos\theta)\right], \tag{9}$$

where

$$\beta_0 \equiv \sum_{l=0}^{\infty}\left[l(1,-1;l-1)^2+(l+1)(-1,-1;l+1)^2\right], \tag{10}$$

$$\beta_2 \equiv -\sum_{l=0}^{\infty}\Bigg[\frac{l(l-1)}{2l+1}(1,-1;l-1)^2 +\frac{(l+1)(l+2)}{(2l+1)}(-1,-1;l+1)^2 -6\cos(\sigma_{l+1}-\sigma_{l-1})\frac{l(l+1)}{(2l+1)} \times(-1,-1;l+1)(1,-1;l-1)\Bigg]. \tag{11}$$

Since an elementary calculation yields immediately the result that

$$(\pm 1,-1;l)=-\frac{4MZ_1Z_2e^2}{\hbar^2(k_2^2-k_1^2)^2}(\pm 1,2,l), \tag{12}$$

it is evident that the β_μ differ from the b_μ only by a factor independent of l. Therefore, the result for dipole bremsstrahlung may be written as

$$\frac{d\sigma}{d\Omega}=\frac{8}{3\pi}\left(\frac{k_2}{k_1}\right)\left(\frac{Z_1Z_2}{k_1k_2}\right)^2\left(\frac{e^2}{\hbar c}\right)^3 \times\left(\frac{Z_1A_2-Z_2A_1}{A_1+A_2}\right)^2\frac{d\omega}{\omega}\left[b_0+\tfrac{1}{2}b_2P_2(\cos\theta)\right], \tag{13}$$

so that the functions $a_0 \equiv (b_0/k_2^2)$ and $a_2 \equiv (b_2/b_0)$ tabulated below are of equal use for bremsstrahlung as for Coulomb excitation.[10]

IV. METHOD OF CALCULATION

The calculation of the b_μ is greatly facilitated by the fact that the radial matrix elements $(\pm 1,2;l)$ can be reduced to a difference of two $(0,1;l)$ functions [see I, Eq. (64)], *viz.*,

$$(\pm m,2;l)=m\left\{\frac{k_2}{l'}\left[(l')^2+\eta_2^2\right]^{\frac{1}{2}}(0,1;l) -\frac{k_1}{l'}\left[(l')^2+\eta_1^2\right]^{\frac{1}{2}}(0,1;l+m)\right\}, \tag{14}$$

[7] A. J. F. Sommerfeld, *Atombau und Spektrallinien* (Ungar, New York, 1953), Vol. 2, Chap. 7, p. 527, Eq. (12).

[8] S. Drell and K. Huang, Phys. Rev. 99, 686 (1955). This type of expansion is also discussed in I.

[9] Mark, McClelland, and Goodman (to be published).

[10] This point has been given already in the paper of C. Mullin and E. Guth, Phys. Rev. 68, 141 (1951).

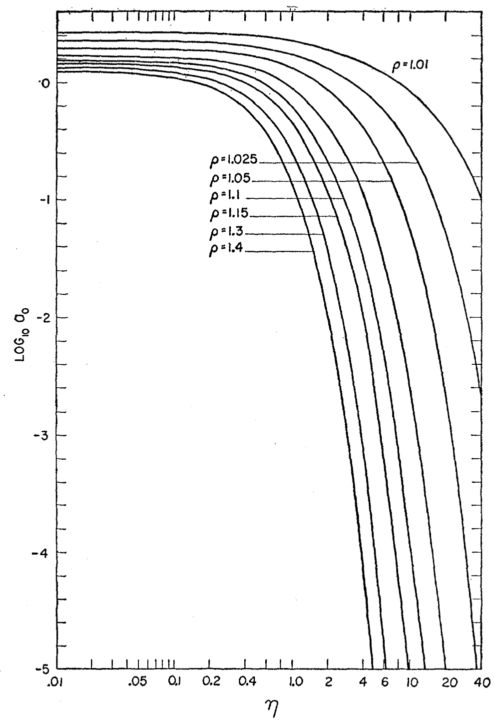

FIG. 1. The quantity $\log_{10}a_0$ *vs* η on a semilog plot for several values of constant ρ.

where $l'=\frac{1}{2}(2l+m+1)$, and $m=\pm1$. The $(0,1;l)$ function has been evaluated by Sommerfeld as an ordinary hypergeometric function whose argument is a function of ρ alone [see I, Eq. (28)], *viz.*,

$$(0,1;l)=\frac{1}{4}\left[\frac{4\rho}{(1+\rho)^2}\right]^{l+1}\exp\left(-\frac{\pi\eta}{2}|\rho-1|\right)$$
$$\times\left|\frac{\rho-1}{\rho+1}\right|^{-2l-2+i\eta(1+\rho)}\frac{|\Gamma(l+1+i\eta)\Gamma(l+1+i\rho\eta)|}{\Gamma(2l+2)}$$
$$\times{}_2F_1\left(l+1-i\eta,\, l+1-i\rho\eta,\, 2l+2;\, -\frac{4\rho}{(\rho-1)^2}\right). \quad (15)$$

The calculation of the required matrix elements is therefore a relatively simple task. The well-known transformation properties of the ordinary hypergeometric function[11] make it possible to find various series for the $(0,1;l)$, such that the calculation is easily carried out for all values of ρ.

The summed field formula of Sommerfeld can be used to calculate the excitation function b_0. However, it proved more convenient in this connection to sum directly the series in angular momentum for b_0.[12] This sum can be performed by using the relation of Eq. (14) in Eq. (1). It is then easily shown by use of the recursion relation, I, Eq. (63), that the expression so obtained may be written as

$$b_0=\sum_{l=1}^{\infty}[Q_{l+1}-Q_l], \quad (16)$$

where

$$lQ_l=2k_1k_2[(l^2+\eta_1^2)(l^2+\eta_2^2)]^{\frac{1}{2}}(0,1;l)(0,1;l-1)$$
$$-[(k_1^2+k_2^2)l^2+2k_1^2\eta_1^2](0,1;l-1)^2. \quad (17)$$

This of course leads to the formula

$$b_0=-Q_1. \quad (18)$$

Unfortunately, no such result was found for the function b_2 so that the summation over angular momenta had to be carried out explicitly for that function. For this reason, both b_0 and b_2 were computed by explicit summation over angular momenta. This summation proceeded until the b_μ were stable to one part in ten thousand. Equation (18) was useful in providing a check on the accuracy of the present work. Moreover, if the sum is carried through $l=L$, then the function $\Delta b_0=-Q_{L+1}$ determines the error in b_0 due to the use of a finite number of terms in the summation. These checks indicate that the numbers in Table I are everywhere accurate to at least three significant figures.

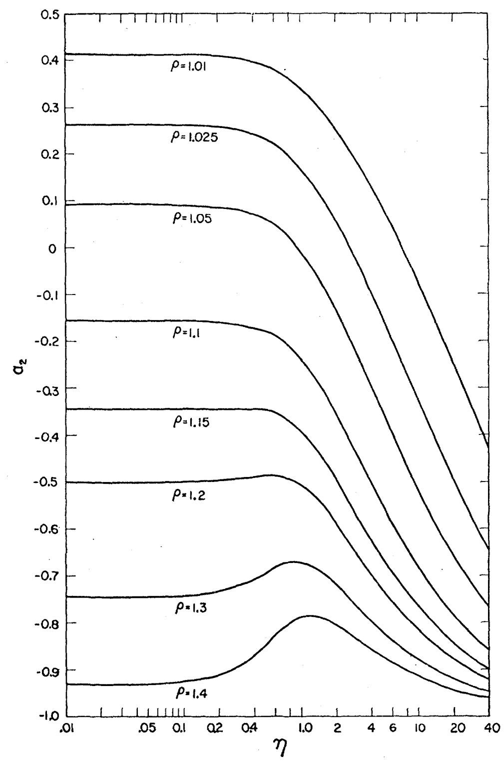

FIG. 2. The particle parameter a_2 *vs* η on a semilog plot for constant values of ρ.

[11] See, e.g., Erdelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1.

[12] L. C. Biedenharn, Phys. Rev. **102**, 262 (1956).

V. RESULTS

Some of the numerical results are presented in the accompanying tables. The coefficients $a_0 \equiv (b_0/k_2^2)$ and $a_2 \equiv (b_2/b_0)$ are given to four figures in Tables I and II, respectively. The arguments ρ, η in these tables have the ranges $1.01 \leq \rho \leq 1.8$, and $0.001 \leq \eta \leq 40$. It is believed that these numbers are accurate to within a few units in the last place.

The tables are supplemented by the accompanying graphs which display some of the qualitative features of these results. In Fig. 1 the dimensionless excitation function a_0 is given as a function of η for constant values of ρ. Because of the rapid variation of a_0 as a function of its arguments, $\log_{10} a_0$ is plotted against η on a semilog

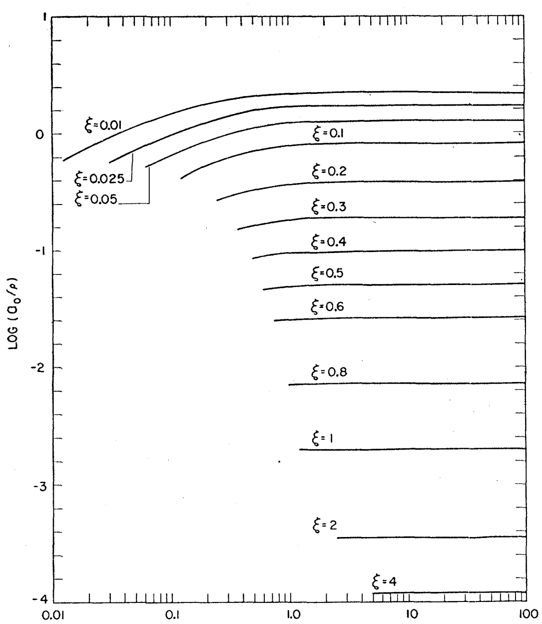

FIG. 3. The quantity $\log_{10}(a_0/\rho)$ *vs* η on a semilog plot for constant values of the classical variable ξ. In the curve for $\xi=2$, the values of (a_0/ρ) have been multiplied by 10^2. In the curves for $\xi=4$, the values of (a_0/ρ) have been multiplied by 10^7.

plot. In Fig. 2 the particle parameter a_2 is plotted against η for constant values of ρ on a semilog plot.

As η becomes large and ρ approaches unity with $\xi = \eta(\rho-1)$ finite, the function $(a_0/\rho) = (b_0/k_1k_2)$ and the particle parameter a_2 become functions of the single classical variable ξ. In the limit, $(a_0/\rho) = a_0$, of course, but it is easily seen that (a_0/ρ) approaches this limit far more rapidly than does a_0 itself. In Fig. 3, $\log_{10}(a_0/\rho)$ is plotted against η, for constant values of ξ, on a semilog plot. In Fig. 4 are shown curves of a_2 *vs* η for constant values of ξ. These plots show the manner in

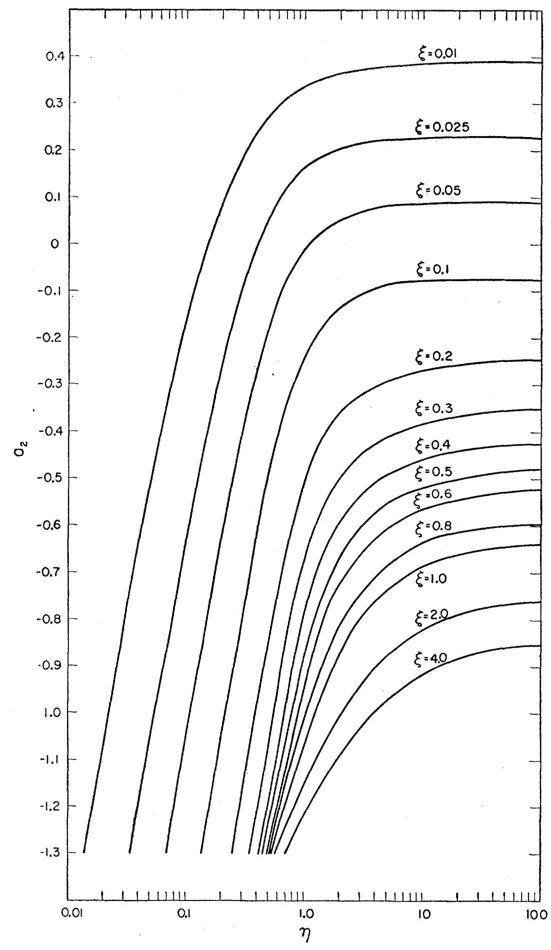

FIG. 4. The particle parameter a_2 *vs* η on a semilog plot for constant values of the classical variable ξ.

which these functions approach their classical limiting values as η increases. The smaller the value of ξ the more rapidly is the limit approached as a function of η, since the classical limit implies both $\rho \to 1$ and $\eta \to \infty$. The regions where a classical calculation determines the functions (a_0/ρ) and a_2 to sufficient accuracy are clear from these figures. The limiting values taken from this quantum-mechanical calculation agree very well with check values calculated classically.[13] In the limit $\rho=1$, $\eta=$finite, $(\xi=0)$, the function a_0 becomes infinite and $a_2=1$. In the limit $\eta=0$, $\rho\to\infty$, $\xi=$finite, the function $a_0 = 2\pi|\xi|/(e^{2\pi|\xi|}-1)$ and $a_2=-2$.

[13] While the manuscript for this paper was in preparation, the authors received a report (K. Alder and A. Winther, Cern/T/KA-AW 4, unpublished) containing calculations in WKB approximation of the b_μ for $\eta \geq 1$. These results are found to be in substantial agreement with the present calculation. The authors would like to express their gratitude for the receipt of this material in advance of publication.