Phase-Shift Analysis of Proton-Proton Scattering at 150 Mev*

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A phase-shift analysis of the 150-Mev proton-proton scattering and polarization data has been carried out assuming that only waves with $J\leqslant 2$ contribute and neglecting ${}^3P_2-{}^3F_2$ coupling. The formulas used to estimate the Coulomb-nuclear interference are essentially nonrelativistic, although they include a kinematic correction. Representative sets of phase shifts which fit the data reasonably are tabulated.

I. INTRODUCTION

DURING the past few years, there have been a number of attempts[1–5] to interpret proton-proton scattering for energies in the vicinity of 150 Mev[6] in terms of phase shifts. The essential assumption that has entered in all of these analyses[7] has been that only S and P waves contribute dominantly to the scattering. However, recent measurements of polarization effects[8] in p-p scattering at 133 Mev and at 210 Mev as obtained at Harwell[9] and Rochester,[10] respectively, clearly show that F waves must be included in any such phase-shift analysis. For the experimental angular distribution of the polarization is not simply proportional to $\sin\theta\cos\theta$ (which would be the case if only triplet P-states were included) but rather assumes the form $\sin\theta\cos\theta\ (b_0+b_2\cos^2\theta)$, the term in b_2 arising from ${}^3P-{}^3F$ interference.[11] One will have such interference even if only one of the ${}^3F$-phases, say, ${}^3F_2$, is nonzero, and it is convenient to assume that this will be the case for all that follows.

We have undertaken to carry out a phase-shift analysis of high-energy nucleon-nucleon scattering that is based on the charge independence of nuclear forces. In this work, all nuclear phase shifts with total angular momenta $J\leqslant 2$ have been retained; also, for the sake of simplicity, the coupling between states of given J and different $L(=J\pm 1)$ has been neglected. In this report we describe the results obtained for the analysis of 150-Mev proton-proton scattering; the extension to the neutron-proton system will be considered in a subsequent paper.

In Sec. II the experimental data used in the present analysis are summarized. It has been found convenient to express the unpolarized and polarized scattering data in terms of expansions in Legendre polynomials and trigonometric functions, respectively. The method of least squares was employed to determine the values of the expansion coefficients and their errors. The various scattering formulas that are needed in the analysis are given in Sec. III. So far as Coulomb effects are concerned, the expressions that we have used are the nonrelativistic ones except for kinematic corrections. In Secs. IV and V, we discuss the procedure employed in the analysis and the results, respectively.

II. EXPERIMENTAL DATA

(A) Unpolarized Scattering

Since the angular distribution for the scattering of unpolarized protons by protons is essentially isotropic (for $\theta\gtrsim 30°$) and energy-independent (from 100- to 350-Mev incident proton energy),[6] the following measurements are relevant in a phase-shift analysis at a nominal energy of 150 Mev: the Rochester data at 240 Mev,[12] the Harwell data at 147 Mev,[13] and the Berkeley data at 164–174 and 250–260 Mev.[14] The experimental points are reproduced in Figs. 1 and 2 for reference.[15]

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[1] R. M. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954).

[2] Thaler, Bengston, and Breit, Phys. Rev. **94**, 683 (1954).

[3] A. Garren, Phys. Rev. **92**, 213 (1953); **92**, 1587 (1953); **96**, 1709 (1954).

[4] C. A. Klein, Nuovo cimento **1**, 581 (1955).

[5] Beretta, Clementel, and Villi, Phys. Rev. **98**, 1526 (1955).

[6] In this paper, energies always refer to the laboratory system, scattering angles to the center-of-momentum system.

[7] Thaler, Bengston, and Breit[2] have made a preliminary survey of the effects on p-p scattering of ${}^3P_2-{}^3F_2$ coupling due to an intermediate state of the two nucleons and have found that the data indeed allow such an interpretation. However, these considerations were not extended to the case of the scattering of polarized protons.

[8] Whenever we speak of polarization effects, we have always in mind the quantity $P(\theta)$ which can be defined in either of two ways [see L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952)]: (a) Assume that the incoming particles move in the z-direction and that both the incident and target protons are unpolarized. Then $P(\theta)$ is twice the expectation value of the y-component of the spin of an outgoing proton when scattered through the polar angle θ, with the azimuth $\phi=0°$. (b) Assume the incident beam to be completely polarized in the positive y-direction. Then $P(\theta)$ is also given by the expression $[I(\phi=0°)-I(\phi=180°)]/[I(\phi=0°)+I(\phi=180°)]$, where $I(\phi)$ is the intensity of the outgoing beam (the scattering angle θ being kept fixed throughout).

[9] J. M. Dickson and D. C. Salter, Nature **173**, 946 (1954); Dickson, Rose, and Salter, Proc. Phys. Soc. (London) **A68**, 361 (1955).

[10] E. Baskir (private communication).

[11] Coulomb-nuclear interference by itself cannot account for the deviation of the polarization from the simple form $\sin\theta\cos\theta$ for the angular region 30°–45°.

[12] C. L. Oxley and R. D. Schamberger, Phys. Rev. **85**, 416 (1952); O. A. Towler, Jr., Phys. Rev. **85**, 1024 (1952).

[13] Cassels, Pickavance, and Stafford, Proc. Roy. Soc. (London) **A214**, 262 (1952).

[14] Chamberlain, Segrè, and Wiegand, Phys. Rev. **83**, 923 (1951); O. Chamberlain and J. D. Garrison, Phys. Rev. **95**, 1349 (1954).

[15] The absolute values of the Rochester and Harwell cross sections as given in Fig. 1 have been renormalized, with the 90° points being set equal to 3.92 and 4.05 mb/sterad, respectively. See the summary discussion of nucleon-nucleon scattering by G. Breit in the *Proceedings of the Fifth Annual Rochester Conference*

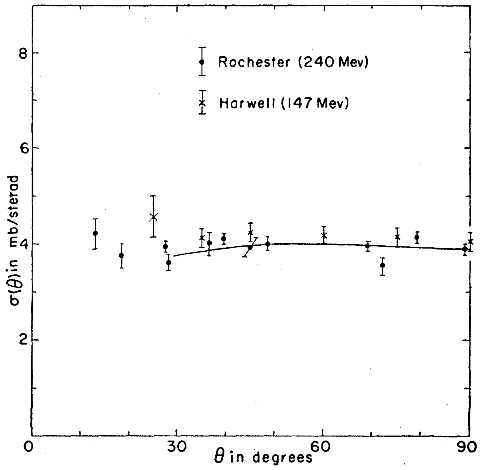

FIG. 1. Unpolarized differential cross section for p-p scattering as measured at Rochester (240 Mev) and at Harwell (147 Mev); the 90° points have been renormalized to give 3.92 and 4.05 mb/sterad, respectively. The solid curve represents the least-square fit of the combined data with σ (90°) being set equal to 3.90 mb/sterad.

Upon using the method of least squares to analyze the experimental angular distribution in terms of Legendre polynomials $P_{2n}(\cos\theta)$, *viz.*,

$$k^2\sigma(\theta)=\sum_{n=0}^{2} a_{2n}P_{2n}(\cos\theta), \qquad (1)$$

where $\hbar k$ is the momentum of either particle in the center-of-momentum system, we obtain the results shown in Table I[16]; we have here adjusted the least-square fit of the Rochester and Harwell data, for the sake of convenience, so as to give $\sigma(90°)=3.90$ mb/sterad—the original Berkeley measurements, on the other hand, yield a least-square value of $\sigma(90°)=3.63$ mb/sterad. Only the data at large angles ($\theta \geqslant 35°$) were taken into account in applying the expansion (1) since Coulomb effects may then be ignored; also, we have retained in (1) only those terms to which one may expect contributions from phase shifts having $J \leqslant 2$. The least-square fits of the Rochester-Harwell and the Berkeley differential cross sections as determined from Table I are plotted in Figs. 1 and 2, respectively.

TABLE I. Least-square analysis of 150 Mev unpolarized p-p scattering in terms of Legendre polynomials [$k^2\sigma(\theta)=\Sigma a_{2n}P_{2n}$, with $k^2=1.81\times10^{26}$ cm^{-2} for $E=150$ Mev].

Data	a_0	a_2	a_4
Rochester	0.700±0.017	−0.037±0.067	−0.044±0.068
Rochester and Harwell	0.700±0.013	−0.036±0.050	−0.045±0.051
Berkeley	0.682±0.014	0.087±0.058	0.040±0.058

It will be noticed that the least-square analysis of the Rochester data is but slightly affected when the Harwell measurements are also included. It is, on the other hand, not clear whether or not the apparent difference between the Rochester-Harwell and the Berkeley angular distributions is significant, particularly in view of the uncertainty of the normalization. We have, therefore, for the sake of definiteness, chosen to use the least-square fit of the combined Rochester and Harwell data in the phase-shift analysis.

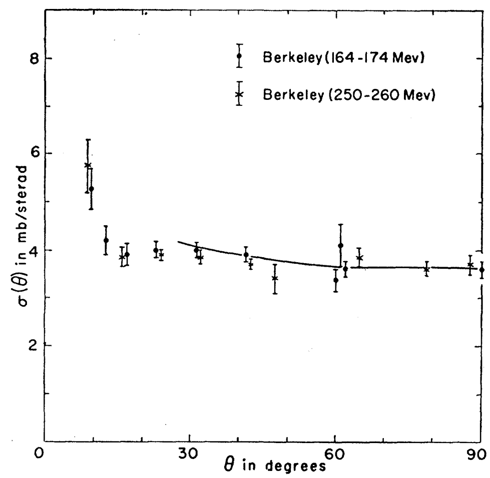

FIG. 2. Unpolarized differential cross section for p-p scattering as measured at Berkeley (164–174 and 250–260 Mev). The solid curve represents the least-square fit.

So far as the measurements at small angles ($\theta \leqslant 35°$) are concerned—here Coulomb effects will be important—, it is convenient to compare the experimental data directly with the theoretical values calculated from those phase shifts which fit the large-angle data reasonably.

(B) Polarized Scattering

At the time that these calculations were begun, the only available measurements of the angular distribution of the scattering of polarized protons by protons at the energies under consideration had been carried out at Harwell[9] (at 133 Mev); the data are plotted in Fig. 3.[17] Upon using the method of least squares to analyze these data, which we take to be representative

on High Energy Physics (Interscience Publishers, Inc., New York, 1955), p. 145; see also B. Rose, *Proceedings of the 1954 Glasgow Conference* (Pergamon Press, London, 1955), p. 32. All errors quoted in this paper are standard deviations.

[16] The errors which are listed in the table are those due to the deviation of the approximating function from the experimental points ("external" errors); the corresponding "internal" errors, which arise from the statistical uncertainties in the experimental observations themselves, are somewhat smaller. From a consideration of the variation of the ratio of these two types of errors as the number of terms in (1) is increased, it becomes evident that the least-square fit of the cross section will be but slightly affected with the inclusion of the $n=3$ term. Indeed, it is already clear from the errors listed in the table that the coefficients a_2 and a_4 contribute very little to the fit.

[17] There is also an additional point at 200 Mev measured at Rochester by Oxley, Cartwright, and Rouvina, Phys. Rev. **93**, 806 (1954).

of the polarization at 150 Mev, in terms of the trigonometric expansion

$$P(\theta)=[k^2\sigma(\theta)]^{-1}\sin\theta\cos\theta\sum_{n=0}^{1}b_{2n}\cos^{2n}\theta, \qquad (2)$$

we find $b_0=0.170\pm0.049$, $b_2=0.277\pm0.104$; we have here set $k^2=1.81\times10^{26}$ cm^{-2} and $\sigma(\theta)=3.90$ mb/sterad,[18] since the experimental data are, in any event, available for large angles only ($\theta>30°$).

The least-square angular distribution of $P(\theta)$ is shown in Fig. 3. Once again, we have kept only those terms in (2) which are compatible with our assumption that only phase shifts with $J\leqslant2$ enter; also Coulomb effects have been neglected. On the other hand, the errors which are quoted here are the internal errors, which are now somewhat larger than the external errors. This last fact is of some importance since it implies that, within the limitations of the experimental data, there is no point in improving upon the approximating series (2) by the inclusion of terms with $n\geqslant2$. The coefficients b_0 and b_2, quoted above, were used in the phase-shift analysis.

Recent measurements at Berkeley[19] of polarization effects in p-p scattering at 170 Mev have led to an angular distribution which is in essential agreement with Harwell. The experimental points and the least-square fit in terms of Eq. (2) are shown in Fig. 3. The expansion coefficients b_0 and b_2, for this case, are given by 0.210 ± 0.056 and 0.207 ± 0.095, respectively; once again, we have taken the differential cross section to be isotropic in the angular region under consideration (in particular, we have set $\sigma(\theta)=3.72$ mb/sterad[18]) and $k^2=1.81\times10^{26}$ cm^{-2} for the sake of comparison with the Harwell data.

It will be noticed that the variation of p-p polarization effects over the energy range 130–170 Mev is slight, particularly in view of the rather large experimental errors involved. One may therefore expect that phase shifts which are based on the polarization measurements at Harwell will explain equally well the corresponding Berkeley data.

There is one final remark which needs to be made with respect to the sign of the polarization. Let us denote by $\mathbf{k}_i$ the propagation vector of the incident proton in the center-of-momentum system; similarly, let $\mathbf{k}_f$ refer to an outgoing proton. We then say that the polarization of the scattered beam is positive or negative according to whether its direction is parallel or antiparallel to $\mathbf{k}_i\times\mathbf{k}_f$. In point of fact, it has been shown at Chicago[20] and at Harwell[21] for 435-Mev and 135-Mev protons, respectively, that the polarization effects are positive in the sense just defined for protons scattered through less than 90°. The data in Fig. 3 are in accord with this choice of sign.

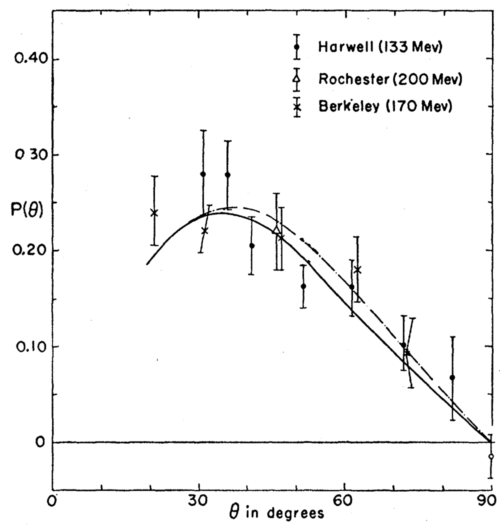

FIG. 3. Angular distribution for the polarization in p-p scattering as measured at Harwell (133 Mev), Rochester (200 Mev), and Berkeley (170 Mev). The solid curve represents the least-square fit of the Harwell data; the broken curve is the corresponding fit of the Berkeley data.

III. THEORETICAL FORMULATION

We introduce the following symbols:

$\mathbf{s}=\sin(\theta/2)$,

$\mathbf{c}=\cos(\theta/2)$,

$v_l=$ velocity of the incident proton in the laboratory system,

$\eta=e^2/\hbar v_l$,

$\delta_L=$ singlet phase shift with orbital angular momentum L,

$\delta_L{}^J=$ triplet phase shift with orbital angular momentum L and total angular momentum J,

$\sigma_L=\arg\Gamma(L+1+i\eta)=$ Coulomb phase shift with orbital angular momentum L,

$$[\delta_L\delta_{L'}]_c=\sin\delta_L\sin\delta_{L'}\sin[\delta_L-\delta_{L'}+2(\sigma_L-\sigma_{L'})],$$

$$(\delta_L\delta_{L'})_c=\sin\delta_L\sin\delta_{L'}\cos[\delta_L-\delta_{L'}+2(\sigma_L-\sigma_{L'})],$$

$$\alpha_0(\theta)=\eta\log(\mathbf{s}^2),$$

$$\alpha_L(\theta)=\eta\log(\mathbf{s}^2)+2\sum_{n=1}^{L}\tan^{-1}(\eta/n)=\alpha_0+2(\sigma_L-\sigma_0),$$

$$\beta_0(\theta)=\eta\log(\mathbf{c}^2),$$

$$\beta_L(\theta)=\eta\log(\mathbf{c}^2)+2\sum_{n=1}^{L}\tan^{-1}(\eta/n)=\beta_0+2(\sigma_L-\sigma_0),$$

$$X_L(\theta)=\mathbf{s}^{-2}\cos\alpha_L+(-1)^L\mathbf{c}^{-2}\cos\beta_L,$$

$$Y_L(\theta)=\mathbf{s}^{-2}\sin\alpha_L+(-1)^L\mathbf{c}^{-2}\sin\beta_L.$$

The analysis of proton-proton scattering in terms of partial waves has been given by many authors.[22] Upon

[18] This value is equal to the average of the least-square distribution of $\sigma(\theta)$ over the angular range $30°\leqslant\theta\leqslant90°$.

[19] D. Fischer and J. Baldwin, Phys. Rev. **100**, 1445 (1955).

[20] L. Marshall and J. Marshall, Phys. Rev. **98**, 1398 (1955).

[21] M. J. Brinkworth and B. Rose, *Proceedings of the Fifth Annual Rochester Conference on High Energy Physics* (Interscience Publishers, Inc., New York, 1955), p. 159.

[22] See, for example, J. Ashkin and T. Y. Wu, Phys. Rev. **73**, 973 (1948); also reference 23, p. 302.

assuming that there exists, in addition to the Coulomb repulsion, a nuclear interaction that does not couple states of different orbital angular momenta, and upon taking into account the identity of the particles, one finds, for the singlet and triplet scattering amplitudes, $S_0(\theta)$ and $S_{m'm}(\theta,\phi)$, respectively, the following expressions, valid in the center-of-momentum system:

$$S_0(\theta)=(1/ik)\sum_{\text{even }L}\{(2L+1)\exp(2i\sigma_L)\times[\exp(2i\delta_L)-1]P_L(\theta)\}+f_c(\theta)+f_c(\pi-\theta), \quad (3a)$$

$$\begin{aligned}S_{m'm}(\theta,\phi)=(1/ik)\sum_{\text{odd }L}\sum_{J=L-1}^{L+1}&\{[4\pi(2L+1)]^{\frac{1}{2}}\\ &\times\exp(2i\sigma_L)[\exp(2i\delta_L{}^J)-1]\\ &\times(LS,m-m',m'|LSJm)\\ &\times(LSJm|LS0m)Y_L{}^{m-m'}(\theta,\phi)\}\\ &+\delta_{mm'}[f_c(\theta)-f_c(\pi-\theta)], \quad (3b)\end{aligned}$$

where $f_c(\theta)$, the Coulomb amplitude, is given by[23]

$$f_c(\theta)=-(\eta/2k)\mathbf{s}^{-2}\exp(-i\eta\log(\mathbf{s}^2)+2i\sigma_0). \quad (4)$$

The spherical harmonics $Y_L{}^m$ and the Clebsch-Gordan coefficients $(LS,\ m-m',\ m'|LSJm)$ are as defined in Condon and Shortley.[24] The differential cross section $\sigma(\theta)$ for the scattering of unpolarized protons by protons can then be expressed in the form

$$\sigma(\theta)=\tfrac{1}{4}|S_0(\theta)|^2+\tfrac{1}{4}\sum_{m,m'}|S_{m'm}(\theta,\phi)|^2. \quad (5)$$

There are two remarks which need to be made with respect to the foregoing formulas. First, it is to be noticed that the phase shifts δ_L and $\delta_L{}^J$ are here *defined* as the difference between the *total* phase shifts that arise in the presence of the combined Coulomb-nuclear interaction and the *pure Coulomb* phase shifts σ_L. It is sometimes convenient to refer to δ_L and $\delta_L{}^J$ as *quasinuclear* phase shifts so as to distinguish them from the corresponding quantities which appear when one has the nuclear interaction only. This distinction will be of some importance when we turn to a consideration of the neutron-proton system in a sequel to this paper.

Second, apart from Coulomb effects, the scattering amplitudes (3a) and (3b) are completely relativistic. For the Coulomb amplitude $f_c(\theta)$, on the other hand, we have taken the nonrelativistic form as given by Eq. (4) with the understanding that η and k are always to be assigned their relativistic values. As we shall see later, this kinematic correction is adequate at the energies and angles under consideration.[25]

[23] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, p. 101.

[24] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), pp. 52, 76.

[25] For a discussion of the relativistic treatment of the Coulomb scattering of protons by protons, see G. Breit, Phys. Rev. **99**, 1581 (1955).

It is convenient, in the analysis, to separate the unpolarized cross section (5) into three parts, *viz.*, a nuclear term $\sigma_n(\theta)$, a Coulomb-nuclear interference term $\sigma_{\text{int}}(\theta)$, and a pure Coulomb term $\sigma_c(\theta)$, where these involve the Coulomb amplitude $f_c(\theta)$ not at all, once, and twice, respectively; then

$$\sigma(\theta)=\sigma_n(\theta)+\sigma_{\text{int}}(\theta)+\sigma_c(\theta). \quad (6)$$

For large angles ($\theta\gtrsim30°$), σ_{int} and σ_c are negligible compared to σ_n. Under these circumstances, the *p-p* differential scattering cross section assumes the form given by Eq. (1) where we have set all quasinuclear phase shifts with $J>2$ equal to zero. The expansion coefficients a_0, a_2, and a_4 will now be given in terms of the remaining quasinuclear phase shifts and associated Coulomb phases; we have, in particular,

$$a_0=\sin^2\delta_0+5\sin^2\delta_2+\textstyle\sum_J(2J+1)\sin^2\delta_1{}^J+5\sin^2\delta_3{}^2, \quad (7a)$$

$$\begin{aligned}a_2=&(50/7)\sin^2\delta_2+10(\delta_0\delta_2)_c+(3/2)\sin^2\delta_1{}^1\\&+(7/2)\sin^2\delta_1{}^2+4(\delta_1{}^0\delta_1{}^2)_c+9(\delta_1{}^1\delta_1{}^2)_c\\&+(32/7)\sin^2\delta_3{}^2+6(\delta_1{}^0\delta_3{}^2)_c+6(\delta_1{}^1\delta_3{}^2)_c\\&+(6/7)(\delta_1{}^2\delta_3{}^2)_c, \quad (7b)\end{aligned}$$

$$a_4=(90/7)\sin^2\delta_2+(10/7)\sin^2\delta_3{}^2+(120/7)(\delta_1{}^2\delta_3{}^2)_c. \quad (7c)$$

In a similar way, for the Coulomb-nuclear interference term, we obtain

$$\sigma_{\text{int}}(\theta)=(\eta/2k^2)\sum_{n=0}^{3}c_n(\theta)P_n(\theta), \quad (8)$$

where

$$c_0(\theta)=-\tfrac{1}{2}X_0\sin(2\delta_0)+Y_0\sin^2\delta_0, \quad (9a)$$

$$c_1(\theta)=-\tfrac{1}{2}X_1\textstyle\sum_J(2J+1)\sin(2\delta_1{}^J)+Y_1\sum_J(2J+1)\sin^2\delta_1{}^J, \quad (9b)$$

$$c_2(\theta)=-(5/2)X_2\sin(2\delta_2)+5Y_2\sin^2\delta_2, \quad (9c)$$

$$c_3(\theta)=-(5/2)X_3\sin(2\delta_3{}^2)+5Y_3\sin^2\delta_3{}^2. \quad (9d)$$

Finally, for the pure Coulomb scattering cross section, we have the usual Mott scattering formula[26]

$$\sigma_c(\theta)=(\eta/2k)^2[\mathbf{s}^{-4}+\mathbf{c}^{-4}-\mathbf{s}^{-2}\mathbf{c}^{-2}\cos(\eta\log\mathbf{s}^2\mathbf{c}^{-2})]. \quad (10)$$

The polarization $P(\theta)$ can also be analyzed in terms of phase shifts by combining the formula[27]

$$P(\theta)=\sqrt{2}\frac{\operatorname{Im}\sum_m S_{m0}{}^*(S_{m1}-S_{m,-1})}{|S_0|^2+\sum_{mm'}|S_{m'm}|^2}, \quad (11)$$

with the expressions (3a) and (3b) given previously for the scattering amplitudes. The resultant $P(\theta)$ has the

[26] See reference 23, p. 104. In the nonrelativistic domain, the factor $(\eta/2k)^2$, which appears in Eq. (10), is usually expressed equivalently as $(e^2/Mv_l{}^2)^2$, where M is the proton mass. When one employs relativistic kinematics, however, the latter form gives a result, at 150 Mev, which is 1.25 times larger than the value predicted by Eq. (10). On the other hand, it is Eq. (10) which agrees with the relativistic Møller formula at small angles.

[27] L. Wolfenstein, Phys. Rev. **76**, 973 (1949).

form given by Eq. (2), where

$$b_0=6[\delta_1^0\delta_1^2]_c+9[\delta_1^1\delta_1^2]_c+6[\delta_3^2\delta_1^0]_c+9[\delta_3^2\delta_1^1]_c -30[\delta_3^2\delta_1^2]_c, \quad (12a)$$

$$b_2=75[\delta_3^2\delta_1^2]_c; \quad (12b)$$

we have neglected the Coulomb amplitude $f_c(\theta)$ entirely since we shall be interested in polarization effects at large angles only.

IV. METHOD OF CALCULATION

Our initial problem is to determine those singlet phase shifts δ_0, δ_2 and triplet phase shifts δ_1^0, δ_1^1, δ_1^2, δ_3^2 which are compatible with the expansion coefficients a_0, a_2, a_4, and b_0, b_2 as determined from the unpolarized and polarized scattering experiments at large angles, respectively ($\theta \gtrsim 30°$). The relations between the expansion coefficients and phase shifts are given by Eqs. (7) and (12); the values assigned to the coefficients are the least-square determinations discussed in Sec. II. Having obtained sets of phase shifts which are in accord with the large-angle scattering, we shall subsequently require that they also predict the small-angle scattering, this time taking Coulomb-nuclear interference into account [Eqs. (8) and (9)]. As we shall see shortly, the latter provides an incisive condition on the phase shifts.

The starting point of the detailed analysis is formula (12b) with b_2 set equal to 0.277. For a given choice of $\delta_1^2>0$, we have only one $\delta_3^2<0$ which satisfies this condition; indeed, this is the case of interest since we cannot have the same sign for both δ_1^2 and δ_3^2 (the resultant a_2 and a_4 would then become too large) and the assignment $\delta_1^2<0$, $\delta_3^2>0$ would violate (12b).[28] Next, for each set of phases δ_1^2, δ_3^2, Eq. (7c) together with $a_4=-0.045$ will fix δ_2 except for its sign; at the same time, Eq. (12a) with $b_0=0.170$ will yield a doubly-valued relation between δ_1^0 and δ_1^1.

The five phase shifts that have been taken into account to this point constitute a two-parameter family with four branches. We can consider, for definiteness, that δ_1^2 and δ_1^0 may be assigned arbitrarily (subject to the conditions that $\delta_1^2>0$ and that, for given δ_1^2, the range of δ_1^0 will be restricted); the remaining three phase shifts δ_2, δ_1^1, δ_3^2 are then determined except for the sign of δ_2 and the fact that there are two possibilities for δ_1^1.

Next, for a given set (δ_2, δ_1^0, δ_1^1, δ_1^2, δ_3^2), one can determine δ_0 (except for its sign) from Eq. (7a) with $a_0=0.700$; the positive-definite character of each term of (7a) also limits the magnitudes of all of the phases. Finally, we use Eq. (7b) with $a_2=-0.036$ to obtain an additional condition on all the six phase shifts so that we have left a one-parameter family of phases (with δ_1^2 as the running index) which can account for the large-angle unpolarized and polarized p-p scattering.

[28] It is assumed, in this investigation, that the magnitudes of the phase shifts do not exceed $\pi/2$.

TABLE II. Representative sets of phase shifts (in degrees) which can account for the 150-Mev p-p scattering and polarization data.

	^1S	^1D	$^3P_0$	$^3P_1$	$^3P_2$	$^3F_2$
A	4.5	−6.0	−4.6	−7.6	20.3	−1.7
B	9.5	−6.8	9.3	−12.5	17.3	−2.2
C	13.0	−6.9	16.3	−12.0	15.3	−2.7
D	19.0	−8.1	27.0	−4.5	12.5	−4.5
E	23.7	−8.5	−22.0	9.5	10.0	−5.3
F	26.5	−8.0	−22.0	10.0	8.0	−6.5
G	−29.5	6.3	−23.0	9.5	6.5	−8.0
H	18.5	−8.1	5.0	19.0	6.0	−9.5
I	20.5	−7.3	−2.5	18.5	5.0	−10.5
J	22.5	−6.5	−4.5	17.0	4.0	−12.0
K	−23.5	5.8	−4.5	17.0	4.0	−12.0

For a given δ_1^2, provided it lies within the interval $1.7° \leqslant \delta_1^2 \leqslant 20°$, there exist two or four sets of solutions (some of which may coincide with one another, however).

It will be noticed that, although the relative sign of the singlet phase shifts, δ_0 and δ_2, is established by Eq. (7b), there still remains an uncertainty about the absolute sign of these phases (this will be fixed later when we consider the Coulomb-nuclear interference). Of course, there is no ambiguity about the sign of the triplet phases since the sign of $P(\theta)$ is known.

The calculation was actually performed for twelve values of δ_1^2 lying within the range 2° to 20° with the magnitude of any given interval never exceeding 2.5°. The procedure outlined in the preceding paragraphs then yields continuous curves for the five phase shifts (δ_0, δ_2, δ_1^0, δ_1^1, δ_3^2) as a function of δ_1^2 (these curves are closed for δ_0, δ_1^0, δ_1^1 and open for δ_2, δ_3^2).

As a final test of the validity of any particular set of phase shifts, we require agreement with the experimental data of the calculated unpolarized differential scattering cross section for small angles, taking Coulomb-nuclear interference into account [Eq. (8)]. The actual condition which we have imposed is

$$3.5 \text{ mb/sterad} \leqslant \sigma(15°) \leqslant 4.5 \text{ mb/sterad}. \quad (13)$$

It will be noticed that, whereas in the consideration of the large-angle scattering we have treated the expansion coefficients a_{2n} and b_{2n} as precisely determined quantities, we have, on the other hand, in Eq. (13), allowed for an uncertainty in $\sigma(15°)$. Clearly, were we to regard $\sigma(15°)$ as fixed exactly, we would thereby rule out most of the possible sets of phase shifts which are in accord with the large-angle scattering, including some that might be physically important. By taking into account condition (13), which is only somewhat weaker than the experimental uncertainty in $\sigma(15°)$ [see Figs. 1 and 2], we hope to include, at least in part, also the effects of the uncertainty of the normalization of the proton-proton scattering cross section as well as the uncertainty in the angular distribution for $\theta \lesssim 35°$.

In testing the sets of phase shifts, predetermined by

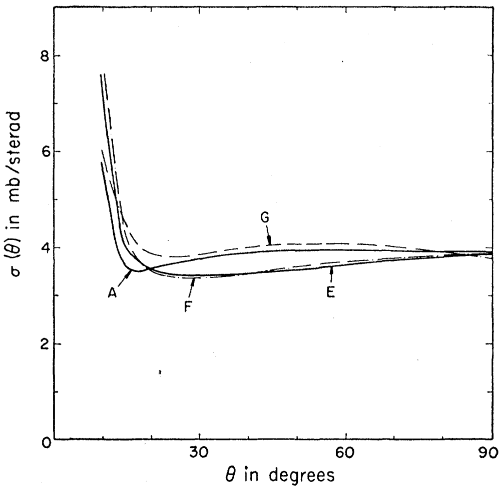

FIG. 4. Unpolarized differential cross section for 150-Mev p-p scattering as given by several sets of phase shifts of Table II.

the large-angle scattering data, for compatibility with the small-angle data, we encountered a number of cases in which (13) was barely violated. Under these circumstances, an attempt was made to adjust the phases so as to satisfy (13); it was then necessary to relax somewhat the agreement with the large-angle scattering. Such sets of phase shifts were considered to be of possible physical interest and were retained.

The results of the analysis are presented in Table II with the solutions being listed in the order of decreasing δ_1^2. It should be emphasized that, because of uncertainties in the experimental data, sets of phase shifts that are in the vicinity of those given in Table II are also in accord with the experimental data. Thus it will be noticed, for example, that sets A, B, C, D constitute, in actuality, a continuously varying set of solutions.

In Figs. 4 and 5, we have plotted $\sigma(\theta)$ and $P(\theta)$ for several of the sets of phase shifts of Table II; the curves corresponding to the remaining sets of phases lie between the extreme cases illustrated in the graphs.

Finally, it should be pointed out that, in calculating $\sigma(\theta)$ at this point, the complete formula (6) was used. On the other hand, in computing $P(\theta)$ [Eqs. (2) and (12)], Coulomb interference was neglected except in the factor $\sigma(\theta)$; this is somewhat inconsistent but has no important practical bearing in view of the large experimental uncertainties of $P(\theta)$ (we shall return to this point in the following section).

V. DISCUSSION

It is evident, from the analysis that has just been presented, that there are, in effect, an infinite number of sets of phase shifts that can account for the scattering of 150 Mev unpolarized and polarized protons by protons; however, these sets all cluster about the representative solutions that have been listed in Table II. In principle, we have six pieces of information [the coefficients a_0, a_2, a_4, b_0, b_2 and $\sigma(15°)$] that can be used to determine the six unknown phase shifts uniquely (except for the possibility of degeneracy). However, the question of uniqueness cannot be settled, at this stage of the analysis, in view of the uncertainty of the experimental data, and we have chosen to retain all sets of phase shifts that can conceivably be of physical interest. We shall see, when we come to the analysis of neutron-proton scattering, that the qualitative behavior of the neutron-proton polarization will provide a very severe test for those phase shifts that are common to the p-p and n-p systems and so will help to reduce the number of possibilities considerably.

There have been several important approximations that have been made in the theoretical analysis which we now consider in greater detail. In the first place, we have retained only those phase shifts with $J \leqslant 2$. The primary motivation, here, has been to include at least one ^3F-phase[29]; this is necessary if one is to be able to account for the angular distribution of $P(\theta)$ and is the essential difference between this analysis and others[1,3–5] which have been carried out at this energy.[30] The exclusion of all phase shifts with $J \geqslant 3$ can be justified only heuristically; the accuracy with which the experimental data are known does not require such phase shifts and the analysis is thereby made much simpler.

Secondly, we have assumed that there is no coupling

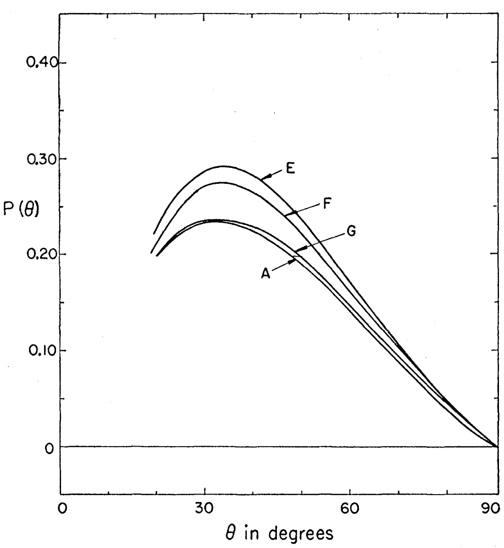

FIG. 5. Angular distribution for the polarization in 150-Mev p-p scattering as given by several sets of phase shifts of Table II.

[29] The choice of δ_3^2 as the phase to be retained simplifies the analysis considerably since there is then no term with $n=3$ in Eq. (1). The inclusion of δ_3^3 would introduce this complication [without leading to an extra term in the expansion of $P(\theta)$] and this is not warranted experimentally. Finally, a nonzero δ_3^4 will complicate both $\sigma(\theta)$ and $P(\theta)$ and, again, the experimental data, in either case, do not require such extra terms.

[30] An analysis of the 310-Mev p-p scattering data has recently been carried out by T. J. Ypsilantis and H. P. Stapp, University of California Radiation Laboratory Reports UCRL-3047 and 3098 (unpublished), which includes all waves with $L \leqslant 3$.

between the $^3P_2$ and $^3F_2$ states. The introduction of such a coupling would not, in any way, alter the number of coefficients in the expansions for $\sigma(\theta)$ and $P(\theta)$ but would lead to one extra parameter. While this additional degree of freedom would undoubtedly lead to a greater variety of solutions, we have chosen to try to fit the data in terms of the smallest number of parameters.

Thirdly, in our treatment of the effect of Coulomb-nuclear interference on the unpolarized differential scattering cross section $\sigma(\theta)$, we used essentially nonrelativistic formulas except for a kinematic correction.[26] In point of fact, an estimate of relativistic effects in the neighborhood of 15°, in which angular region these effects may be expected to be most important, leads to a fractional change in the cross section which is not in excess of 2%.[31]

Fourthly, we have ignored completely the effects of Coulomb interference in the calculation of $P(\theta)\sigma(\theta)$ [see Eqs. (2) and (12)] since we have tried to fit the corresponding data at large angles only ($\theta \gtrsim 30°$). The resultant error in $P(30°)$ may be as large as 10%; there will also be an additional correction of the order of 3% due to relativistic effects. While the neglect of both of these corrections is not so important at 30°, this would certainly not be the case at smaller angles (Coulomb interference may change $P(\theta)$ at 15° by 25%, and relativistic effects are equally large).[31]

So far as the phase shifts themselves are concerned, we have already emphasized the importance of δ_3^2 for the angular distribution of the polarization; as noted earlier, the signs of δ_1^2 and δ_3^2 are fixed by the isotropy of $\sigma(\theta)$ at large angles and the sign and peaking of $P(\theta)$ at smaller angles. The ^1D phase also plays an important role in the analysis. In the first place, the interference between the ^1S and ^1D states can give a large contribution to a_2 [Eq. (7b)]; in point of fact, it turns out that this contribution must always be negative so that δ_0 and δ_2 cannot have the same sign. Secondly, the ^1D phase affects the scattering cross section considerably in the Coulomb-nuclear interference region; this last property is characteristic of the higher angular momentum phase shifts, even when small.

It is of interest that all of the sets of phases but one violate the condition $\delta_1^2 > \delta_1^1 > \delta_1^0$, which corresponds to the level inversion of the nuclear shell model; the set E barely satisfies this inequality. The over-all sign of the singlet phases is determined by Coulomb-nuclear interference [Eq. (13)]; it will be noticed that there are two cases, G and K, which we can conceivably interpret in terms of a hard core in the singlet states—reversing the signs of the singlet phases in the remaining cases would lead to too large a destructive Coulomb-nuclear interference. The net Coulomb-nuclear interference for all of the sets of solutions is always negative or destructive.

The possible sets of phase shifts which are in agreement with the 150-Mev proton-proton scattering data and which are tabulated in Table II admit of a great variety and it is desirable to reduce the number in some way before attempting to draw any further detailed conclusions from them. This reduction in number will be accomplished when we come to the analysis of the high-energy neutron-proton scattering data and invoke charge independence.

[31] A. Garren, Phys. Rev. **101**, 419 (1956).