Energy and Angle Distribution of Electrons in Bremsstrahlung*

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The cross section $\sigma(E,\alpha)$ for an electron of incident energy E_0 to radiate in the presence of a nucleus with atomic number Z and to come out with energy E, at the angle α has been recalculated, checking the result of Racah. $\sigma(E,\alpha)$ has been tabulated for various outgoing energies and angles of the electron corresponding to incident momenta of $2mc$, $4mc$, $6mc$, $10mc$ and $20mc$. For the case of large α, and large incident and final energies, $\sigma(E,\alpha)$ may be represented by a simple expression and the coefficients in this expression have been tabulated. This paper also collects the various formulas which are involved in this particular aspect of bremsstrahlung and which do not seem to be conveniently available in the literature.

I. INTRODUCTION

The cross section for an electron of energy E_0 to radiate in the presence of a nucleus of atomic number Z and to come out with energy E at the angle α has been calculated by Racah,[1] who integrated the Bethe-Heitler[2] formula over the directions of the radiated photon. We have repeated the calculations of Racah and have checked his result. We have tabulated this cross section for various outgoing energies and angles of the electron corresponding to incident momenta of $2mc$, $4mc$, $6mc$, $10mc$, and $20mc$. In the case of very large incident energies ($E_0 \gg mc^2$), there exist simpler expressions for large α and for small α. For the case of large α ($\alpha \gg mc^2/E$ for an unscreened nucleus) and for large incident energies, we have tabulated the coefficients in the expression for the cross section. As we have not included the effect of screening, the results have not been computed for the very small angles and the formulas given are not applicable at the small angles.

The results for this particular aspect of bremsstrahlung, the distribution in energy and angle of the electron, do not seem to be conveniently available in the literature and we hope that this paper will collect them and present them in a usable form.

Table I. Values of R for electrons of incident energy corresponding to momentum $2mc$ and for various scattering angles and energies of the electron. R is defined in Eq. (2). α gives the direction of the scattered electron. $\gamma = p/p_0$, where p is the outgoing momentum and p_0 is the incident momentum.

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.0	3.780	3.780	3.780	3.780	3.780
0.2	8.835	5.410	3.190	2.052	1.532
0.4	16.26	5.490	2.230	1.108	0.6912
0.6	21.10	4.708	1.581	0.6764	0.3694
0.8	21.61	3.979	1.213	0.4726	0.2325
1.0	20.39	3.479	1.005	0.3692	0.1680

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[1] G. Racah, Nuovo cimento **11**, 476 (1934).

[2] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1944), second edition.

II. RESULTS FOR INTERMEDIATE ENERGIES

Let $\sigma(E,\alpha)dEd\Omega$ be the cross section for the electron with incident energy E_0 to undergo bremsstrahlung and to come out with energy E, at the angle α with the incident direction and in the solid angle $d\Omega$. The expression for $\sigma(E,\alpha)$ was obtained by Racah by integrating the Bethe-Heitler formula over the directions of the radiated photon. We repeated his calculations and

Table II. Values of R for electrons of incident energy corresponding to momentum $4mc$ and for various scattering angles and energies of the electrons.

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.0	4.634	4.634	4.634	4.634	4.634
0.2	16.75	6.216	2.507	1.227	0.7621
0.4	21.09	3.780	1.072	0.3961	0.1874
0.6	17.21	2.432	0.6103	0.1953	0.07334
0.8	14.18	1.831	0.4346	0.1285	0.04063
1.0	12.64	1.548	0.3560	0.1012	0.02905

Table III. Values of R for electrons of incident energy corresponding to momentum $6mc$ and for various scattering angles and energies of the electron.

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.0	6.410	6.410	6.410	6.410	6.410
0.2	27.11	6.569	2.090	0.8749	0.4859
0.4	21.10	2.742	0.6653	0.2160	0.08740
0.6	13.62	1.541	0.3456	0.09902	0.03045
0.8	10.26	1.102	0.2386	0.06422	0.01634
1.0	8.829	0.9133	0.1938	0.05077	0.01184

Table IV. Values of R for electrons of incident energy corresponding to momentum $10mc$ and for various scattering angles and energies of the electron.

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.0	10.30	10.30	10.30	10.30	10.30
0.2	38.47	5.448	1.368	0.4884	0.2371
0.4	16.15	1.569	0.3357	0.09761	0.03348
0.6	8.577	0.7915	0.1624	0.04284	0.01100
0.8	5.988	0.5424	0.1093	0.02754	0.005895
1.0	4.974	0.4409	0.08789	0.02177	0.004358

checked his results. As there were several misprints in Racah's result and for the sake of completeness, we shall give the result here. $\sigma(E,\alpha)$ is given by ($\hbar=c=1$ throughout)

$$\sigma(E,\alpha)=\frac{1}{2\pi}\frac{Z^2}{137}\left(\frac{e^2}{m}\right)^2\frac{p}{p_0}\frac{1}{k}R, \tag{1}$$

$$\begin{aligned}\frac{1}{2m^2}R=&\frac{2m^2E_0E+(E_0^2+E^2-m^2)\lambda-\lambda^2}{\lambda^2[\lambda(\lambda+2m^2)]^{\frac{1}{2}}}\log\frac{\lambda+m^2+[\lambda(\lambda+2m^2)]^{\frac{1}{2}}}{m^2}-\frac{2E_0E-\lambda}{\lambda^2}-\frac{3k^2(E_0^2E^2-m^4)^2}{\lambda^2p_0^4p^4}\\&+k^2\frac{4(E_0^2E^2-E_0Em^2+m^4)-E_0E(p_0^2+p^2)+\lambda(E_0^2+E^2+E_0E-m^2)}{2\lambda^2p_0^2p^2}+m^2k^2\left(2\frac{E_0E-m^2}{\lambda^3}-\frac{k^2m^2}{\lambda^4}\right)\\&\times\frac{2(E_0^2+E^2-E_0E)p_0^2p^2+3k^2m^2(E_0+E)^2}{p_0^4p^4}+k^2\Bigg\{\frac{m^4E+2m^2E^3}{p^2\lambda^2}+\frac{2E^4+2p_0^2p^2+m^2E_0(E_0+E)-\lambda(E_0E+p^2)}{2k\lambda^2}\\&+m^2E\left(2\frac{E_0E-m^2}{\lambda^3}-\frac{k^2m^2}{\lambda^4}\right)\frac{2E_0p^2-3kE^2}{kp^2}\Bigg\}\frac{1}{p^3}\log\frac{E+p}{m}+k^2\Bigg\{\frac{m^4E_0+2m^2E_0^3}{p_0^2\lambda^2}\\&-\frac{2E_0^4+2p_0^2p^2+m^2E(E+E_0)-\lambda(E_0E+p_0^2)}{2k\lambda^2}-m^2E_0\left(2\frac{E_0E-m^2}{\lambda^3}-\frac{k^2m^2}{\lambda^4}\right)\frac{2Ep_0^2+3kE_0^2}{kp_0^2}\Bigg\}\frac{1}{p_0^3}\log\frac{E_0+p_0}{m},\end{aligned} \tag{2}$$

$\lambda=E_0E-p_0p\cos\alpha-m^2$, p_0 and p are the incident and outgoing momenta of the electron, and k is the energy of the photon emitted.

For intermediate incident energies $E_0\simeq mc^2$, there are no approximate results available and the general result, Eqs. (1) and (2), must be used. We have tabulated the dimensionless quantity R for various outgoing energies and angles of the electron corresponding to incident momenta $p_0=2mc$, $4mc$, $6mc$, $10mc$, and $20mc$. Screening is not included in Eqs. (1) and (2) and they are therefore not valid at small angles. However, for the energies and angles we have tabulated, we expect screening to introduce little error. The results are given in Tables I–V.

To illustrate the dependence of $\sigma(E,\alpha)$ on E and α, we have plotted in Fig. 1 $\sigma(E,\alpha)$ against E, holding α fixed, corresponding to an incident energy of $4mc^2$. The peak which appears has been discussed in another paper.[3]

III. HIGH-ENERGY LARGE-ANGLE RESULTS

For the case of large angles ($\alpha\gg mc^2/E_0$ for an unscreened nucleus), and for large incident and final energies (E_0, $E\gg mc^2$), the general result of Eqs. (1) and (2) can be considerably simplified.[4] In this limit, $\sigma(E,\alpha)$ is given by

$$\sigma(E,\alpha)=\frac{1}{2\pi}\frac{Z^2}{137}\left(\frac{e^2}{m}\right)^2\frac{p}{p_0}\frac{1}{k}R', \tag{3}$$

where

$$R'=\left(\frac{m}{p_0}\right)^2\left\{A(\gamma,\alpha)\log\frac{2E_0}{m}+B(\gamma,\alpha)\right\}, \tag{4}$$

where $\gamma=p/p_0$. The functions $A(\gamma,\alpha)$ and $B(\gamma,\alpha)$

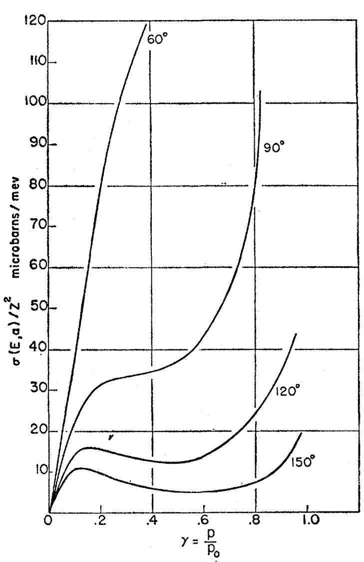

FIG. 1. $\sigma(E,\alpha)/Z^2$ as a function of $\gamma=p/p_0$, α being held fixed at various values. p and p_0 are the outgoing and incident moment of the electron. $p_0=4\ mc$ and σ/Z^2 is measured in microbarns/Mev.

TABLE V. Values of R for electrons of incident energy corresponding to momentum $20mc$ and for various scattering angles and energies of the electron.

γ \ α	30°	60°	90°	120°	150°
0.0	20.27	20.27	20.27	20.27	20.27
0.2	31.53	2.764	0.5790	0.1791	0.07392
0.4	7.751	0.6066	0.1187	0.03172	0.009299
0.7	3.606	0.2836	0.05460	0.01361	0.003062
0.8	2.386	0.1881	0.03594	0.008685	0.001677
1.0	1.925	0.1504	0.02858	0.006836	0.001259

[3] D. G. Keiffer and G. Parzen, Phys. Rev. **101**, 1244 (1956).

[4] This result was obtained by H. Brysk [National Bureau of Standards Report 2277 (unpublished)] using Racah's general result. This report also contains the other formulas in this paper. However, the general result contains the misprints in Racah's paper. His high-energy results were not affected by this.

depend on the energy E_0 only through $\gamma=p/p_0$, so that they can be easily tabulated for all E_0 and E. A and B are given by,

$$A=\frac{1}{2}\frac{(1+\gamma^2)^2}{\gamma^3}\frac{\cos^2\frac{1}{2}\alpha}{\sin^4\frac{1}{2}\alpha}, \tag{5}$$

$$B=\frac{1}{2}\frac{1+\gamma^2}{\gamma^3}\left\{\frac{\cos^2\frac{1}{2}\alpha}{\sin^4\frac{1}{2}\alpha}\log\gamma+\frac{\gamma}{\sin^4\frac{1}{2}\alpha}\log(\sin^2\tfrac{1}{2}\alpha)\right.$$
$$-\frac{1+\gamma^2}{2\sin^4\frac{1}{2}\alpha}+\frac{1}{\sin^2\frac{1}{2}\alpha}(1-\gamma+\gamma^2)$$
$$\left.-\frac{2\gamma^2}{1+\gamma^2}\frac{1}{\sin^2\frac{1}{2}\alpha}\log(\sin^2\tfrac{1}{2}\alpha)\right\}. \tag{6}$$

In Eqs. (3) to (6), it has been assumed that both E_0 and E are large compared to mc^2.

In Tables VI and VII we have tabulated $A(\gamma,\alpha)$ and $B(\gamma,\alpha)$.

TABLE VI. Values of $A(\gamma,\alpha)$ defined in Eq. (5).

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.2	14056	811.2	135.2	30.04	5.202
0.4	2186	126.2	21.02	4.672	0.8090
0.6	890.2	51.38	8.563	1.903	0.3295
0.8	546.1	31.52	5.253	1.167	0.2021
1.0	415.8	24.00	4.000	0.8889	0.1539

TABLE VII. Values of $B(\gamma,\alpha)$ defined in Eq. (6).

$\gamma \backslash \alpha$	30°	60°	90°	120°	150°
0.2	−36098	−1838	−264.3	−38.51	10.98
0.4	−4878	−222.7	−30.44	−4.747	0.6020
0.6	−1846	−76.64	−9.921	−1.657	−0.0452
0.8	−1068	−41.41	−5.097	−0.8756	−0.1034
1.0	−770.0	−28.64	−3.386	−0.5723	−0.0823

In order to get some idea of how the exact result, Eqs. (1) and (2), and the approximate result, Eqs. (3) and (4) compare, we have plotted $(p_0/m)^2R$ and $(p_0/m)^2R'$ in Fig. 2. If we plot $(p_0/m)^2R'$ against $\log(p_0/m)$ for a fixed α and a fixed $\gamma=p/p_0$, we should get a straight line. In Fig. 2 the straight lines representing $(p_0/m)^2R'$ are dotted; the solid lines are the exact R calculated from Eqs. (1) and (2). α was fixed at 90° and we have drawn curves for various $\gamma=p/p_0$.

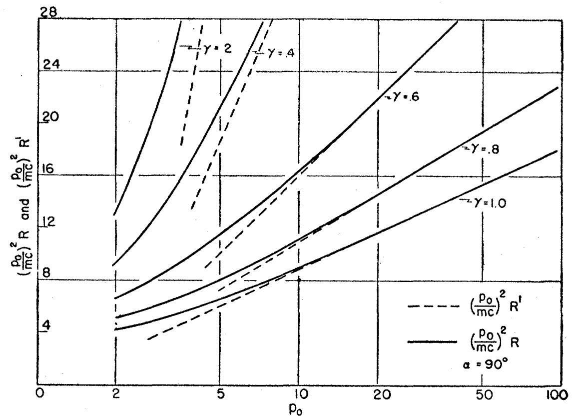

FIG. 2. $(p_0/m)^2R$ and $(p_0/m)^2R'$ plotted against $\log p_0$, γ and α being held fixed at various values. The dotted straight lines correspond to R', the approximate high-energy large-angle result. α is 90° for each curve. $\gamma=p/p_0$, where p and p_0 are the outgoing and incident momenta of the electron.

IV. HIGH-ENERGY SMALL-ANGLE RESULT

For the case of small angles ($\alpha\ll mc^2/E_0$) and for large incident and final energies (E_0, $E\gg mc^2$), the general result Eqs. (1) and (2) can again be simplified. The result so obtained[4] does not seem to have any region of application as screening has not been included. For this reason no numerical results were computed. We would nevertheless like to give this result for the sake of completeness. In this limit, $\sigma(E,\alpha)$ is given by

$$\sigma(E,\alpha)=\frac{1}{2\pi}\frac{Z^2}{137}\left(\frac{e^2}{m}\right)^2\frac{p}{p_0}\frac{1}{k}R'',$$

$$R''=\frac{2m^2}{\lambda^2}\left\{\frac{2m^2E_0E+(E_0^2+E^2)\lambda}{[\lambda(\lambda+2m^2)]^{\frac{1}{2}}}\right.$$
$$\times\log\frac{\lambda+m^2+[\lambda(\lambda+2m^2)]^{\frac{1}{2}}}{m^2}-\frac{(E_0^2+E^2)^2}{2E_0E}$$
$$+\frac{2k^2m^2}{\lambda^2E_0^2E^2}(2E_0E\lambda-k^2m^2)(k^2+E_0E)$$
$$+\frac{k}{E^2\lambda^2}[\lambda^2E(E_0^2+E^2)+m^2(2E_0E\lambda-k^2m^2)$$
$$\times(3E-E_0)]\log\frac{2E}{m}-\frac{k}{E_0^2\lambda^2}[\lambda^2E_0(E_0^2+E^2)$$
$$\left.+m^2(2E_0E\lambda-k^2m^2)(3E_0-E)]\log\frac{2E_0}{m}\right\}, \tag{7}$$

where

$$\lambda=\frac{k^2m^2}{2E_0E}(1+\chi^2), \tag{8}$$

and $\chi=(E_0E/mk)\alpha$.