Spin Exchange Effects in Ferromagnetic Resonance

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A discussion of theoretical and experimental work concerned with spin exchange effects in ferromagnetic resonance is given. Qualitative and quantitative results of the author's exchange theory are summarized together with work of Ament and Rado in the same field. An approximate solution of the exchange problem due to the latter authors is used to analyze the recent ferromagnetic resonance results of Hoskins and Wiener on permalloys. Such analysis indicates that some of the observed effects may have arisen from exchange but that it cannot explain the observed experimental g shifts from the free-spin value.

INTEREST in the effects of exchange interactions in ferromagnetic resonance has been enhanced by experimental observations of Rado and Weertman[1] suggesting that such effects can be observed under the proper conditions at room temperature. Landau and Lifshitz[2] first showed how the exchange effect might be included in the equation of motion of the magnetization, and the solution of this equation for rather general boundary conditions was the subject of a detailed investigation by the author.[3,4] Kittel and Herring[5] have also carried out a perturbation analysis of the problem. The conclusion was drawn from these treatments that at X-band or K-band exchange effects were most likely to be observable in metals at very low temperatures where exchange damping might be expected to dominate other damping mechanisms.*

By carrying out resonance measurements on a 66% nickel, 34% iron permalloy of extremely low magnetocrystalline anisotropy, Rado and Weertman hoped to achieve the foregoing condition at room temperature. Their conclusion that they have done so is based on their derivation from the data of a reasonable value of the exchange coupling constant A and on fairly good agreement between experimental and theoretical resonance absorption curve shapes. Concomitant with this experimental work, Ament and Rado[6] independently developed a mathematical machinery for the calculation of exchange effects.

Although the RW data could be and was adequately analyzed by Rado and Weertman using a simplified approximate form of the AR solution of the equation of motion, cases of importance may arise where the conditions necessary for such simplification are inapplicable. Since the author's solution is more general than that of AR and has not been previously published, it has seemed worthwhile to summarize some of its results.[7] In addition, a brief discussion of other resonance data which may indicate the presence of observable room-temperature exchange effects will be given.

There are two principal differences between the two theories. First, the author treated the case of resonance in a plane-parallel metallic sheet of arbitrary thickness with material of arbitrary impedance abutting its rear surface, while AR considered a magnetic sample of infinite thickness. In the former case, reflection effects at the back surface must be taken into account. Because of triple dispersion arising from exchange damping, there are then six electromagnetic waves present, three going into the material from the front surface and three traveling in the opposite direction. There are twelve magnetic field components, six electric field components, and twelve magnetization components present in the sample. Solution of the equations leads to a general expression for the surface impedance for arbitrary thickness and, in addition, to a formula for transmitted power through the sheet. If the sheet is sufficiently thin, of the order of a skin depth or two, enough power may be transmitted through it so that a transmission curve can be measured which has a minimum near (but not at) absorption resonance, where the effective loss factor $\sqrt{\mu_R}$ reaches a maximum.

The second difference is in the methods of treating the complicated equations of the problem. Both treatments use the same boundary conditions and lead to exact[8] expressions for the surface impedance which differ in form but presumably not in content in the infinite thickness case where the solutions overlap.[9] Because of the considerable complexity of these results, their exact comparison, which would have to be done

[1] G. T. Rado and J. R. Weertman, Phys. Rev. **94**, 1386 (1954). (Referred to in the text as RW.)
[2] L. Landau and E. Lifshitz, Physik. Z. Sowjetunion **8**, 153 (1935).
[3] J. R. Macdonald, Ph.D. thesis, Oxford, 1950 (unpublished).
[4] J. R. Macdonald, Proc. Phys. Soc. (London) **A64**, 968 (1951). Some of the details of the work of reference 3 are given in this paper.
[5] C. Kittel and C. Herring, Phys. Rev. **77**, 725 (1950).
* *Note added in proof.*—K. H. Reich, Phys. Rev. **101**, 1647 (1956) has apparently succeeded in observing such predicted exchange anisotropy effects in nickel at 4°K for both K- and X-band wavelengths.
[6] W. S. Ament and G. T. Rado, Phys. Rev. **97**, 1558 (1955). This work will be referred to by the designation AR.
[7] The pertinent mathematical results of reference 3 are summarized in the Appendix.
[8] Exact except for the usual neglect of second and higher harmonics in the equations.
[9] There is, actually, a second-order difference in the treatment of phenomenological damping. This difference arises from the present author's inclusion of the spin-exchange contribution to the effective internal field in the phenomenological damping term of the equation of motion.

numerically, has not been carried out by either the author or by Ament and Rado.

In spite of the complexity of the author's solution, sufficient numerical calculations were originally made using it to allow the following conclusions to be drawn.[3]

(a) Exchange interaction leads to damping of the resonance absorption curves even in the absence of other types of damping.

(b) Exchange leads to a shift in g-values in such a sense that values calculated without taking exchange into account are reduced when exchange is present and properly accounted for in calculating g. Thus, if exchange is present but not recognized in calculating g (giving the apparent g, g_a), g_a will be larger than the true g value of the material.

(c) When exchange is sufficiently strong, the resonant frequency (or resonant magnetic field strength) and apparent g value depend on film thickness in thin films, with g_a rising rapidly from the true value to a larger value in the neighborhood of a film thickness related to the skin depth.

(d) Because of a difference in the weighting of exchange effects for absorption and transmission of electromagnetic power in a thin sheet, the maximum of transmitted power occurs at a greater magnetic field strength than does the maximum of absorbed power. It is the latter field strength which is used to calculate g in usual resonance experiments.

(e) With exchange damping only, one of the electromagnetic wave propagation factors goes to zero at antiresonance (the low-field minimum of the $\sqrt{\mu_R}$ curve,[10] where $b \simeq G$), giving an undamped wave which can be reflected at the far side of a sample no matter how thick it is. Solutions of the infinite thickness case yield zero effective surface impedance for this field strength. The more complete thin-film solution shows, however, that the necessary nonzero value is obtained even in this case. The approximate Ament-Rado solution does not apply in those experimental cases where antiresonance is observable since it holds only when $b \cong 1 \gg G^2$.

(f) When exchange is appreciable, it was found that plotting the real and imaginary parts,[11] $\sqrt{\mu_R}$ and $\sqrt{|\mu_L|}$ of the theoretical normalized surface impedance against one another in the manner of a circle diagram gave an excellent circle in the infinite thickness case provided the region around antiresonance was calculated from the complete solution to avoid the behavior discussed in (e). As stated in the earlier work, such a result shows that the complicated exact formula for the surface impedance may often be well approximated by a far simpler formula which represents a bilinear transformation between the static magnetic field axis and the complex surface impedance plane.

[10] Normalized quantities such as b and G are defined near the beginning of the Appendix.

[11] The absolute value sign in $\sqrt{|\mu_L|}$ is required to keep this quantity real. $\sqrt{|\mu_L|}$ takes the sign of μ_L.

Ament and Rado[6] have presented a useful approximate form of their exact solution of the equation of motion which holds well when h, G^2, $\sqrt{k_0}$, and P_0 are all negligible compared to unity. These quantities are defined in the Appendix. Since these conditions were satisfied for the RW experiments, the simpler approximate solution was used for their analysis. Since Ament and Rado did not consider the case of a ferromagnetic sheet of arbitrary thickness, it has seemed worthwhile to apply their approximations to the author's solution for such sheets. As shown in the second part of the Appendix, considerable simplification occurs, and easily applied formulas for effective surface impedance and the ratio of transmitted to absorbed power in the sheet are obtained.

As a partial check between the Macdonald and Ament-Rado solutions, the possibility of reduction of the simplified arbitrary-thickness solution for the effective surface impedance to the AR approximate result[12] was investigated by letting the thickness become infinite. This limit is carried out in the Appendix and does indeed yield the AAR solution. Such reduction was not carried out in the original work, however, because some of the above conditions were not satisfied in the author's experiments to which the theory was applied. In particular, the condition $h \ll 1$ requires that the static magnetic field strength H_0 be much smaller than $4\pi M_s$ in the region of interest around resonance. In the author's experiments and in most others except those of Rado and Weertman, this quantity has been of the order of unity near resonance. Rado and Weertman noted that the use of materials with extremely low magnetocrystalline anisotropy allows experiments to be made at sufficiently low frequencies that h is small near resonance but saturation is still achieved. Operation in the $h \ll 1$ region not only makes possible comparison of experimental results with the AAR solution but also increases any resonant field shift arising from exchange relative to the resonant field itself. Such an increase, of course, increases the probability of observation of exchange effects. Whereas earlier workers[3,5] pointed out that an increase in $\Delta H_{(\text{exchange})}/H_{0(\text{resonance})}$ could be achieved by increasing metal conductivity by going to low temperatures of measurement, Rado and Weertman[1] noted that this quantity could also be increased at room temperature by using sufficiently low frequencies.

It is of interest to point out that the AAR solution is of almost the form required by the bilinear transformation mentioned in (f). In addition, Rado and Weertman[1] have noted that the AAR solution yields negative values of μ_2, the imaginary component of the equivalent permeability,[13] $\mu = \mu_1 - i\mu_2$ for low static

[12] The approximate Ament-Rado [their Equation (31)] result will hereafter be denoted by AAR.

[13] The equivalent permeability, determined from measurements of effective surface impedance, is not necessarily the intrinsic permeability of the material, but may include exchange and other

magnetic field strengths below resonance. Such a negative μ_2 was observed experimentally by RW. As evidence that this negative μ_2 is not an artifact introduced by the approximations leading to the AAR solution, it may be mentioned that numerical calculations[3] using the author's exact solution showed that exchange effects can lead to a region on the low-field side of resonance where $|\mu_L|>\mu_R$. Since $\mu_2=[\mu_R-|\mu_L|]/2$, this condition implies a negative μ_2. Unlike the AAR solution which predicts that μ_2 will remain negative down to zero magnetic field strength once it goes negative, the above calculations show that μ_2 becomes positive again as it should for sufficiently low fields. This difference is probably not very significant since even low-anisotropy materials will not be saturated at very low fields.

Recently, Hoskins and Wiener[14] have presented resonance data on a range of permalloys in the 36- to 48-percent nickel range for wavelengths of 3.15, 1.2, and 0.6 cm. As in the RW experiments, very narrow resonance lines were obtained and both line width and apparent g values were found to be frequency dependent but little dependent on composition. It is of interest to inquire whether some of these Permalloy results are explicable by exchange damping.

The line widths quoted by Hoskins and Wiener were obtained by calculation from the derivative of the $\sqrt{\mu_R}$ absorption curve assuming Lorentzian line shape. The widths are, therefore, somewhat approximate and refer to the $\sqrt{\mu_R}$ curve, not to the μ_2 curve. First, it is of interest to compare the line width obtained by Rado and Weertman at a 10-cm wavelength with that of Hoskins and Wiener. To do this, the RW data, given in terms of the derived permeability μ, must be converted back to give the corresponding $\sqrt{\mu_R}$ curve. We may carry out such conversion by first fitting the permeability curves theoretically by means of the approximate AR exchange result. Such a procedure yields more accurate $|\mu|$ and μ_2 values than can be obtained by reading directly off the μ_1 and μ_2 curves presented by RW. Next, these values may be combined[15] to calculate μ_R and $\sqrt{\mu_R}$, and the widths then obtained by plotting the results. We find $\mu_{2\,\max}=1330$, $\Delta H_{\mu_2}=28.5$ oersteds; $\mu_{R\,\max}=2670$, $\Delta H_{\mu_R}=36.3$ oersteds; and $\sqrt{\mu_R}_{\max}=51.7$, $\Delta H_{\sqrt{\mu_R}}=61.2$ oersteds. Since this conversion was based on the approximate theoretical formula, which does not fit the actual data very closely appreciably off resonance, the width of the measured $\sqrt{\mu_R}$ curve may have been as large as 75 oersteds. On the other hand, Hoskins and Wiener found a width of 90 oersteds on 44 and 48% alloys at a 3-cm wavelength. Since the width apparently decreases with increasing wavelength, a larger value is indeed to be expected at 3 cm as compared to a 10-cm wavelength.

Next, we may investigate, in a preliminary fashion, to what extent the wavelength dependence of Hoskins and Wiener's line widths and apparent g values may arise from exchange effects. Because of the considerable complexity of the exact solutions, we shall make use of the AR approximate form even though the approximations on which it is based are not strictly valid over the whole wavelength range investigated by HW. A more exact analysis would require the use of an exact solution but is not warranted at this time because Hoskins and Wiener do not present entire resonance curves.

A simple calculation shows that the AAR solution predicts that the widths at half-maximum of the μ_R, $\sqrt{\mu_R}$ and μ_2 resonance curves are proportional to $(\sqrt{A})/M_s\delta$ where A is a constant measuring the strength of exchange coupling,[16] M_s is the saturation magnetization, and δ is the skin depth for unity permeability. Since δ is proportional to $\omega^{-\frac{1}{2}}$, the width is therefore inversely proportional to the square root of the free-space wavelength. The widths given for 44 and 48% nickel Permalloys are 90, 125, and 250 oersteds at 3.15, 1.2, and 0.6 cm wavelengths, respectively. When these widths are multiplied by the square root of their corresponding wavelengths, a constant value should be obtained if the dependence is actually of the above form. Instead, we obtain 156, 137, and 193. Although these values differ, the original variation is somewhat reduced. In view of the approximation made in applying the above result for the width dependence to the HW width values, it seems possible that exchange effects were playing an appreciable role in these experiments even though other effects also involving the skin depth cannot be ruled out on the present evidence.

Finally, it is pertinent to investigate quantitatively how exchange effects may affect g values. First, we define the apparent, uncorrected g value, g_a, in terms of the usual resonance expression,[16,17] such as $\omega=\gamma(B_sH_{0r})^{\frac{1}{2}}$ for parallel field orientation, where $\gamma=g_ae/2mc$, H_{0r} is the static resonance field, and it and B_s are corrected for shape demagnetization effects.[4] Next, we can define a new exchange-corrected g value in terms of the new resonance condition which takes explicit account of exchange. This condition is conventionally defined in terms of the field which makes $\sqrt{\mu_R}$ a maximum. This is not necessarily the field at which μ_2 is maximum or that for which μ_L and μ_1 are zero, although the differences between the three possible field values will usually be small when the damping is small. As an illustration, these differences and the resulting values of μ_R, μ_2, and μ_1 for the different resonance conditions are summarized for the AAR solu-

contributions. The relation between μ and $\sqrt{\mu_R}$ and $\sqrt{|\mu_L|}$ is $(2i\mu)^{\frac{1}{2}}=\sqrt{\mu_R}+i\sqrt{|\mu_L|}$.

[14] R. Hoskins and G. Wiener, Phys. Rev. **96**, 1153 (1954). (Referred to as HW.)

[15] The usual relation $\mu_R=(|\mu|+\mu_2)$ is used. The corresponding relation for μ_L is $\mu_L=\pm(|\mu|-\mu_2)$.

[16] C. Kittel, Revs. Modern Phys. **21**, 541 (1949). See pp. 550–552.

[17] C. Kittel, Phys. Rev. **73**, 155 (1948).

tion at the end of the Appendix. Using these results, we may write the following relation for the exchange-corrected g value in terms of the apparent g value without exchange correction:

$$g^{-2}-g_a^{-2}=3.84\times10^{11}aM_s\omega^{\frac{1}{2}}(A/\rho)^{\frac{1}{2}},$$

where ρ is the resistivity in ohm-centimeters. The constant a is 0.817, 0.785, and 0.704 for resonance defined by $\sqrt{\mu_R}$ a maximum, μ_2 a maximum, and $\mu_1=0$, respectively. Note that these results hold well only when the approximations of the AAR solution are valid.

When the above formula is applied to the results of Rado and Weertman, it is found that the apparent g value without exchange correction, 2.87, is reduced by such correction to 2.42, the value quoted by these authors. Here, even though exchange produces a very significant change in g, the magnitude of the final corrected value indicates that other factors as well are influential in determining g.†

The above formula is not applicable to the HW results except possibly at the longest wavelength, where the approximate solution may be reasonably applicable. Its frequency dependence cannot explain the magnitude of the strong increase in g_a with wavelength found by Hoskins and Wiener, although the above dependence is in the right direction to agree with experiment. If we use a value of A of 3×10^{-5} ergs/cm, an order of magnitude or more larger than the value likely or that derived from the RW data, it is found that exchange reduces the value of g_a of 2.3 found by Hoskins and Wiener at a 3-cm wavelength by less than 1%, within the limits of applicability of the foregoing formula. It therefore here again appears that although exchange may be able to explain some of the curve-shape features of these resonance experiments reasonably well, other processes may have to be invoked to explain the magnitude and dependence of their g values. Kittel and Mitchell[18] have suggested, as one such process, exchange coupling between $3d$ electrons and $4s$ conduction electrons. If an explicit theoretical account of this process proves possible, it may perhaps explain the HW results and, in addition, be applicable as well to the Rado-Weertman results.

The question of how well exchange effects alone are capable of explaining the precise shapes of experimental absorption and dispersion curves in pure metals and alloys is one of considerable interest. We have already seen that even in those cases where exchange is apparently important, it is incapable of explaining the appreciable frequency-dependent deviation of g from the free-spin value. Judging from the RW results, the AAR solution does a reasonably good job of predicting resonance curve shapes where it is applicable. At any rate, it yields closer agreement with the RW results than does a theory of simple phenomenological damping alone.

Since the approximations leading to the AAR solution are inapplicable to most previous resonance experiments with metals and alloys, investigation of possible spin exchange effects for these experiments must be made using either the exact AR solution or that of the author. The choice, for either hand or machine calculation, can be made on the basis of relative ease of calculation. The author's solution is given as a function of the complex roots of a cubic equation while the AR solution is a function of two quantities which satisfy quartic equations. In either case, the cubic roots or the roots of the two quartics must be recalculated for each value of h considered. Although all these quantities are complex, the cubic roots can be calculated directly from the exact root expressions given in the Appendix. On the other hand, exact expressions for the quartic roots are likely to be more complicated than those for the cubic roots. In practice, Ament and Rado have calculated the quartic roots only by means of successive approximations using a digital computer,[6] while a number of calculations have been made by hand using the author's result.[3]

We have thus far considered only resonance experiments with alloys having exceedingly low magnetocrystalline anisotropy and little damping. For these materials, it is possible that many of the observed curve features may be explained by exchange. On the other hand, materials such as supermalloy and especially nickel[19] show considerably greater damping and larger half-power line widths when measured in the wavelength range of 1 to 3 cm. Further, these curve shapes can be well fitted on the basis of ordinary phenomenological damping alone, whereas this is not the case in the RW experiment.

A calculation has been made to see whether the earlier exact exchange formula can explain any of the resonance curve shape features found for nickel at $\lambda=1.25$ cm. It was found that reasonable agreement between theory and experiment could be obtained using exchange damping alone only if a value of the exchange constant A of about 6×10^{-4} ergs/cm were used. This value is at least two orders of magnitude larger than expected for this material.[16] In addition, transmitted-power resonance curves obtained with thin sheets of nickel showed no sign of exchange effects although a decrease of transmitted power by a factor of forty was observed at resonance.[3] It is thus extremely unlikely that exchange effects play an appreciable role in nickel. Supermalloy is an intermediate case, however, and it is possible that measurements on this material at relatively low frequencies and low resonance field strengths in the manner

† *Note added in proof.*—Rado and Weertman (private communication) in work to be published show that more accurate correction for static demagnetization together with exchange-shift correction yields an S-band g value in good accord with values obtained at X and K band.

[18] C. Kittel and A. H. Mitchell, Phys. Rev. **101**, 1611 (1956).

[19] K. W. H. Stevens, Proc. Phys. Soc. (London) **A65**, 149 (1952).

of Rado and Weertman might show the influence of exchange.

In conclusion, we should like to suggest that whenever exchange effects are suspected in resonance experiments, measurements over an appreciable range of temperatures might give an unequivocal test of their presence or absence. Variation of temperature will primarily affect the resistivity and so the skin depth and will thus strongly influence the region within the material in which exchange effects can be of importance. If exchange is indeed of importance, both the damping (as reflected in the maximum value of $\sqrt{\mu_R}$) and the g value will depend appreciably on temperature.

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APPENDIX

I. Mathematical Results of the Author's Theory

The mathematical results of the author's exchange theory[3] are summarized in this section. We have designated the region in front of the plane metal surface by the superscript (1), that within the metal by (2), and that behind it as (3). A plane electromagnetic wave is assumed normally incident on the metal surface at the 1–2 interface. The effective surface impedance of the metal is then $Z^{(2)}(0)$. We also designate the ratio of power leaving the material at its rear surface, $P(d)$, to that absorbed at the front surface of the metal, $P(0)$, as Y. The impedance of the material adjoining the metal at the 2–3 interface is defined as Z_3.

Since the ferromagnetic metal is triply refracting when exchange is present, the boundary-value problem for a sheet of arbitrary thickness d is complicated and its explicit solution correspondingly complex. To present it in a concise form, we use a large number of interlocking simplifying definitions. The solution is

$$Z^{(2)}(0)=\left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}}\frac{\phi_{123}}{\theta_{123}-(\pi_{123}/\Omega_{123})\Phi_{123}+F_{123}}$$

$$\equiv\left(\frac{\omega}{8\pi\sigma}\right)^{\frac{1}{2}}[(\mu_R)^{\frac{1}{2}}+i(|\mu_L|)^{\frac{1}{2}}],$$

$$\underset{(d\to\infty)}{{}^{\infty}Z^{(2)}}(0)=\left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}}\frac{\phi_{123}}{\theta_{123}},$$

$$Y=\frac{\mathrm{Re}(Z_3)\,|T|^2}{\mathrm{Re}[Z^{(2)}(0)]},$$

$$T=[Z^{(2)}(0)/Z_3](\pi_{123}/\Omega_{123}).$$

The quantities used in the above equations are defined as follows:

$$\phi_{123}=\phi_1\tau_{32}+\phi_2\tau_{13}+\phi_3\tau_{21},\quad \phi_j=1/(t_j-ik_0),$$

$$\tau_{ij}=\tau_i-\tau_j,\quad G=\omega/(4\pi M_s\gamma),\quad h=H_0/(4\pi M_s),$$

$$\tau_j=\frac{iG-P_0[t_j-ik_0\phi_j-h]}{b-t_j}\equiv\frac{ik_0+(t_j-ik_0)(t_j-h)}{[iG+P_0(b-t_j)][t_j-ik_0]},$$

$$b=B_s/4\pi M_s=1+h,\quad k_0=A/\pi\delta^2M_s^2.$$

P_0 is a phenomenological damping constant. In the AR notation it is given by $\lambda/\gamma M_s$. The three t_j's are the roots of the cubic equation given later. H_0 is the static magnetic field.

All the above quantities with three subscripts are formed with the τ_{ij}'s in the same manner as ϕ_{123}. Hence, it is only necessary to give the basic term of each. These are

$$\theta_j=\phi_j/\sqrt{t_j},\quad \Phi_j=\theta_j r_j e^{-\eta_j},\quad \eta_j=\gamma_j d,$$

$$\gamma_j=M_s(2\pi t_j/A)^{\frac{1}{2}},\quad \pi_j=\theta_j\,\mathrm{csch}\eta_j,\quad \Omega_j=q_j\pi_j,$$

$$F_j=\textstyle\sum_j-(\pi_{123}/\Omega_{123})\xi_j,\quad q_j=\cosh\eta_j+{}^iZ_{23}\sinh\eta_j,$$

$$\textstyle\sum_j=\pi_j e^{-\eta_j},\quad \xi_j=\Omega_j e^{-\eta_j},\quad r_j=1-{}^iZ_{23},$$

$${}^iZ_{23}=\gamma_j/[4\pi\sigma Z_3/c].$$

The cubic equation having the three t_j roots may be written

$$t^3+C_1t^2+C_2t+C_3=0,$$

where

$$C_1=-\left[m+i\left(k_0+\frac{2P_0G}{1+P_0^2}\right)\right],$$

$$C_2=\left\{\left[bh-\frac{G^2}{1+P_0^2}-\frac{2k_0P_0G}{1+P_0^2}\right]+i\left[2bk_0+\frac{P_0mG}{1+P_0^2}\right]\right\},$$

$$C_3=\frac{2k_0P_0bG}{1+P_0^2}-ik_0\left[b^2-\frac{G^2}{1+P_0^2}\right],$$

and $m=h+b$.

The following approximate roots were given in the earlier work.

$$t_{1,3}\cong\frac{-C_2}{2C_1}\pm\left[\left(\frac{C_2}{2C_1}\right)^2-\frac{C_3}{C_1}\right]^{\frac{1}{2}},$$

$$t_2\cong(C_2/C_1)-C_1.$$

From these approximate values, the AAR solution may be derived as shown by these authors when the usual approximations employed by AR are used to simplify the roots further.

For completeness, and since they will be of especial use in digital computer calculations using the exact solution, the following exact expressions for the roots

may also be given.

$$t_j = Z_j - C_1/3,$$
$$Z_{1,3} = -P^{\frac{1}{3}}\{\cos(\phi/3) \pm \sqrt{3}\sin(\phi/3)\},$$
$$Z_2 = 2P^{\frac{1}{3}}\cos(\phi/3),$$
$$P = (C_1/3)^2 - C_2/3,$$
$$\cos\phi = -k^3/P^{\frac{3}{2}},$$
$$k^3 = (C_1/3)^3 - C_1C_2/3 + C_3/2.$$

II. Simplification of the Theory when the AR Approximations Apply

In this section, we shall show how the result for the effective surface impedance $Z^{(2)}(0)$ of a ferromagnetic film of arbitrary thickness simplifies when the AR approximations apply and, from this simplified result, derive the AAR solution by going to the limit of infinite thickness. The Ament-Rado approximations require that the normalized quantities h, G^2, $\sqrt{k_0}$, and P_0 all be negligible compared to unity. For simplicity, we shall take the phenomenological damping constant P_0 zero so that only exchange damping is present. In addition, we shall take $Z_3 = 1$, implying that the material at the rear of the ferromagnetic sheet is air or vacuum. This assumption is unnecessary, but it corresponds to most experimental situations and simplifies the results considerably.

The AR approximations allow us to write the following approximate equations for the coefficients of the cubic equation.

$$C_1 \cong -1,$$
$$C_2 \cong (h - G^2) + 2ik_0,$$
$$C_3 \cong -ik_0.$$

The approximate cubic roots given in the last section then become

$$t_{1,3} \cong \frac{h - G^2}{2} \pm \left[\left\{\frac{h - G^2}{2}\right\}^2 - ik_0\right]^{\frac{1}{2}},$$

$$t_2 \cong 1.$$

In addition, it is readily shown that

$$t_1 t_2 t_3 \cong t_1 t_3 \cong ik_0,$$
$$t_1 + t_3 \cong (h - G^2) + 2ik_0.$$

The magnitude of the $2ik_0$ term will generally be small compared to $(h - G^2)$ except when $h \cong G^2$. It has been neglected in the expression for $t_{1,3}$ and, as we shall see, may also be neglected in $(t_1 + t_3)$ as well as far as any contribution to the final results is concerned.

The condition $Z_3 = 1$ allows us to neglect iZ_{23} compared to unity for physical cases of interest. Then, $r_j \cong 1$ and $q_j \cong \cosh\eta_i$. On evaluating some of the quantities appearing in $Z^{(2)}(0)$, we find

$$\phi_j \cong t_j^{-1}, \qquad \tau_{21} \cong (iG)^{-1},$$
$$\tau_{32} \cong -(iG)^{-1}, \qquad \tau_{31} \ll \tau_{32}, \tau_{21}.$$

When these results are used to calculate $Z^{(2)}(0)$, we find, after considerable cancellation and simplification

$$Z^{(2)}(0) \cong \left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}} \times \frac{(t_3^{-1} - t_1^{-1})(t_3^{-\frac{3}{2}}\coth\eta_3 - t_1^{-\frac{3}{2}}\coth\eta_1)}{t_3^{-3} - 2t_1^{-\frac{3}{2}}t_3^{-\frac{3}{2}}(\coth\eta_1\coth\eta_3 + \operatorname{csch}\eta_1\operatorname{csch}\eta_3) + t_1^{-3}}.$$

We may write the decay factor γ_j as

$$\gamma_j = \left[\frac{4\pi\omega\sigma t_j}{k_0 c^2}\right]^{\frac{1}{2}} = \left[\frac{2t_j}{k_0\delta^2}\right]^{\frac{1}{2}},$$

where $\delta = c/(2\pi\omega\sigma)^{\frac{1}{2}}$ is the skin depth for unity permeability. Then η_1 and η_3 become

$$\eta_{1,3} \cong \frac{d}{\delta}\left(\frac{2}{k_0}\right)^{\frac{1}{2}}\left\{\frac{h - G^2}{2} \pm \left[\left(\frac{h - G^2}{2}\right)^2 - ik_0\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

This result, together with the earlier expression for $t_{1,3}$ allows $Z^{(2)}(0)$ to be calculated for any experimental conditions for which the approximations used hold. Note than when $Z^{(2)}(0)$ is written as $(\omega/8\pi\sigma)^{\frac{1}{2}}[\sqrt{\mu_R} + i\sqrt{|\mu_L|}]$, the quantities $\sqrt{\mu_R}$ and $\sqrt{|\mu_L|}$ are only effective values for any $d < \infty$ and are not the true values characteristic of the material itself.

Next, a small amount of further calculation leads to the transmission factor T, which is found to be

$$T \cong \left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}} \times \frac{(t_3^{-1} - t_1^{-1})[t_3^{-\frac{3}{2}}\operatorname{csch}\eta_3 - t_1^{-\frac{3}{2}}\operatorname{csch}\eta_1]}{t_3^{-3} - 2t_1^{-\frac{3}{2}}t_3^{-\frac{3}{2}}(\coth\eta_1\coth\eta_3 + \operatorname{csch}\eta_1\operatorname{csch}\eta_3) + t_1^{-3}}.$$

For any specific choice of thickness d, this result together with that for Y in the last section allows the ratio of transmitted to absorbed power to be calculated.

Next, we shall carry out the limit $d \to \infty$. We then obtain

$${}^{\infty}Z^{(2)}(0) \cong \left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}} \frac{(t_3^{-1} - t_1^{-1})}{(t_3^{-\frac{3}{2}} - t_1^{-\frac{3}{2}})} = \left(\frac{\omega}{4\pi\sigma k_0}\right)^{\frac{1}{2}} \frac{(t_1 t_3)^{\frac{1}{2}}[\sqrt{t_1} + \sqrt{t_3}]}{t_1 + (t_1 t_3)^{\frac{1}{2}} + t_3},$$

$${}^{\infty}Y = 0.$$

Although the above expression for ${}^{\infty}Z^{(2)}(0)$ simplifies

TABLE I. Summary of results.

Resonance condition	$(h-G^2)/\sqrt{k_0}$	$(\sqrt{k_0})\mu_R$	$(\sqrt{k_0})\mu_2$	$(\sqrt{k_0})\mu_1$
μ_R max	−0.57771	6.31239	3.13269	−0.54473
μ_2 max	−0.55490	6.30266	3.13958	−0.42136
$\mu_1=0$	−0.49809	6.19846	3.09923	0

somewhat when the earlier expressions for the roots are substituted, better simplification occurs when

$$\mu=(2i)^{-1}[\sqrt{\mu_R}+i\sqrt{|\mu_L|}]^2$$
$$=(4\pi\sigma/i\omega)[Z^{(2)}(0)]^2$$

is calculated instead. We find

$$\mu\cong(ik_0)^{-1}\frac{t_1t_3[t_1+2(t_1t_3)^{\frac{1}{2}}+t_3]}{[t_1+(t_1t_3)^{\frac{1}{2}}+t_3]^2}.$$

On substituting the roots and neglecting k_0 compared to $\sqrt{k_0}$, this expression becomes

$$\mu\cong\frac{(h-G^2)+2(ik_0)^{\frac{1}{2}}}{[(h-G^2)+(ik_0)^{\frac{1}{2}}]^2}.$$

This is the AAR result for zero phenomenological damping expressed in terms of the present notation. The quantity k_0 equals $2\epsilon^2$ in the AR notation.

Finally, using the above expression we may investigate the various resonance conditions. Ament and Rado have already given the condition which makes $\mu_1=0$; in addition, the conditions for μ_R and μ_2 to reach their maximum values are of interest, together with the actual values of μ_R, μ_2, and μ_1 in the three cases. All these quantities may be expressed in terms of $\sqrt{k_0}$, and the desired results are summarized in Table I. In applying these results, it must be remembered that they are valid only when the AR approximations are well satisfied. Since one of these approximations requires that $\sqrt{k_0}\ll1$, the results in Table I may only be used when μ_R max and μ_2 max are of the order of 10^2 or greater.

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Trapped Flux in Impure Superconductive Tin*†

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We have measured the percentage of trapped flux in cylindrical tin samples containing antimony, bismuth, or indium, with particular regard to the effect of annealing. The flux was trapped by the application and removal of a large transverse magnetic field. Our results do not confirm Pippard's conclusion that a marked change in flux trapping behavior occurs at a critical concentration of impurity. Instead we find that all samples, whether of high- or of low-impurity concentration, follow the same behavior, namely, that the percentage of trapped flux rises steeply near the transition temperature, and that this rise decreases monotonically with annealing time. In all cases in which this rise is discernible its temperature dependence is linear with the function $(1-t^4)^{-\frac{1}{2}}$ up to $t\gtrsim0.98$, where $t=T/T_c$. We believe that the binary specimens used by us as well as those used by Pippard have a substructure of filaments of different concentration, which trap flux when the sample is insufficiently annealed. In support of this view we cite extensive metallurgical evidence for the existence of such a substructure, and a crude measurement on two of our annealed specimens which showed that the magnetic field needed to restore resistance was much higher than the threshold field of the bulk material, and that this transition was quite broad.

I. INTRODUCTION

WHEN a superconductive substance passes from the normal to the superconducting state in the presence of an external magnetic field, it becomes ideally a perfect diamagnetic, excluding the applied field entirely except in a thin surface layer. In practice, however, during the course of transition multiply-connected parts within the specimen may develop which have the general form of closed superconducting regions surrounding cores of normal metal. Such cores will have magnetic flux running through them. The perfect conductivity of the enclosing superconducting regions now makes it impossible for this flux to change. Thus the specimen retains a small magnetic moment proportional to the amount of flux trapped in this fashion even after the external field has been reduced to zero.

Although the phenomenon of flux trapping has been known since the first discovery of the diamagnetic nature of superconductors, the few studies of this effect

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