Formal Theory of Nuclear Models*

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The formal theory of nuclear model operators discussed by Eden and Francis is presented from a somewhat different point of view. Our way of approach is, first, to introduce the model space $\mathfrak{M}$ of arbitrary dimension which is composed of simple known wave functions; second, to transform the original Schrödinger equation into that in the model space, the model wave function and the model Hamiltonian being defined in an unambiguous way; third, to obtain the integral equation for the model operator which also determines the model Hamiltonian. This equation no longer contains the exact eigenvalues as in the work of Eden and Francis and is shown to have a unique solution in some cases. The formal relation of this theory to the nuclear scattering problem is discussed.

1. INTRODUCTION

As is well known, quantum mechanical many-body problems are so complicated that it is generally very difficult to obtain even reliable approximate solutions, let alone exact ones. Especially in the theory of the nucleus, conventional approximation methods may not work well, since the nuclear forces are very strong and singular and this necessarily causes strong correlation effects in the nuclear wave functions. However, in recent years, a number of nuclear models have been developed which successfully describe many aspects of nuclear structure, and the complicated nuclear problems have been replaced by some simpler ones. The most striking successes have been obtained by the Mayer-Jensen[1] shell model and by the Bohr-Mottelson[2] collective model, and it is clear that for low energies there must be a close correspondence between these models and the actual nucleus. It seems quite surprising that these models, though rather contradictory to each other in many cases, are successful in reproducing the nucleus itself in some respects.

It is usually accepted that the nucleus can be described in terms of a nonrelativistic many-body Schrödinger equation including some appropriate static potentials, insofar as the low-energy phenomena are concerned. Whether these potentials will be given by the sum of the ordinary two-body potentials, or whether many-body forces play a role in the heavy nucleus, is quite open to question at this stage, but the recent success of meson theory seems to suggest that the latter forces may be safely neglected, at least at low energies.[3] In fact, the properties of the nuclear forces are intimately connected with the success of nuclear models, and will play an essential role in deriving the models in a deductive way from many-body nuclear problems.

Brueckner and co-workers[4] have tried to derive the weak-coupling nuclear models (the optical model and the shell model) by applying the coherent approximation developed by Watson and Francis,[5] but it has been pointed out recently by the author and Newton[6] that a serious difficulty arises in connection with the boundary condition on the scattering equation they derived. Tomonaga[7] and Coester[8] have succeeded in separating the Hamiltonian of the collective, surface oscillations of the nucleus from the internal motion, but the stability of these oscillations has never been theoretically demonstrated.

Eden and Francis[9] have taken a quite different approach and present a formal theory of nuclear models, in which they assume that the wave functions for different nuclear models are in general different from the actual nuclear wave functions and that the former will be obtained by transformations applied to the latter wave functions. Thus they provided a basis for explaining the success of weak-coupling nuclear models and showing that they are not in conflict with the assumption that nucleons have very strong mutual interactions. They investigated the properties which transformation operators (model operators) must have to change a model wave function into the nuclear wave function, and obtained the integral equation to determine the model operator.

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[1] M. Goeppert-Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).

[2] A. Bohr, Phys. Rev. **81**, 134 (1951); A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[3] It is shown by S. D. Drell and K. Huang [Phys. Rev. **91**, 1527 (1953)] that the many-body forces will play an important role for the saturation problem, if one applies the pseudoscalar meson theory with pseudoscalar coupling in perturbation theory, but this theory clearly contradicts experiment, giving rise to a strong S-wave meson scattering. If one assumes phenomenologically that the S-wave interaction is sufficiently small so as to fit experiment, the many-body forces turn out to be negligibly small. See Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).

[4] Brueckner, Levinson, and Mahmoud, reference 3. K. A. Brueckner, Phys. Rev. **96**, 508 (1954); **97**, 1353 (1955). K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955).

[5] K. M. Watson, Phys. Rev. **89**, 115 (1953); N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

[6] N. Fukuda, Progr. Theoret. Phys. (to be published); N. Fukuda and R. Newton, (to be published).

[7] S. Tomonaga, Progr. Theoret. Phys. **13**, 467 (1955); **13**, 482 (1955).

[8] F. Coester, Phys. Rev. **99**, 170 (1955).

[9] R. J. Eden and N. C. Francis, Phys. Rev. **97**, 1366 (1955).

They also set up self-consistent equations for a model having a product wave function in the particle variables and discussed their approximate solutions. However, there seems to be no justification for the existence or the uniqueness of the solution of these equations. Furthermore, the integral equation for the model operator contains the total energy of the nucleus to be determined, which will make it difficult to treat the excited states of the nucleus.

The purpose of the present paper is to consider a formal construction of model operators and model Hamiltonians which is independent of the nature of nuclear forces. We do not intend to justify any special nuclear model, but to reduce the complicated nuclear problems to the formally simpler ones in the model space, which may be chosen quite arbitrarily at the beginning according to the type of the model to be considered. Since our methods are quite general and apply to any nuclear model, a complete presentation of this theory would require detailed consideration of various aspects of the relation of nuclear models to experiment. Our theory has some formal advantages over that of Eden and Francis in that the exact energy level or other quantities related directly to the real nuclear wave function have been completely eliminated from the integral equations to determine model operators and model Hamiltonians, and that the excited states of the nucleus may be equally well treated in this formalism.

In Sec. 2, a mathematical demonstration is given to show that there may exist various kinds of models which will duplicate some properties of the nucleus, especially the level structure. However, the model wave function may be, in general, quite different from the real nuclear wave function. Therefore it is necessary to know the transformation, namely, the model operator, which connects the model functions and real functions to each other in order to calculate properties of the nucleus such as magnetic or quadrupole moments. In Sec. 3, the integral equation to determine the model operator is given and the uniqueness of the solution is discussed. In Sec. 4, it is shown that this equation is formally identical with that of the wave matrix in the scattering problem where there occur finite shifts of the energy levels during the scattering process,[10] as discussed by Gell-Mann and Goldberger[11] and by Tanaka.[12] The boundary condition is, however, entirely different. In Sec. 5 the independent-particle model is considered as an illustration and the conventional approximation method is re-examined from this standpoint. The self-consistent problem, applicable even to excited states, is set up.

2. NUCLEAR MODELS AND MODEL OPERATORS

In this section we shall consider some subset of the eigenstates of the nucleus, e.g., the ground states and some excited states, and introduce another subspace which has the same dimensionality as this subset. This subspace may be chosen according to the model with which we are concerned (say some combination of specified product functions if we are going to discuss the independent-particle model) and will be denoted as the model space $\mathfrak{M}$ in the following. We will discuss how to find a linear operator, or its inverse, which transforms the above subset of eigenstates into the model space such that scalar products are invariant. Then the model Hamiltonian which is Hermitian in the model space is defined by operating with this transformation on the total Hamiltonian of the nucleus.

Denoting the total Hamiltonian of the nucleus by H, its eigenvalues by E_a, and the corresponding eigenfunctions by ψ_a (which are assumed to be orthonormal), the Schrödinger equation is given by

$$(E_a-H)\psi_a=0. \tag{2.1}$$

What we actually want to know about the nucleus is not all the eigenstates, but for instance, the ground state or several excited states. Assuming the number of states to be m, we consider only those states $\{\psi_a\}$ with $a=1,2,\cdots,m$. These states span an m-dimensional linear subspace $\mathfrak{R}$ in the whole Hilbert space of the nucleus.

Next, we consider another m-dimensional subspace, the model space $\mathfrak{M}$, in this Hilbert space. A variety of models may occur according to the choice of this model space, which is quite arbitrary in principle as long as the condition imposed just below is satisfied. Denoting the projection operator to the model space $\mathfrak{M}$ by P, we put

$$P\psi_a=\chi_a, \quad a=1,2,\cdots,m. \tag{2.2}$$

Here we assume that the χ_a's are linearly independent. This is the only condition which the model space $\mathfrak{M}$ should satisfy. Then the χ_a's span the whole model space, so that there exists a linear operator J which transforms $\mathfrak{M}$ into $\mathfrak{R}$ such that

$$\psi_a=J\chi_a, \quad a=1,2,\cdots,m. \tag{2.3}$$

Denoting the Hermitian conjugate to J by $J^\dagger$, it follows that $J^\dagger J$ is a positive definite Hermitian operator in the model space and so it has an inverse operator $(J^\dagger J)^{-1}$ in the model space. It is also convenient to generalize J to the operator in the whole Hilbert space such that it leads to 0 when operating on any vectors orthogonal to the model space. Then we may write

$$J=JP. \tag{2.4}$$

[10] In scattering by a potential of finite range, the energy shifts are always infinitesimal, going over to zero as $1/R$, as R, the radius of the normalization volume, tends to infinity, and are usually neglected. However, these shifts should be taken seriously in connection with the problem of boundary conditions, when one considers the scattering in a finite volume such as a nucleus. See reference 6.

[11] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[12] H. Tanaka, Progr. Theoret. Phys. **13**, 497 (1955).

Operating with P from the left on Eq. (2.1), we get

$$(E_a-\langle HJ\rangle)\chi_a=0, \tag{2.5}$$

where we used the bracket in the sense

$$\langle HJ\rangle=PHJP. \tag{2.6}$$

Equation (2.5) seems at first sight to be an eigenvalue problem in the model space which gives the same energy-level structure as Eq. (2.1), but this is not the case, since the operator $\langle HJ\rangle$ is not Hermitian in general and the χ_a's do not form an orthogonal set.

However, if we perform the transformation[13] in the model space,

$$\chi_a \longrightarrow \varphi_a=(J^\dagger J)^{\frac{1}{2}}\chi_a, \tag{2.7}$$

the φ_a's still remain in the model space and are orthonormal since

$$\begin{aligned}(\varphi_a,\varphi_b)&=((J^\dagger J)^{\frac{1}{2}}\chi_a,(J^\dagger J)^{\frac{1}{2}}\chi_b)=(\chi_a,J^\dagger J\chi_b)\\&=(J\chi_a,J\chi_b)=(\chi_a,\psi_b)=\delta_{ab}.\end{aligned} \tag{2.8}$$

Then Eq. (2.5) is transformed into

$$(E_a-H_M)\varphi_a=0, \tag{2.9}$$

where

$$H_M=(J^\dagger J)^{\frac{1}{2}}\langle HJ\rangle(J^\dagger J)^{-\frac{1}{2}}. \tag{2.10}$$

We may call H_M and φ_a the model Hamiltonian and the model wave function, respectively. The model operator M which transforms the model wave function φ_a into the real nuclear wave function ψ_a is given by

$$\psi_a=M\varphi_a, \quad M=J(J^\dagger J)^{-\frac{1}{2}}. \tag{2.11}$$

We shall show in the next section that H_M can be obtained directly by transforming H by means of the model operator.

It should be noted here that there is still some ambiguity in the definition of φ_a and H_M, since one may further apply any kind of unitary transformation to Eq. (2.7) through Eq. (2.10) without losing the Hermitian character of H_M. This ambiguity will be reduced by imposing the additional condition on the model operator that it should commute with the total angular momentum, the total isotopic spin, and the parity.[14]

3. INTEGRAL EQUATION FOR THE MODEL OPERATOR

Next, we shall set up the integral equation to determine the model operator and the model Hamiltonian. To this end, it is more convenient to consider the operator J, Eq. (2.3), than to directly treat the model operator. While Eden and Francis derived the integral equation of the model operator using the self-consistency requirement, in which the total energy of the system appears explicitly, our equation is exact and all quantities related to the eigensubspace $\mathfrak{R}$ are completely eliminated.

Operating with J from the left on Eq. (2.5), we get

$$E_a\psi_a=J\langle HJ\rangle\chi_a. \tag{3.1}$$

When we compare this equation with Eq. (2.1) and use Eq. (2.3), it follows that

$$(HJ-J\langle HJ\rangle)\chi_a=0. \tag{3.2}$$

Since the χ_a's are assumed to form a complete set in the model space, Eq. (3.2) is equivalent to

$$(HJ-J\langle HJ\rangle)P=0. \tag{3.3}$$

On the other hand, using Eqs. (2.2) and (2.3),

$$PJ=P, \tag{3.4}$$

so that the operator J must have the following form:

$$J=P+(1-P)JP. \tag{3.5}$$

Equation (3.5) may be considered as the boundary condition for the operator J which satisfies Eq. (3.3). If we take into account this boundary condition, we may omit the bracket and the projection operator in Eq. (3.3); thus it may be written as

$$HJ-JHJ=0. \tag{3.6}$$

In order to obtain the symmetric form of H_M, we have only to multiply by $J^\dagger$ from the left on Eq. (3.3), and then divide it by $J^\dagger J$. Then

$$\langle HJ\rangle=\frac{1}{J^\dagger J}(J^\dagger HJ). \tag{3.7}$$

Inserting this into Eq. (2.10), we get

$$H_M=(J^\dagger J)^{-\frac{1}{2}}(J^\dagger HJ)(J^\dagger J)^{-\frac{1}{2}}=M^\dagger HM, \tag{3.8}$$

which clearly shows that H_M is an Hermitian operator.

It is convenient in actual applications to define the model space in terms of some eigenstates of a prescribed Hamiltonian H_0 which is generally different from the model Hamiltonian H_M. In this case P commutes with H_0:

$$[P,H_0]=0. \tag{3.9}$$

If we write

$$H=H_0+W, \tag{3.10}$$

then Eq. (3.6) becomes, when we use the boundary condition (3.5),

$$WJ+[H_0,J]-J\langle WJ\rangle=0. \tag{3.11}$$

Since the diagonal part of Eq. (3.6) or (3.11) is automatically satisfied, we have only to consider its nondiagonal part.[15]

[13] Such a transformation was first considered by Fukuda, Sawada, and Taketani in connection with the nuclear forces in meson theory. See Progr. Theoret. Phys. **12**, 156 (1954). See also S. Okubo, Progr. Theoret. Phys. **12**, 603 (1954).

[14] See reference 8. This condition may be weakened in this case so that $[J_i,M]P=0$, where J_i is any component of the angular momentum or the isotopic spin.

[15] The diagonal part of an arbitrary operator of A is here defined by PAP and the nondiagonal part by $(1-P)AP$.

If we assume that the solution of (3.11) can be obtained by a power series of W, we may write

$$J=\sum_{n=0}^{\infty} J^{(n)}, \tag{3.12}$$

where $J^{(n)}$ are of nth order in W and

$$J^{(0)}=P \tag{3.13}$$

since in the limit $W\rightarrow 0$, J necessarily becomes equal to P. All $J^{(n)}$ with $n>0$ have zero diagonal part on account of the boundary condition (3.6). Inserting Eq. (3.12) into Eq. (3.11), we obtain the following recurrence formula for $J^{(n)}$:

$$[H_0,J^{(n)}]=F^{(n)}(J^{(1)},\cdots,J^{(n-1)}), \tag{3.14}$$

where $F^{(n)}$ is a function of the $J^{(i)}$ with $i<n$. Since it is sufficient to consider only the nondiagonal part of this equation, we can obtain the unique solution of $J^{(n)}$ in terms of $J^{(i)}$ (with $i<n$) only if the eigenvalues of H_0 belonging to the model space are different from the eigenvalues belonging to eigenfunctions outside the model space. This seems to suggest the uniqueness of the solution of Eq. (3.6) under the boundary condition (3.5).

4. FORMAL RELATION TO THE SCATTERING PROBLEM

Here we shall discuss the relation of the integral equation derived in the last section to the scattering problem. Let us assume that the total Hamiltonian of the nucleus is divided into two parts just as in Eq. (3.10) and that the eigenvalues and eigenfunctions of the "unperturbed" Hamiltonian H_0 are denoted by ϵ_a and ϕ_a, respectively. If we enclose the whole system in a sphere of radius R, the unperturbed energy levels are always shifted compared to the exact eigenvalues by some amount in the scattering states.

If Δ_a is defined by

$$\Delta_a=E_a-\epsilon_a, \tag{4.1}$$

and the operator Δ by

$$\Delta=\sum_a \Delta_a P_a, \tag{4.2}$$

where P_a is the projection operator to ϕ_a, then

$$E_a\phi_a=(H_0+\Delta)\phi_a. \tag{4.3}$$

We may now write

$$H=(H_0+\Delta)+(W-\Delta), \tag{4.4}$$

and applying the general method of Lippmann and Schwinger,[16] we obtain the outgoing wave solution as follows:

$$\psi_a^{(+)}=\phi_a+\frac{1}{E_a-H_0-\Delta+i\eta}(W-\Delta)\psi_a^{(+)}. \tag{4.5}$$

[16] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

If we define the wave matrix Ω as

$$\psi_a^{(+)}=\Omega\phi_a, \tag{4.6}$$

and operate with Ω from the left on Eq. (4.3), we have

$$E_a\psi_a^{(+)}=\Omega(H_0+\Delta)=\phi_a. \tag{4.7}$$

Comparing this with the original Schrödinger equation

$$E_a\psi_a^{(+)}=\{(H_0+\Delta)+(W-\Delta)\}\Omega\phi_a, \tag{4.8}$$

we get the following integral equation for Ω:

$$(W-\Delta)\Omega+[(H_0+\Delta),\Omega]=0,$$

or

$$W\Omega+[H_0,\Omega]-\Omega\Delta=0, \tag{4.9}$$

since $\{\phi_a\}$ may be assumed to be complete.

On the other hand, if we take the scalar products of Eqs. (4.7) and (4.8) with ϕ_a and subtract the equations, we have

$$\Delta_a(\phi_a,\Omega\phi_a)=(\phi_a,W\Omega\phi_a) \tag{4.10}$$

The diagonal elements of the wave matrix can be estimated from (4.5) and are shown to be equal to 1, if one disregards the terms of the order of $1/R^2$ which come from the second term.[17] Therefore,[18]

$$\Delta=\sum_a\Delta_a P_a=\sum_a P_a W\Omega P_a=\langle W\Omega\rangle. \tag{4.11}$$

It should be noted that Eqs. (4.9) and (4.11) were first derived by Tanaka[12] from the time-dependent formulation of scattering theory.

In actual application the left-hand side of Eq. (4.9) may be assumed to operate on a particular state ϕ_a, so that the one term of Δ corresponding to this state contributes to the integral equation. Alternatively, we may omit the bracket in the expression (4.11) for Δ in Eq. (4.9) since the bracket automatically follows from the boundary condition on Ω. Thus it is shown that the integral equation (3.11) is formally equivalent to the scattering Eq. (4.9), though different boundary conditions are specified in each case.

5. INDEPENDENT-PARTICLE MODEL

As an illustration of our formalism, we will consider a generalization of the Hartree-Fock approximation. The model space $\mathfrak{M}$ is defined in this case by the m-dimensional eigensubspace which belongs to the lowest m eigenvalues of the Hamiltonian

$$H_0=T+U,\quad T=\sum_{i=1}^{A} T_i,\quad U=\sum_{i=1}^{A} U_i, \tag{5.1}$$

where A denotes the number of nucleons present, T_i

[17] As is well known, the asymptotic form of the second term of Eq. (4.5) is $[\exp(ikr)/r]f(\theta,\varphi)$, which is sufficient to determine the R dependence of its scalar product with ϕ_a.

[18] If we note that $\Omega^\dagger\Omega=1$ and operate with $\Omega^\dagger$ from the left on Eq. (4.9), we get $\Delta=\langle\Omega^\dagger H\Omega\rangle-H_0$, which clearly shows the Hermitian character of the self-energy operator.

and U_i, the kinetic and potential energy of the ith particle, respectively; the form of U_i being common to all particles. This potential energy may be specified *arbitrarily*; for instance, we may use the phenomenological one which is associated with the shell model. This is in contrast to the Hartree-Fock approximation where the potential must be determined in a self-consistent way by applying the variational principle.

We denote the eigenvalues and orthonormal eigenfunctions of H_0 by ϵ_a and ϕ_a, respectively, which lie in the model space, i.e.,

$$(\epsilon_a - H_0)\phi_a = 0, \quad a = 1,2,\cdots,m, \tag{5.2}$$

where ϕ_a is given by a Slater determinant of single-particle wave functions, or some linear combination of them so as to be an eigenstate of the total angular momentum and isotopic spin. Then the χ_a defined by Eq. (2.2) may be expanded as[19]

$$\chi_a = \sum_{b=1}^{m} \alpha_{ba}\phi_b, \quad a = 1,2,\cdots,m \tag{5.3}$$

with expansion coefficients α_{ba}. Substituting Eq. (5.3) into Eq. (2.5) and taking the scalar product with ϕ_b, we have

$$E_a\alpha_{ba} - \sum_c \langle b|HJ|c\rangle \alpha_{ca} = 0. \tag{5.4}$$

If m tends to infinity and the model space coincides with the whole Hilbert space, then J becomes necessarily equal to 1. We may get, therefore, a good approximation by putting J equal to P, if we choose m sufficiently large. In this case Eq. (5.4) may be written as

$$E_a\alpha_{ba} - \sum_c \langle b|H|c\rangle \alpha_{ca} = 0, \tag{5.5}$$

which is nothing but the equation of perturbation theory for the degenerate case. Note in particular that if we choose m equal to 1 (ground state problem) and replace J by P, the diagonal element of H gives the usual perturbed energy. This approximation appears to be poor. We can, however, improve the situation by choosing the most appropriate H_0 (or corresponding wave function ϕ) by minimizing the expectation value of H, i.e., by applying the variational principle

$$\delta \frac{(\phi, H\phi)}{(\phi,\phi)} = 0, \tag{5.6}$$

with ϕ being an arbitrary antisymmetrized product function. This is nothing but the conventional Hartree-Foch approximation. Eden[20] obtained a generalized Hartree-Foch approximation by applying this principle to the model Hamiltonian; however, it was not the H_M of Eq. (2.10) or Eq. (3.8), but corresponds to $\langle HJ\rangle$ of Eq. (2.6) which is not Hermitian in general.

Great simplification is attained if $\langle HJ\rangle$ turns out to be Hermitian, which is equivalent to the condition that $J^\dagger J$ commutes with $(J^\dagger HJ)$, as is clear from Eq. (3.7). This is always the case if, for instance, the model space is one-dimensional. Then $\langle HJ\rangle$ is equal to H_M and Eq. (3.6) may be written as

$$H_0J + WJ - JH_M = 0. \tag{5.7}$$

H_M is in general different from H_0, but if we impose the condition $H_M = H_0$, the following self-consistent equation is obtained:

$$[H_0,J] + WJ = 0, \quad \langle WJ\rangle = 0. \tag{5.8}$$

The second equation follows from the first, since

$$\langle [H_0,J]\rangle = 0. \tag{5.9}$$

W is also written as

$$W = V - \textstyle\sum_i U_i, \tag{5.10}$$

where we use the real nuclear potential V, so that Eq. (5.8) may be taken as a generalization of the self-consistent equation of Eden and Francis.[9]

If J operates on a particular state χ_a, Eq. (5.7) may be written in the form

$$J = 1 + \frac{1}{E_a - H_0}(1-P)WJ, \tag{5.11}$$

which is always correct if m is equal to 1. Thus it is shown that the theory of Eden and Francis is a special case of ours.

6. CONCLUDING REMARKS

A formal theory of nuclear model operators has been presented which makes no explicit use of any quantities connected with the real nuclear wave functions. The model space may be chosen arbitrarily, depending on the model we wish to consider. However, it seems to be convenient in actual applications to define this space in terms of a phenomenological Hamiltonian which is well established in the corresponding nuclear model. The difference between this Hamiltonian and the exact model Hamiltonian may be treated by perturbation theory, as long as this model works well with respect to the level structure of the nucleus.

It was also shown that the theory of Eden and Francis is derived as a special case, i.e., when the model space is one dimensional. Their theory will not be suitable, however, for dealing with the excited states of the nucleus, since the self-consistent formulation usually depends on the variational principle which is applicable only to the ground state. In our theory the model wave function is always simple, whereas the model Hamiltonian may be of a complicated form in

[19] It should be noted that only those terms will appear in Eq. (5.3) which have the same transformation property as ψ_a with respect to rotations in ordinary space and isotopic spin space.

[20] R. J. Eden, Phys. Rev. **99**, 1418 (1955).

order to reproduce the level structure of the real nucleus; in our opinion, nuclear models should thus be described in terms of the wave function.

A concrete calculation has not been made in this paper, since our primary concern was how to define and construct the model operators in various nuclear models. Such a calculation will be done at another opportunity.

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Relativistic Theory of Polarization Phenomena

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A covariant form of the polarization formalism of Wolfenstein and Ashkin is developed. After the hole-theory boundary conditions are incorporated, the theory may be transformed into a form in which the positive- and negative-energy components are separated. This form involves two-by-two matrices of the Pauli type and its similarity to the form of the nonrelativistic equations allows the relativistic contributions to be extracted. It is concluded that, with suitable interpretations, the nonrelativistic formalism may be used if an additional rotation of the polarization vector is added at each scattering. The relativistic forms of the Wolfenstein equations for various polarization parameters are then derived.

INTRODUCTION

IN recent experiments at Berkeley[1] and elsewhere,[2] the spin dependence of the nucleon-nucleon interaction has been investigated by scattering beams of nucleons in which the spin directions have been partially aligned by means of previous scattering processes. The analysis of these polarization experiments has been discussed by several authors,[3–5] and the more recent of these discussions are based on the polarization formalism introduced by Wolfenstein and Ashkin. Except for a brief note by Michel and Wightman[6] and the early work of Mott[7] on the double scattering of electrons by fixed source centers, the treatments have been based upon the nonrelativistic Pauli approximation. However, since the present cyclotron energies are within the relativistic range and because of the increased energies now becoming available, it is of importance to extend the polarization formalism into the relativistic region. Such an extension is the object of this paper, and a completely covariant formalism for the description of polarization phenomena in the collisions of relativistic particles is developed.

In the first two sections, the covariant forms of the S matrix and the density matrix for the collision of a Dirac particle with a finite-mass spin-zero particle are obtained and the restrictions upon these forms implied by the hole-theory boundary conditions are imposed. In the third section, these forms are used in the manner developed by Wolfenstein and Ashkin for the nonrelativistic treatment of polarization phenomena and a covariant polarization formalism is obtained. This formalism may be transformed into a relativistic but noncovariant form which separates the scattering of positive and negative energy particles into two distinct parts, each of which is expressed in terms of a form which involves two-by-two matrices and which is quite similar to that obtained in the nonrelativistic treatment. The relativistic effects may be extracted by comparing the theory in this form to the nonrelativistic theory. It is shown that the relativistic effects modify the nonrelativistic formulas by the effects of an additional rotation of the polarization vector, a rotation whose magnitude depends upon the scattering angle in a manner which is explicitly exhibited. The effects are of order $(\gamma-1)$, where γ is the relativistic contraction factor for the Dirac particle as seen in the center-of-mass frame.

In the fourth section the method is extended to the

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