Perturbation Calculation of the Inelastic Scattering of Electrons by Hydrogen Atoms*

SIDNEY BOROWITZ AND MILTON M. KLEIN†
Department of Physics, College of Engineering, New York University, New York, New York
(Received December 14, 1955)

A perturbation calculation has been made of the inelastic scattering of fast electrons by hydrogen atoms to the $2S$ and $2P$ states, using for the perturbation only the interaction between the incident electron and the bound electron. The results are then compared, to leading order terms in the energy, with calculations by the more customary perturbation scheme. For direct scattering, the results for forward scattering amplitudes are identical using either procedure. The angular distribution and energy dependence are different. For exchange scattering, the symmetric and asymmetric perturbations give different results both in the forward direction and at angles other than zero. However, in the former scheme exchange scattering is negligible compared to the direct scattering. The calculations for direct scattering are shown to be simplified by using a somewhat different perturbation scheme (method of altered states).

1. INTRODUCTION

RECENTLY a comparison has been made between the results obtained for the elastic scattering of electrons by hydrogen atoms at high energies using two different Born approximations: (1) the interaction between the electrons is taken as the perturbation (symmetric perturbation),[1] and (2) the interaction between the electrons and the atom is the perturbation (asymmetric perturbation).[2,3] However, the difference between these results was found to be too small for a decisive experiment to be possible. In this paper we have undertaken a similar theoretical comparison of the results obtained by the two schemes for inelastic scattering to the $n=2$ state of the hydrogen atom. As anticipated, the difference in these results is much larger, and an experimental measurement of differential scattering cross section is possible in principle and could determine which perturbation procedure is more accurate.

In Sec. 2, we give the integrals which must be evaluated. In Secs. 3 and 5, we discuss the direct and exchange scattering to the $2S$ state. Sections 6 and 7 are devoted to the scattering to the $2P$ state. Section 4 is concerned with a simplified method, which we call the method of altered states for evaluating the direct scattering cross sections.

2. FORMULATION OF THE PROBLEM

The notation and system of units is the same as in those used in I. However, since we now have two possible final states, we shall denote the direct- and exchange-scattered amplitudes by f and g, respectively, for $2S$ scattering, and by f' and g' for $2P$ scattering.

The scattered amplitudes for $2S$ scattering are given by[4]

$$f_a=-\frac{\sqrt{2}}{16\pi^2}\int \exp[i(\mathbf{k}_0-\mathbf{k}_n)\cdot\mathbf{r}_1]\times\exp[-\tfrac{3}{2}r_2](2-r_2)\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right)d\tau_1 d\tau_2, \quad (2.1)$$

$$g_a=-\frac{\sqrt{2}}{16\pi^2}\int \exp[i\mathbf{k}_0\cdot\mathbf{r}_1-i\mathbf{k}_n\cdot\mathbf{r}_2]\times\exp[-\tfrac{1}{2}r_1+r_2](2-r_1)\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right)d\tau_1 d\tau_2, \quad (2.2)$$

and

$$f_s=\frac{\sqrt{2}}{16\pi^2}\int \exp[i(\mathbf{k}_0-\mathbf{k}_n)\cdot\mathbf{r}_1]\times\exp[-\tfrac{3}{2}r_2](2-r_2)\ {}_1F_1(-n_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1))\frac{1}{r_{12}}\times{}_1F_1(-n_2,1,i(k_nr_2+\mathbf{k}_n\cdot\mathbf{r}_2))d\tau_1 d\tau_2, \quad (2.3)$$

$$g_s=\frac{\sqrt{2}}{16\pi^2}\int \exp[i\mathbf{k}_0\cdot\mathbf{r}_1-i\mathbf{k}_n\cdot\mathbf{r}_2]\exp[-(\tfrac{1}{2}r_1+r_2)]\times(2-r_1)\frac{1}{r_{12}}\ {}_1F_1(-n_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1))\times{}_1F_1(-n_2,1,i(k_nr_2+\mathbf{k}_n\cdot\mathbf{r}_2))d\tau_1 d\tau_2, \quad (2.4)$$

where $\mathbf{k}_0$ and $\mathbf{k}_n$ are the propagation vectors of the electrons in the initial and final states, and

$$n_1=1/ik_0, \quad n_2=1/ik_n.$$

The ${}_1F_1$ are confluent hypergeometric functions.[5] The magnitude of k_n is obtained for given k_0, from conservation of energy

$$k_0^2-k_n^2=\tfrac{3}{4}. \quad (2.5)$$

* The research reported in this article was done at the Institute of Mathematical Sciences, New York University under the sponsorship of the Geophysics Research Directorate, Air Force Cambridge Research Center, and in part at the Physics Department, College of Engineering, New York University, under sponsorship of the Office of Naval Research.
† Present address: General Electric Company, Schenectady, New York.
[1] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1949), second edition, Chap. VIII, Sec. 2.
[2] S. Borowitz, Phys. Rev. **96**, 1523 (1954), hereafter referred to as I.
[3] E. Corinaldesi and L. Trainor, Nuovo cimento **9**, 940 (1952).
[4] We have taken the normalization of the continuum Coulomb functions as 1 since that is their limit for $k\to\infty$, which is the only case we discuss.
[5] Reference 1, Chap. III.

For the case of $2P$ scattering, there are three $2P$ states corresponding to the values $m=0, \pm 1$, where m is the magnetic quantum number. Only the states corresponding to the value $m=0$ need be considered because for $m=\pm 1$ the scattered amplitude is zero. The direct and exchange scattered amplitudes for $2P$ scattering are then given by

$$f_a'=-\frac{\sqrt{2}}{16\pi^2}\int \exp[i(\mathbf{k}_0-\mathbf{k}_n)\cdot\mathbf{r}_1] \times\exp[-\tfrac{3}{2}r_2]r_2\cos\theta_2\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right)d\tau_1 d\tau_2, \quad (2.6)$$

$$g_a'=-\frac{\sqrt{2}}{16\pi^2}\int \exp[i\mathbf{k}_0\cdot\mathbf{r}_1-i\mathbf{k}_n\cdot\mathbf{r}_2] \times\exp[-(\tfrac{1}{2}r_1+r_2)]r_1\cos\theta_1\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right)d\tau_1 d\tau_2, \quad (2.7)$$

and

$$f_s'=\frac{\sqrt{2}}{16\pi^2}\int \exp[i(\mathbf{k}_0-\mathbf{k}_n)\cdot\mathbf{r}_1]\exp[-\tfrac{3}{2}r_2] \times r_2\cos\theta_2\frac{1}{r_{12}}\,{}_1F_1(-n_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1)) \times{}_1F_1(-n_2,1,i(k_nr_1+\mathbf{k}_n\cdot\mathbf{r}_1))d\tau_1 d\tau_2, \quad (2.8)$$

$$g_s'=\frac{\sqrt{2}}{16\pi^2}\int \exp[i\mathbf{k}_0\cdot\mathbf{r}_1-i\mathbf{k}_n\cdot\mathbf{r}_2]\exp[-(\tfrac{1}{2}r_1+r_2)] \times r_1\cos\theta_1\frac{1}{r_{12}}\,{}_1F_1(-n_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1)) \times{}_1F_1(-n_2,1,i(k_nr_2+\mathbf{k}_n\cdot\mathbf{r}_2))d\tau_1 d\tau_2. \quad (2.9)$$

3. DIRECT 2S SCATTERING

We consider now the evaluation of the integral (2.3) for f_s. Expanding $1/r_{12}$ in a series of Legendre polynomials[6] and integrating with respect to $d\tau_2$, we obtain

$$f_s=-\frac{2\sqrt{2}}{27\pi}\frac{\partial I}{\partial\lambda}+\frac{\sqrt{2}}{9\pi}\frac{\partial^2 I}{\partial\lambda^2}\bigg|_{\lambda=\frac{3}{2}}, \quad (3.1)$$

where

$$I=\int \exp[i\mathbf{q}\cdot\mathbf{r}_1]\exp[-\lambda r_1]\frac{1}{r_1} \times{}_1F_1(ia_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1)) \times{}_1F_1(ia_2,1,i(k_nr_1+\mathbf{k}_n\cdot\mathbf{r}_1))d\tau_1, \quad (3.2)$$

$$a_1=1/ik_0, \quad a_2=1/ik_n,$$

and $\mathbf{q}=\mathbf{k}_0-\mathbf{k}_n$. The integral I has been evaluated by Nordsieck[7]:

$$I=\frac{2\pi}{\alpha}\exp[-\pi a_1]\left(\frac{\alpha}{\gamma}\right)^{ia_1}\left(\frac{\gamma+\delta}{\gamma}\right)^{-ia_2} \times F(1-ia_1, ia_2,1,x), \quad (3.3)$$

where

$$\alpha=\tfrac{1}{2}(q^2+\lambda^2), \qquad \beta=\mathbf{k}_n\cdot\mathbf{q}-i\lambda k_n,$$
$$\gamma=\mathbf{k}_0\cdot\mathbf{q}+i\lambda k_0-\alpha, \quad \delta=k_0k_n+\mathbf{k}_0\cdot\mathbf{k}_n-\beta, \quad (3.4)$$

$$x=\frac{\alpha\delta-\beta\gamma}{\alpha(\gamma+\delta)}, \quad (3.5)$$

and F is the hypergeometric function of its argument.

The quantity x in (3.5) will generally be of order of magnitude unity in our calculations. It will be convenient, therefore, to define a new variable

$$y=1-x=\gamma(\alpha+\beta)/\alpha(\gamma+\delta), \quad (3.6)$$

which will generally be small compared to unity. Using (3.4), we may express the quantity y in the form

$$y=[(k_0-k_n)^2+\lambda^2]/(q^2+\lambda^2). \quad (3.7)$$

Inspection of (3.7) shows y is generally of order $1/k_0^2$ except when θ, the angle between $\mathbf{k}_0$ and $\mathbf{k}_n$, is close to zero. The case $\theta=0$ will be considered separately.

The hypergeometric functions F of the variable x can now be expressed in terms of hypergeometric functions of the variable y.[8] We shall be only interested in the approximate form of the hypergeometric function for large k_0. Noting that $a_1\sim 1/k_0$, $a_1-a_2\sim 1/k_0^3$, $\Gamma(z)\cong 1/z$ for small z, we obtain

$$F=-\frac{a_1}{b}F_1+\left(1+\frac{a_1}{b}-ib\ln y-ia_1\ln y\right)F_2, \quad (3.8)$$

where $b=a_1-a_2$. Since the argument y is small, the hypergeometric functions on the right-hand side of (3.8) may be expanded by means of the standard series for the hypergeometric function.[9] Equation (3.8) then becomes, if we retain only leading order terms in k,

$$F=1-a_1a_2y-ia_2\ln y-a_1a_2y\ln y-a_1a_2y^2 +\tfrac{1}{2}(a_1a_2)y^2\ln y. \quad (3.9)$$

We see that the leading order terms in $\partial F/\partial\lambda$ and $\partial^2F/\partial\lambda^2$ come from the first term in $\ln y$; hence we may write (3.9) in the approximate form

$$F=1-ia_2\ln y. \quad (3.10)$$

We now evaluate the first and second derivatives of F with respect to λ. The calculation shows that the terms involving the derivatives of the coefficient of F

[6] L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), first edition, p. 173.

[7] A. Nordsieck, Phys. Rev. **93**, 785 (1954).

[8] E. J. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1946), fourth edition, p. 281.

[9] See reference 8, p. 281.

are small compared to the derivatives of F. Neglecting these terms, we obtain[10] for f_s:

$$f_s = c\frac{32\sqrt{2}ik_0(1-\cos\theta)}{81\pi}\frac{(q^2+9/2)}{(q^2+9/4)^2}, \tag{3.11}$$

where $c=2\pi/\alpha \exp[-\pi a_1](\alpha/\gamma)^{ia_1}(\gamma+\delta/\gamma)^{-ia_2}$. For high energies

$$c \cong \frac{2\pi}{\alpha} = \frac{4\pi}{(q^2+9/4)},$$

$$q^2 \cong 2k_0^2(1-\cos\theta) = (2k_0 \sin\tfrac{1}{2}\theta)^2,$$

so that f_s has the limiting form

$$f_s = \frac{16}{81}\left(\frac{\sqrt{2}i}{k_0^3\mu}\right), \tag{3.12}$$

with $\mu = \sin^2(\frac{1}{2}\theta)$.

The scattering amplitude for $2S$ scattering f_a has been evaluated by Corinaldesi and Trainor,[3] who give

$$f_a = \frac{8\sqrt{2}}{(q^2+9/4)^3} = \frac{1}{8}\left(\frac{\sqrt{2}}{k_0^6\mu^3}\right). \tag{3.13}$$

This differs from f_s in both energy and angular dependence.

Small Scattering Angles

The calculation of the scattering amplitude for small scattering angle θ requires an additional analysis because the orders of magnitude of the various terms involved change. This is most easily done by specializing (3.3) to the case $\theta=0$. The principal contribution to the scattered amplitude comes from the terms in I involving the derivatives of $2\pi/\alpha$. For $x=0$, the function $F=1$ and the coefficients other than $2\pi/\alpha$ are approximately equal to unity. The scattered amplitude may therefore be obtained from the approximate form of (3.3):

$$I = \frac{2\pi}{\alpha} = \frac{4\pi}{q^2+\lambda^2} \cong \frac{4\pi}{\lambda^2}. \tag{3.14}$$

Substituting the values of $\partial I/\partial\lambda$ and $\partial^2 I/\partial\lambda^2$ into (3.1) yields

$$f_s = \frac{8\sqrt{2}}{(q^2+9/4)^3} \cong \frac{512\sqrt{2}}{729}.$$

This value of f_s, with q^2 retained, is exactly the same as that given by (3.13) for f_a. It is of mathematical interest to note that the approximate from (3.14) of the fundamental integral I is the same result obtained by the use of the asymmetric perturbation.

4. METHOD OF ALTERED STATES

The symmetric perturbation scheme involves considerable mathematical difficulties because the integrals to be evaluated contain the product of two hypergeometric functions. It is desirable, therefore, to consider possible approximations by which the calculations may be simplified without introducing too much error.

The work of Schwebel[11] suggests that an excellent approximation of the symmetric perturbation scheme would be to replace the final-state wave function by a plane wave. That this approximation should not essentially change the results of the more exact solution may be seen as follows.

The solution of the Schrödinger equation is the sum of the unperturbed wave function Ψ_0 and the perturbed or scattered wave Ψ_s. The function Ψ_0 satisfies the equation

$$H_0\Psi_0 = 0, \tag{4.1}$$

where

$$H_0 = \Delta_1 + \Delta_2 + 2\left(E + \frac{1}{r_1} + \frac{1}{r_2}\right).$$

Ψ_s then satisfies the equation

$$H_0\Psi_s = \frac{2}{r_{12}}(\Psi_0 + \Psi_s). \tag{4.2}$$

We now rewrite (4.2) as follows:

$$H'\Psi_s = \frac{2}{r_{12}}\Psi_0 + 2\left(\frac{1}{r_{12}} - \frac{1}{r_1}\right)\Psi_s, \tag{4.3}$$

where $H' = H_0 - (2/r_1)$. Neglecting the term in Ψ_s on the right-hand side of (4.3) and using the Green's function appropriate to the operator H', we obtain integrals similar to the integrals I_1, I_2, except that the Coulomb wave function for the final state has been replaced by plane waves.[12] Since our approximation involves only the neglect of a term containing the perturbation Ψ_s, the method of altered states should give the same order of accuracy as the more exact solution.

We consider now the application of the method of altered states to $2S$ direct scattering. The integral to be evaluated is

$$f_s = \frac{\sqrt{2}}{16\pi^2}\int \exp[i\mathbf{q}\cdot\mathbf{r}_1]\exp[-\tfrac{3}{2}r_2](2-r_2) \times {}_1F_1(n_1, 1, i(k_0 r_1 - \mathbf{k}_0\cdot\mathbf{r}_1))\frac{1}{r_{12}}d\tau_1 d\tau_2. \tag{4.4}$$

The integration with respect to τ_2 can be done immediately. The integration with respect to τ_1 can be done

[10] A more detailed evaluation of this and subsequent integrals can be found in S. Borowitz and M. M. Klein, Research Report No. CX-22, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research (unpublished).

[11] S. L. Schwebel, Research Report No. CX-15, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research.

[12] S. Borowitz and B. Friedman, Phys. Rev. 89, 441 (1953).

by expressing ${}_1F_1$ as a contour integral in the complex plane[8]:

$$ {}_1F_1(-n_1, 1, i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1)) $$
$$ =\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\exp[-i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1)v]\frac{dv}{v}, \quad (4.5) $$

where the contour in the v-plane encloses the point 0 and -1.

Using this method, we can write (4.4) as

$$ f_s=(8/9)\sqrt{2}(\tfrac{2}{3}ik_0\,{}_0L_2+9\,{}_1L_3+12k_0\,{}_0L_3-4k_0^2\,{}_{-1}L_3), \quad (4.6) $$

where

$$ {}_rL_s=\frac{1}{2\pi i}\oint\frac{[1+(1/v)]^{n_1}dv}{(A+Bv)^s v^r}. \quad (4.7) $$

In (4.7),

$$ A=q^2+9/4,\quad B=2(k_0^2-\mathbf{k}_0\cdot\mathbf{k}_n+\tfrac{3}{2}ik_0). $$

The integrals ${}_rL_s$ may be evaluated by replacing the contour around the points $0, -1$ by a new contour enclosing the singularity of the denominator $(A+Bv)$. This procedure is justified because the integrals ${}_rL_s$ are of order $1/v^2$ or smaller in the neighborhood of infinity. We utilize Cauchy's theorem to evaluate our integrals, and note that a minus sign is introduced by use of the new contour. For large k_0, a common factor $(1-B/A)^{n_1}$ which occurs is taken equal to unity.[13] The principal terms in the expression (4.6) for f_s are contributed by ${}_0L_2$ and ${}_{-1}L_3$. We evaluate f_s neglecting all but these terms; we use the values of A and B in (4.7), noting that $n_1=1/ik_0$, and retaining only leading order terms; then we have as the limiting form

$$ f_s=\frac{8}{81}\left(\frac{\sqrt{2}i}{k_0^3\mu}\right), \quad (4.8) $$

a result in good agreement with (3.12).

It is of interest to see whether the method of altered states will yield the correct result for $\theta=0$. The principal contribution to f_s comes from the term in ${}_1L_3$ which does not contain $(A-B)$. Neglecting all but this term in (4.6) we have

$$ f_s=\frac{8\sqrt{2}}{A^3}=\frac{8\sqrt{2}}{(q^2+9/4)^3}, $$

in agreement with (3.15).

5. EXCHANGE 2S SCATTERING

The exchange-scattered amplitude g_s is given by (2.4). In order to evaluate this integral, we first express $1/r_{12}$ as a Fourier integral:

$$ \frac{1}{r_{12}}=\frac{1}{2\pi^2}\int\frac{\exp[i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)]}{p^2}d\tau_p. \quad (5.1) $$

The hypergeometric functions occurring in (2.4) are then replaced by contour integrals in the complex plane [see (4.5)], and (2.4) takes the form

$$ g_s=\frac{\sqrt{2}}{32\pi^4}\int\frac{d\tau_p}{p^2}J^{(1)}J^{(2)}, \quad (5.2) $$

where

$$ J^{(m)}=\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}J_m, \quad (5.3) $$

and

$$ J_1=\int\exp[-(\tfrac{1}{2}+ik_0v)r_1]\exp[i\mathbf{K}_1\cdot\mathbf{r}_1](2-r_1)d\tau_1, $$
$$ J_2=\int\exp[-(1+ik_nu)r_1]\exp[-i\mathbf{K}_2\cdot\mathbf{r}_2]d\tau_2; \quad (5.4) $$

here

$$ \mathbf{K}_1=(1+v)\mathbf{k}_0+\mathbf{p},\quad \mathbf{K}_2=(1+u)\mathbf{k}_n+\mathbf{p}. \quad (5.5) $$

The contour variable u has been used to correspond to $\mathbf{r}_2$ and the contour variable v to correspond to $\mathbf{r}_1$.

The integrals J_1 and J_2 may be simply evaluated:

$$ J_1=16\pi\left[\frac{(1+ik_0v)}{(A_1+B_1v)^2}-\frac{2(\tfrac{1}{2}+ik_0v)^2}{(A_1+B_1v)^3}\right], $$
$$ J_2=8\pi\frac{(1+ik_nu)}{(A_2+B_2u)^2}, \quad (5.6) $$

where

$$ A_1=p^2+k_0^2+2\mathbf{k}_0\cdot\mathbf{p}+\tfrac{1}{4},\quad B_1=2(k_0^2+\mathbf{k}_0\cdot\mathbf{p}+\tfrac{1}{2}ik_0), \quad (5.7) $$
$$ A_2=p^2+2\mathbf{k}_n\cdot\mathbf{p}+k_n^2+1,\quad B_2=2(k_n^2+\mathbf{k}_n\cdot\mathbf{p}+ik_n). \quad (5.8) $$

Substitution of (5.6) and (5.3) yields

$$ J^{(1)}=16\pi({}_1L_1'+ik_0\,{}_0L_2'-\tfrac{1}{2}\,{}_1L_3'-2ik_0\,{}_0L_3'+2k_0^2\,{}_{-1}L_5'), $$
$$ J^{(2)}=8\pi({}_1L_1''+ik_n\,{}_0L_2''), \quad (5.9) $$

where the single prime is used to denote the fact that the quantities A, B, and n_1 occurring in the ${}_1L_3$ integrals of (4.7) are A_1, B_1, and n_1; the double prime means that A, B, and n_1 are replaced by A_2, B_2, and n_2. The integrals are evaluated by the method of Sec.

[13] S. Borowitz, Phys. Rev. 96, 1527 (1954).

4, and give for g_s in (5.2):

$$g_s=\frac{\sqrt{2}}{\pi^2}\{(1+n_2)[4(1+n_1)E_{00}{}^{22}-(2+3n_1+n_1{}^2)E_{00}{}^{32}$$
$$+4(1-n_1)E_{10}{}^{12}-2(1+n_1)(2-n_1)E_{10}{}^{22}$$
$$+(4ik_0-8+5n_1-n_1{}^2)E_{20}{}^{12}]+(1-n_2)$$
$$\times[4(1+n_1)E_{01}{}^{21}-(2+3n_1+n_1{}^2)E_{01}{}^{31}$$
$$+4(1-n_1)E_{11}{}^{11}-2(1+n_1)(2-n_1)E_{11}{}^{21}$$
$$+(4ik_0-8+5n_1-n_1{}^2)E_{21}{}^{11}]\}, \quad (5.10)$$

where

$$E_{rs}{}^{pq}=\int\frac{d\tau_p/p^2}{A_1{}^pA_2{}^q(A_1-B_1)^r(A_2-B_2)^s}. \quad (5.11)$$

We shall evaluate (5.10) for high values of k_0 and will therefore consider only leading order terms. Because of singularities occurring in the integrals in (5.11), their order of magnitude with respect to k_0 can not be obtained from simple dimensional considerations. Experience gained in working with these integrals shows that the leading order terms are contributed by the integrals $E_{10}{}^{12}$, $E_{20}{}^{12}$, $E_{01}{}^{21}$, and $E_{01}{}^{31}$. Equations (5.10) may then be written

$$g_s=(\sqrt{2}/\pi^2)(4E_{10}{}^{12}+4ik_0E_{20}{}^{12}+4E_{01}{}^{21}-2E_{01}{}^{31}). \quad (5.12)$$

The integrals in (5.12) may be evaluated by a method due to Feynman.[14,2] The results for the integrals for g_s are

$$g_s=\frac{173\sqrt{2}i}{16k_0{}^3q^2}\cong\frac{173\sqrt{2}i}{64k_0{}^5\mu}. \quad (5.13)$$

The exchange scattering amplitude g_a [see (2.2)] has been evaluated exactly by Corinaldesi and Trainor.[3] The approximate form of their result for large k_0, expressed in our notation, is

$$g_a=-\frac{\sqrt{2}}{2k_0{}^6}\left(8-\frac{1}{\mu^2}\right). \quad (5.14)$$

This result differs from that for the symmetric case in both energy dependence, k_0 and angular dependence, μ.

It is of interest to note that the part of the asymmetric scattering g_a corresponding to $1/r_{12}$ is furnished by the integrals $E_{00}{}^{22}$ and $E_{00}{}^{32}$, since these integrals are only ones which would occur if plane-wave functions were used with the symmetric perturbation. Direct evaluation of these integrals shows, however, that they are of order $1/k_0{}^6\mu^2$ [see (5.14)] and are therefore negligible compared to the terms we have evaluated.

If the method of altered states is used for $2S$ exchange scattering, we do not obtain all of the leading order terms. Because of the separation of variables occurring in exchange scattering, no essential simplification is obtained, but only a reduction in the number of integrals to be computed. The method of altered states will not, therefore, be considered further in exchange scattering.

Small Scattering Angles

An examination of the orders of magnitude of the integrals occurring in (5.10) for $\theta=0$ shows that now $E_{00}{}^{22}$ and $E_{00}{}^{32}$ are of order $1/k_0{}^2$ while the integrals previously considered are now of order $1/k_0{}^3$. It would thus appear that, as in direct scattering, the plane wave terms become dominant for small θ. A further examination of $E_{00}{}^{22}$ and $E_{00}{}^{32}$ shows, however, that these terms cancel to order $1/k_0{}^2$ and that the next leading order term is of order $1/k_0{}^4$. The plane-wave terms thus appear to play a negligible role in exchange scattering for both large and small scattering angles.

Examination of the integrals in (5.11) for $\theta=0$ shows that the four integrals previously considered are leading order terms but that, in addition, the integral $E_{10}{}^{22}$ becomes of comparable order of magnitude. The appropriate form of (5.10) for $\theta=0$ is thus

$$g_s=(\sqrt{2}/\pi^2)(4E_{10}{}^{12}-4E_{10}{}^{22}+4ik_0E_{20}{}^{12}+4E_{01}{}^{21}-2E_{01}{}^{31}). \quad (5.15)$$

This gives

$$g_s|_{\theta=0}=(47\sqrt{2}/16)(i/k_0{}^3). \quad (5.16)$$

The result given by Corinaldesi and Trainor for the asymmetric perturbation, when evaluated for the limiting case $\theta=0$, has the form

$$g_a|_{\theta=0}=(40\sqrt{2}/81)(1/k_0{}^4). \quad (5.17)$$

The scattering amplitude given by the asymmetric method for $2S$ exchange scattering is therefore negligible compared to that given by the symmetric method for small as well as large scattering angles. This is in contrast to direct scattering where the two methods give similar results for small scattering angles.

6. DIRECT 2P SCATTERING

The scattering amplitude f_s' for direct $2P$ scattering is given by (2.8). Because of the good results obtained with the method of altered states (Sec. 4) for direct $2S$ scattering, we shall use this method for the present case. The wave function for the final state is replaced, therefore, by a plane-wave function and (2.8) becomes

$$f_s'=\frac{\sqrt{2}}{16\pi^2}\int\exp[i\mathbf{q}\cdot\mathbf{r}_1]\exp[-3r_2/2]r_2\cos\theta_2\frac{1}{r_{12}}\times{}_1F_1(-n_1,1,i(k_0r_1-\mathbf{k}_0\cdot\mathbf{r}_1))d\tau_1d\tau_2. \quad (6.1)$$

Integrating with respect to $d\tau_2$ and expressing the hypergeometric function ${}_1F_1$ as a contour integral

[14] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[see (4.5)] yields

$$f_s'=\frac{\sqrt{2}}{9\pi}\left[\frac{64}{27}\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\int \exp[i\mathbf{K}\cdot\mathbf{r}_1]\right.$$
$$\times(\exp[-ik_0vr_1][\exp[-K_0r_1])\frac{\cos\theta_1}{r_1^2}d\tau_1 \tag{6.2}$$
$$-\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\int \exp[i\mathbf{K}\cdot\mathbf{r}_1)$$
$$\left.\times\exp[-K_0r_1]\left(\frac{32}{9}\frac{1}{r_1}+\frac{8}{3}+r_1\right)\cos\theta_1 d\tau_1\right],$$

where

$$\mathbf{K}=\mathbf{q}+\mathbf{k}_0v,\quad K_0=\tfrac{3}{2}+ik_0v.$$

The integration with respect to $d\tau_1$ may be carried out directly, but, because of the factor $\cos\theta$, branch points occur which lead to difficulties in carrying out the v integration. The factor $\cos\theta_1$ therefore is eliminated by means of

$$\cos\theta_1\exp[i\mathbf{K}\cdot\mathbf{r}_1]=\frac{\partial}{\partial k}\left(\frac{\exp[i\mathbf{K}'\cdot\mathbf{r}_1]}{ir_1}\right)\Bigg|_{k=k_0}, \tag{6.3}$$

where $\mathbf{K}'=\mathbf{k}+v\mathbf{k}_0-\mathbf{k}_n$ and obviously $\mathbf{K}'=\mathbf{K}$ for $k=k_0$. The form of $\mathbf{K}'$ has been chosen to avoid a factor of v in the differentiation. Equation (6.2) now takes the form

$$f_s'=\frac{\sqrt{2}}{2\pi i}\left[\frac{64}{27}M_1-\frac{32}{9}M^{(2)}-\frac{8}{3}M^{(1)}-M^{(0)}\right], \tag{6.4}$$

where

$$M_1=\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int \exp[i\mathbf{K}'\cdot\mathbf{r}_1]$$
$$\times(\exp[-ik_0vr_1]-\exp[-K_0r_1])\frac{d\tau_1}{r_1^3},$$

and

$$M^{(n)}=\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int \exp[iK'\cdot r_1]$$
$$\times\exp|-K_0r_1|\frac{d\tau_1}{r_1^n}; \tag{6.5}$$

k is to be set equal to k_0 after the differentiation.

The integrals with respect to τ_1 may be readily carried out if we use the integral

$$\int_0^\infty(e^{-ax}-e^{-bx})\frac{dx}{x}=\ln\left(\frac{b}{a}\right).$$

It is convenient to eliminate the logarithmic form occurring in M_1 and $M^{(2)}$ after the integration by use of

$$\int_\lambda^\infty\frac{d\lambda}{\lambda^2+a^2}=-\frac{1}{2ia}\ln\left(\frac{\lambda-ia}{\lambda+ia}\right);$$
$$\int_\lambda^\infty\frac{\lambda d\lambda}{\lambda^2+b^2}-\int_\lambda^\infty\frac{\lambda d\lambda}{\lambda^2+a^2}=\tfrac{1}{2}\ln\left(\frac{\lambda^2+a^2}{\lambda^2+b^2}\right).$$

Performing the differentiation with respect to k in (6.5), and noting that

$$\frac{\partial}{\partial k}(K^2)\Bigg|_{k=k_0}=2(k_0-k_n\cos\theta+k_0v),$$

we obtain

$$M_1-\tfrac{3}{2}M^{(2)}=-8\pi k_0\int_0^{\frac{3}{2}}\lambda d\lambda[\epsilon\,{}_1L_1(\lambda)+{}_0L_2(\lambda)],$$
$$M^{(1)}=-8\pi k_0(\epsilon\,{}_1L_1+{}_0L_2), \tag{6.6}$$
$$M^{(0)}=-32\pi k_0[\tfrac{3}{2}\epsilon\,{}_1L_3+(\tfrac{3}{2}+ik_0\epsilon)\,{}_0L_3+ik_0\,{}_{-1}L_5)],$$

where $\epsilon=(1/ik_0)(k_0-k_n\cos\theta)$. The notation $L(\lambda)$ indicates that the quantities A and B occurring in L are replaced by A_λ and B_λ. We evaluate the L integrals in (6.6) as before and, retaining only the terms of highest order, obtain

$$f_s'=\frac{16}{27}\frac{\sqrt{2}i}{q^2k_0}(1-\epsilon)\cong\frac{8\sqrt{2}i}{27k_0^3}\frac{\cos\theta}{(1-\cos\theta)}$$
$$=\frac{4\sqrt{2}i}{27k_0^3}\left(\frac{1-2\mu}{\mu}\right). \tag{6.7}$$

The scattering amplitude f_a' [see (2.6)] has been evaluated exactly by Corinaldesi and Trainor; their result is

$$f_a'=\frac{12\sqrt{2}i(k_0-k_n\cos\theta)}{q^2(q^2+9/4)^3}\cong\frac{3\sqrt{2}}{32}\frac{i}{k_0^7\mu^3}. \tag{6.7}$$

Equation (6.7) differs considerably from our result; the energy variation is of order $1/k_0^7$ instead of $1/k_0^3$, and in addition, the angular variation with the asymmetric perturbation is opposite to that given by the symmetric perturbation.

A detailed analysis[10] of f_s' for $\theta=0$ indicates that the principal contribution comes only from the plane-wave terms. The result, then, is in exact agreement with (6.7) with $\theta=0$, namely,

$$f_s'=\frac{12\sqrt{2}i(k_0-k_n\cos\theta)}{q^2(q^2+9/4)^3}\Bigg|_{\theta=0}\cong\frac{1536}{728}ik_0. \tag{6.8}$$

7. EXCHANGE 2P SCATTERING

The scattering amplitude g_s' for $2P$ exchange scattering is given by (2.9). The procedure for evaluating the

integrals is very similar to that used in $2S$ exchange scattering. Replacing $1/r_{12}$ by its Fourier integral and the hypergeometric functions by contour integrals [see (5.1) and (4.8)] yields

$$g_s'=\frac{\sqrt{2}}{32\pi^4}\int\frac{d\tau_p}{p^2}J^{(1)\prime}J^{(2)}, \tag{7.1}$$

where

$$J^{(1)\prime}=\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}J_1', \tag{7.2}$$

$$J_1'=\int \exp[-(\tfrac{1}{2}+ik_0v)r_1]\times\exp[i\mathbf{K}_1\cdot\mathbf{r}_1]r_1\cos\theta_1 d\tau_1. \tag{7.3}$$

$J^{(2)}$ is defined by (5.3). The integral $J^{(2)}$ has been evaluated previously. To evaluate $J^{(1)\prime}$ we first remove the factor $\cos\theta_1$ in J_1' by the method explained in Sec. 6 [see (6.3)]. We obtain for J_1':

$$J_1'=\frac{32\pi i(\frac{1}{2}+ik_0v)[k_0^2(1+v)+\mathbf{k}_0\cdot\mathbf{p}]}{k_0(A_1+B_1v)^3}. \tag{7.4}$$

The integral $J^{(1)\prime}$ then takes the form

$$J^{(1)\prime}=(32\pi i/k_0)[\tfrac{1}{2}(k_0^2+\mathbf{k}_0\cdot\mathbf{p})\,{}_1L_3' +ik_0(k_0^2+\mathbf{k}_0\cdot\mathbf{p}-\tfrac{1}{2}ik_0)\,{}_0L_3'+ik_0^3\,{}_{-1}L_5'], \tag{7.5}$$

where the L integrals are identical with those in (5.9). Evaluating the L integrals, we obtain for g_s'

$$\begin{aligned}g_s'=\frac{\sqrt{2}}{\pi^2}\frac{i}{k_0}\{&(1+n_2)[(1+n_1)(2+n_1)(F_{00}{}^{32})'\\&-2(1-n_1^2)F_{00}{}^{22}+3n_1(1-n_1)F_{10}{}^{12}+2(1+n_1)\\&\times(1-2ik_0)F_{10}{}^{22}+(1-n_1)(2-n_1)F_{20}{}^{02}-(1-n_1)\\&\times(4k_0^2+4ik_0-1)F_{20}{}^{12}]+(1-n_2)[(1+n_1)(2+n_1)\\&\times(F_{01}{}^{31})'-2(1-n_1^2)F_{01}{}^{21}+3n_1(1-n_1)F_{11}{}^{11}\\&+2(1+n_1)(1-2ik_0)F_{11}{}^{21}+(1-n_1)(2-n_1)F_{11}{}^{01}\\&-(1-n_1)(4k_0^2+4ik_0-1)F_{21}{}^{11}]\},\end{aligned} \tag{7.6}$$

where

$$\begin{aligned}F_{rs}{}^{pq}&=\int\frac{d\tau_p/p^2}{A_1{}^pA_2{}^q(A_1-B_1)^r(A_2-B_2)^s},\\(F_{rs}{}^{pq})'&=\int\frac{2(k_0^2+k_0\cdot p)d\tau_p/p^2}{A_1{}^pA_2{}^q(A_1-B_1)^r(A_2-B_2)^s}.\end{aligned} \tag{7.7}$$

A consideration of these integrals shows that the leading order terms in (7.6) are contributed by the integrals $F_{20}{}^{12}$ and $(F_{01}{}^{31})'$. Equation (7.6) has, accordingly, the approximate form

$$g_s'=\frac{\sqrt{2}}{\pi^2}\frac{i}{k_0}(-4k_0^2F_{20}{}^{12}+2(F_{01}{}^{31})'). \tag{7.8}$$

The integral $F_{20}{}^{12}$ is identical with the integral $E_{20}{}^{12}$. To evaluate $(F_{01}{}^{31})'$, we first remove the factor $\mathbf{k}_0\cdot\mathbf{p}$ from the numerator by means of

$$\frac{2(k_0^2+\mathbf{k}_0\cdot\mathbf{p})}{A_1^3}=-\frac{k_0}{2}\frac{\partial}{\partial k_0}\left(\frac{1}{A_1^2}\right). \tag{7.9}$$

Then we have

$$(F_{01}{}^{31})'=-\frac{k_0}{2}\frac{\partial}{\partial k_0}\int\frac{d\tau_p}{p^2A_1^2A_2(A_2-B_2)}=-\frac{k_0}{2}\frac{\partial}{\partial k_0}F_{01}{}^{21}. \tag{7.10}$$

We can therefore evaluate $(F_{01}{}^{31})'$, using Feynman's method.[14] The results are

$$\begin{aligned}F_{20}{}^{12}&=\tfrac{3}{2}(\pi^2/k_0^4q^2),\\(F_{01}{}^{31})'&=\tfrac{3}{16}\pi^2(k_0^2/k_n^4q^2).\end{aligned} \tag{7.11}$$

Substituting these values into (7.8) for the scattered amplitude, we have

$$g_s'=-\frac{45}{8}\frac{\sqrt{2}i}{k_0^3q^2}\cong-\frac{45}{32}\frac{\sqrt{2}i}{k_0^5\mu}. \tag{7.12}$$

The exchange-scattered amplitude g_a', that is, (2.7), has been evaluated by Corinaldesi and Trainor. The approximate form of their result, for large k_0, is

$$g_a'=-\frac{\sqrt{2}i}{8k_0^7}\left(32-\frac{5}{\mu^2}\right). \tag{7.13}$$

This result differs from that for the symmetric method in both energy and angular dependence.

Small Scattering Angles

Examination of (7.6) for $\theta=0$ shows that $F_{20}{}^{12}$ and $(F_{01}{}^{31})'$ remain leading order terms but that, in addition, the integrals $(F_{00}{}^{32})'$, and $F_{10}{}^{22}$ and $F_{10}{}^{22}$ become of comparable order of magnitude. The appropriate form of (7.6) for $\theta=0$ is then

$$g_s'=\frac{\sqrt{2}}{\pi^2}\frac{i}{k_0}2(F_{00}{}^{32})' -2(F_{00}{}^{22}-4ik_0F_{10}{}^{22}-4k_0^2F_{20}{}^{12}+2(F_{01}{}^{31})'). \tag{7.14}$$

These integrals can again be performed by Feynman's method,[14] and we have

$$\begin{aligned}(F_{00}{}^{32})'&=(32/125)(8/5)^{\frac{1}{2}}(\pi^2/k_0^2),\\F_{00}{}^{22}&=(8/35)(8/5)^{\frac{1}{2}}(\pi^2/k_0^2),\\F_{10}{}^{22}&=(10/3)(\pi^2i/k_0^3),\\F_{20}{}^{12}&=(5/2)(\pi^2/k_0^4),\\F_{01}{}^{31\prime}&=(5/48)(\pi^2k_0^2/k_n^4).\end{aligned} \tag{7.15}$$

The integrals $(F_{00}{}^{32})'$ and $F_{00}{}^{22}$ contribute an amount

TABLE I. Summary of results.

Type of scattering		Symmetric method $k^2\mu \gg 1$	$\mu=0$		Asymmetric method $k^2\mu \gg 1$	$\mu=0$
$2S$ direct	f_s	$\frac{8}{81}\frac{\sqrt{2}i}{k_0^3\mu}$	$\frac{512}{729}\sqrt{2}$	f_a	$\frac{1}{8}\frac{\sqrt{2}}{k_0^6\mu^3}$	$\frac{512}{729}\sqrt{2}$
$2S$ exchange	g_s	$\frac{173}{64}\frac{\sqrt{2}i}{k_0^5\mu}$	$\frac{47}{16}\frac{\sqrt{2}i}{k_0^3}$	g_a	$-\frac{\sqrt{2}}{2k_0^6}\left(8-\frac{1}{\mu^2}\right)$	$\frac{40\sqrt{2}}{81}\frac{1}{k_0^4}$
$2P$ direct	f_s'	$\frac{4}{27}\frac{\sqrt{2}i(1-2u)}{k_0^3\mu}$	$\frac{1536}{729}ik_0$	f_a'	$\frac{3\sqrt{2}}{32}\frac{i}{k_0^7\mu^3}$	$\frac{1536}{729}ik_0$
$2P$ exchange	g_s'	$-\frac{45}{32}\frac{\sqrt{2}i}{k_0^5\mu}$	$\frac{85}{24}\frac{\sqrt{2}i}{k_0^3}$	g_a'	$-\frac{\sqrt{2}i}{8k_0^7}\left(32-\frac{5}{\mu^2}\right)$	$\frac{32}{27}\frac{\sqrt{2}i}{k_0^3}$

small compared to the other integrals and will be neglected. Substituting the values of the other integrals into (7.14) yields

$$g_s'=\frac{85}{24}\left(\frac{\sqrt{2}i}{k_0^3}\right). \tag{7.16}$$

The result given by Corinaldesi and Trainor for g_a' when evaluated for the limiting case $\theta=0$ has the form

$$g_a'=\frac{32}{27}\left(\frac{\sqrt{2}i}{k_0^3}\right). \tag{7.17}$$

Equations (7.16) and (7.17) agree with regard to energy dependence for $\theta=0$ in $2P$ exchange scattering. This is in contrast to the result for $\theta=0$ in $2S$ exchange scattering.

8. SUMMARY AND DISCUSSION

The results obtained in the present investigation are summarized in Table I.

In contrast to the results obtained using the asymmetric perturbation, we see that the present results for direct scattering are much more important than exchange scattering for all energies. This arises from the fact that the direct-scattered amplitude is much larger when one uses the symmetric perturbation than when one uses the asymmetric perturbation. Since the results are so different, it is conceivable that some experiment may be done which would tell us which of the perturbation procedures is the more accurate. In doing such an experiment, one would have to measure the differential scattering cross section. Measurement of the total cross section would not be useful, since it would be dominated by the scattering in the forward direction, and for this scattering there is nothing to distinguish the two cases.

The identity of the results in the forward direction for direct scattering leads us to the conclusion that the Born approximation using the asymmetric perturbation gives the correct amplitude but an incorrect phase for the direct-scattered wave. For exchange scattering, however, the symmetric perturbation scheme is a more natural one.