Nuclear Binding Energies in the Shell Model

I. TALMI AND R. THIEBERGER
Department of Physics, The Weizmann Institute of Science, Rehovoth, Israel
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Nuclear binding energies are calculated from a general two-body interaction by using jj-coupling shell model wave functions for the nucleons. The good agreement obtained indicates that the effective central field in which the nucleons move is the same for all nuclei in which the same shell is being filled.

RECENTLY some evidence has been gathered showing that the potential well of the nucleus is constant, to a good approximation, at least in nuclei where the same shell is being filled.[1-3] This encouraged us to try and fit nuclear binding energies on the basis of such an assumption. As neither the interaction between nucleons nor their wave functions are known, we adopted a procedure which does not involve detailed knowledge of these. In the present work only the following assumptions are made: I. The wave function describing the nucleus is that of independent nucleons moving in a central field (shell model wave function), being the same for all nuclei with the same unfilled shell; the spin j of each nucleon is a good quantum number (jj-coupling). II. The interaction between nucleons is a charge-independent two-body force (this may include central forces, any mutual spin-orbit interaction, and tensor forces). Thus, the states are characterized by the configuration, the total spin J, and the total isotopic spin T. If more than one state with these quantum numbers occur, we characterize the states by additional quantum numbers of the generalized seniority.[4,5]

The expectation value of any two-body interaction in one shell can be expressed as a linear combination of the energies in a two-nucleon configuration. To carry out this calculation, fractional parentage coefficients[6,7] may be used. Instead we use an elegant method due to Racah.[8] He considers simple two-body operators, the eigenvalues of which characterize the states [for example, $\sum(\mathbf{t}_i\cdot\mathbf{t}_j)=T(T+1)-\frac{3}{4}n$]. Clearly, if there are enough operators with different eigenvalues for each state of the two-nucleon configuration, the energies of this configuration can be uniquely expressed as a linear combination of the eigenvalues. If this is the case, then the energies of other configurations are also given by a linear combination of the operators with the same coefficients. Since usually this is not the case, it is only possible to calculate average energies of groups of states which belong to the same eigenvalues of the known operators. Fortunately, almost every ground state is the only member of its group and Racah's method can therefore be used. In this way the energies (or rather average energies) in the j^n configuration are given by

$$nA+\tfrac{1}{2}n(n-1)a+[T(T+1)-\tfrac{3}{4}n]b+[g(W)-2n(j+1)]c. \quad (1)$$

In the first term, A is the single-nucleon energy (its kinetic energy and its interaction with the closed shells), while the other terms express the mutual interaction. The quantity $g(W)$ is the eigenvalue of Casimir's operator[4,8] (of the symplectic group in $2j+1$ dimensions) characterizing the group of states which belong to the same irreducible representation specified by $W\equiv(w_1w_2\cdots)$. This last term essentially represents the pairing energy. For even n, the ground states with $J=0$ belong to $W=(00)$ and since $g(W)=w_1(w_1+2j+1)+w_2(w_2+2j-1)+\cdots$ we have in these cases $g(00)=0$. For odd n, the ground states (and sometimes excited levels) with $J=j$ belong to $W=(10)$ and therefore in these cases $g(10)=2(j+1)$. If we specify the interaction, the coefficients a, b, and c can be calculated in terms of radial integrals of the potentials.[8] However, in order to have the procedure as general as possible we keep these coefficients (as well as the coefficient A) as free parameters. We check the agreement of our assumptions with the data by fitting all available energies in a definite shell to the linear combination (1) with the same coefficients; the best values of these are determined by a least squares fit. This is in accordance with the standard procedure in atomic spectroscopy. In the $p_{1/2}$ and $s_{1/2}$ shells, a smaller number of operators is sufficient for the calculation and the parameter c was left out.

Binding energies were taken from the review articles by Wapstra[9] and others.[10] In order to obtain the energy within a shell, we subtracted from the binding energy of every nucleus the binding energy of the preceding doubly closed shells nucleus. These energies contain the contributions from the Coulomb energy which should

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TABLE I. Experimental and calculated binding energies (in Mev).

Nucleus	J of state	Binding energies: Experimental	Binding energies: Present model
$p_{3/2}$ shell B.E. minus that of He^4			
$_2He_3^5$	3/2	−0.95	−0.74
$_2He_4^6$	0	0.93	3.99
$_3Li_3^6$	0	0.13	2.90
$_3Li_3^6$	1, 3(20)	2.17	4.24
$_3Li_4^7$	3/2	10.95	13.62
$_3Li_5^8$	2(11)	12.98	13.59
$_3Li_6^9$	3/2	16.53	16.05
$_4Be_3^7$	3/2	9.30	11.92
$_4Be_4^8$	0	28.19	28.12
$_4Be_5^9$	3/2	29.85	29.76
$_4Be_6^{10}$	0	36.66	36.86
$_5B_4^9$	3/2	28.00	27.69
$_5B_5^{10}$	0	34.70	34.80
$_5B_5^{10}$	1, 3(20)	36.22	36.14
$_5B_6^{11}$	3/2	47.90	47.90
$_6C_4^{10}$	0	32.02	32.13
$_6C_5^{11}$	3/2	45.14	45.23
$_6C_6^{12}$	0	63.85	63.80
$p_{1/2}$ shell B.E. minus that of C^{12}			
$_6C_7^{13}$	1/2	4.95	5.34
$_6C_8^{14}$	0	13.12	13.04
$_7N_6^{13}$	1/2	1.95	2.31
$_7N_7^{14}$	0	10.18	10.02
$_7N_7^{14}$	1	12.49	12.19
$_7N_8^{15}$	1/2	23.32	23.35
$_8O_6^{14}$	0	6.54	6.48
$_8O_7^{15}$	1/2	19.79	19.81
$_8O_8^{16}$	0	35.43	35.51
$d_{5/2}$ shell B.E. minus that of O^{16}			
$_8O_9^{17}$	5/2	4.14	4.56
$_8O_{10}^{18}$	0	12.21	13.65
$_8O_{11}^{19}$	5/2	16.16	15.61
$_8O_{12}^{20}$	0	?	22.10
$_9F_8^{17}$	5/2	0.59	0.95
$_9F_{10}^{19}$	5/2	19.97	21.62
$_{10}Ne_9^{19}$	5/2	15.85	17.49
$_{10}Ne_{10}^{20}$	0	33.05	33.59
$_{10}Ne_{11}^{21}$*	5/2	39.46(?)	39.36
$_{10}Ne_{12}^{22}$	0	50.17	49.65
$_{10}Ne_{13}^{23}$*	5/2	55.36(?)	52.83
$_{11}Na_{10}^{21}$	5/2	?	34.95
$_{11}Na_{12}^{23}$*	5/2	58.53(?)	58.03
$_{11}Na_{14}^{25}$	5/2	?	72.71
$_{12}Mg_{11}^{23}$	5/2	?	53.11
$_{12}Mg_{12}^{24}$	0	70.66	70.41
$_{12}Mg_{13}^{25}$	5/2	77.99	77.40
$_{12}Mg_{14}^{26}$	0	89.10	88.90
$_{13}Al_{12}^{25}$	5/2	72.96	72.18
$_{13}Al_{13}^{26}$	0	84.09	83.69
$_{13}Al_{13}^{26}$	1, 3, 5(20)	84.03	83.44
$_{13}Al_{14}^{27}$	5/2	97.36	97.68
$_{14}Si_{13}^{27}$	5/2	91.75	91.96
$_{14}Si_{14}^{28}$	0	108.95	110.47
$s_{1/2}$ shell B.E. minus that of Si^{28}			
$_{14}Si_{15}^{29}$	1/2	8.47	8.61
$_{14}Si_{16}^{30}$	0	19.09	19.03
$_{15}P_{14}^{29}$	1/2	2.79	2.81
$_{15}P_{15}^{30}$	0	13.28	13.24
$_{15}P_{15}^{30}$	1	13.97	13.87
$_{15}P_{16}^{31}$	1/2	26.38	26.43
$_{16}S_{15}^{31}$	1/2	20.16	20.21
$_{16}S_{16}^{32}$	0	35.24	35.22
$d_{3/2}$ shell B.E. minus that of S^{32}			
$_{16}S_{17}^{33}$	3/2	8.65	8.66
$_{16}S_{18}^{34}$	0	20.05	20.04
$_{16}S_{19}^{35}$	3/2	27.07	27.12
$_{16}S_{20}^{36}$	0	36.97	36.93
$_{17}Cl_{16}^{33}$	3/2	2.42	2.38
$_{17}Cl_{17}^{34}$	0	13.75	13.77
$_{17}Cl_{18}^{35}$	3/2	26.46	26.29
$_{17}Cl_{19}^{36}$	2(11)	35.03	35.10
$_{17}Cl_{20}^{37}$	3/2	45.39	45.23
$_{18}A_{17}^{35}$	3/2	19.69	19.58
$_{18}A_{18}^{36}$	0	34.96	34.83
$_{18}A_{19}^{37}$	3/2	43.79	43.96
$_{18}A_{20}^{38}$	0	55.54	55.82
$_{19}K_{18}^{37}$	3/2	36.88	37.00
$_{19}K_{19}^{38}$	0	48.88	48.86
$_{19}K_{20}^{39}$	3/2	61.96	61.85
$_{20}Ca_{19}^{39}$	3/2	54.32	54.44
$_{20}Ca_{20}^{40}$	0	70.29	70.17
$f_{7/2}$ shell B.E. minus that of Ca^{40}			
$_{20}Ca_{21}^{41}$	7/2	8.37	8.68
$_{20}Ca_{22}^{42}$	0	19.85	21.03
$_{20}Ca_{23}^{42}$	7/2	27.78	28.84
$_{20}Ca_{24}^{44}$	0	38.96	40.31
$_{20}Ca_{25}^{45}$*	7/2	46.38 or 46.20	47.24
$_{20}Ca_{26}^{46}$	0	57.17	57.84
$_{20}Ca_{27}^{47}$	7/2	63.87	63.90
$_{20}Ca_{28}^{48}$	0	73.77	73.63
$_{21}Sc_{20}^{41}$	7/2	1.63	2.03
$_{21}Sc_{22}^{43}$	7/2	24.79	25.74
$_{21}Sc_{24}^{45}$	7/2	45.85	46.52
$_{21}Sc_{26}^{47}$	7/2	65.23	65.55
$_{21}Sc_{28}^{49}$	7/2	83.27	82.83
$_{22}Ti_{22}^{44}$*	0	35.13	33.73
$_{22}Ti_{23}^{45}$	7/2	43.03	43.03
$_{22}Ti_{24}^{46}$	0	56.32	56.00
$_{22}Ti_{25}^{47}$	7/2	64.84	64.43
$_{22}Ti_{26}^{48}$	0	76.50	76.53
$_{22}Ti_{27}^{49}$	7/2	84.62	84.08
$_{22}Ti_{28}^{50}$	0	95.54	95.30
$_{23}V_{23}^{46}$*	0	48.30	48.76
$_{23}V_{24}^{47}$	7/2	61.41	60.74
$_{23}V_{26}^{49}$	7/2	83.23	82.76
$_{23}V_{28}^{51}$	7/2	103.44	103.03
$_{24}Cr_{24}^{48}$*	0	69.50	68.76
$_{24}Cr_{25}^{49}$	7/2	?	78.69
$_{24}Cr_{26}^{50}$	0	92.87	92.28
$_{24}Cr_{27}^{51}$	7/2	102.02	101.33
$_{24}Cr_{28}^{52}$	0	114.25	114.04
$_{25}Mn_{25}^{50}$*	0	84.60	84.44
$_{25}Mn_{26}^{51}$	7/2	?	97.04
$_{25}Mn_{28}^{53}$	7/2	120.42	120.30
$_{26}Fe_{26}^{52}$	0	?	105.09
$_{26}Fe_{27}^{53}$	7/2	116.17	115.63
$_{26}Fe_{28}^{54}$	0	129.76	129.85
$_{27}Co_{27}^{54}$*	0	120.10	121.42
$_{27}Co_{28}^{55}$	7/2	134.63	134.64
$_{28}Ni_{28}^{56}$	0	141.19	142.72

be subtracted. This latter energy can be taken from mirror nuclei, but we used a more consistent and satisfactory way of subtracting. We utilized the Coulomb energy in the harmonic-oscillator model[1] given in terms of $e^2(\nu/\pi)^{\frac{1}{2}}$ which was considered as an additional parameter. In this case a greater number of nuclei could be taken into account in the least squares fit. This is the only possibility in cases where no information on mirror nuclei is available (e.g., in the $f_{7/2}$ shell). The $e^2(\nu/\pi)^{\frac{1}{2}}$ thus derived for various shells is in a very good agreement with previous results.[1] Our results are presented in Table I. In this table are included also some nuclei (marked with an asterisk) whose energies are not accurately known; these were not considered in the least squares fit. The binding energies listed include the Coulomb energies. Where averages of states were considered all the spins are given; the brackets contain the numbers W.

The agreement is excellent for the $s_{1/2}$ and $d_{3/2}$ shells. Also in the other shells the root-mean-square deviation is less than 1% of the width of the energy range considered. The rms deviation is defined in the usual way as $[\sum_{i=1}^{n}\Delta_i^2/(N-k)]^{\frac{1}{2}}$, where the Δ_i are differences between the experimental and calculated energies, N is the number of data and k is the number of parameters. The agreement is poor for the first nuclei up to

TABLE II. Energy parameters of the present model (in Mev).

Shell	A	a	b	$2(j+1)c$	$e^2(\nu/\pi)^{\frac{1}{2}}$	rms deviation	Percent rms deviation
$p_{3/2}$	−0.738	0.594	−5.137	−3.721	0.364	0.33	0.66%
$p_{1/2}$	5.337	2.912	−1.085		0.349	0.29	0.86%
$d_{5/2}$	4.565	0.302	−3.205	−2.909	0.349	0.92	0.85%
$s_{1/2}$	8.606	1.978	−0.316		0.320	0.10	0.31%
$d_{3/2}$	8.658	0.118	−1.815	−1.759	0.312	0.14	0.21%
$f_{7/2}$	8.681	0.155	−1.185	−2.053	0.289	0.75	0.57%

$A=7$, indicating breakdown of the shell model (or only of jj-coupling). These cases were excluded from the least squares fit and were not taken into account in the calculation of the rms deviation. The best values of the coefficients for the various shells are listed in Table II along with the rms deviations. The large difference in the values of a and b between the $p_{1/2}$, $s_{1/2}$ shells and the other shells is not surprising in view of the difference in their meaning in the two cases. It is worth while to note that the deviations in the $d_{5/2}$ shell (and also in the $f_{7/2}$ shell) show a marked regularity. In the middle of the shell all the experimental binding energies are bigger than those calculated whereas both in the beginning and the end the situation is reversed. This may be associated with the effects of deformation (or possibly of configuration interaction) not considered here. Positions of some excited levels (usually only position of averages) can be calculated with the parameters obtained, and are found to be in fair agreement with the experimental data. It is hoped to report soon on this and related problems. It seems to us that it is interesting and rather surprising that the simple jj-coupling shell model is so adequate in this region of nuclei.

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Neutron Scattering from Iron and Carbon by Time-of-Flight*

C. O. MUEHLHAUSE, S. D. BLOOM, H. E. WEGNER,† AND G. N. GLASOE
Brookhaven National Laboratory, Upton, New York

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A fixed-frequency cyclotron in conjunction with millimicrosecond time-of-flight techniques has been used to study elastic and inelastic neutron scattering from iron and carbon in the Mev range. The elastic angular distribution from iron exhibits an optical diffraction type pattern, and the inelastic angular distribution proves to be primarily isotropic except for a slight asymmetry around 90 degrees. The elastic angular distribution from carbon is in essential agreement with previous work.

INTRODUCTION

THE dynamics of excitation of low-lying nuclear levels is of interest in the study of nuclear structure. Some details of the structure may be investigated by bombarding the nucleus with neutrons and studying the reaction and scattering processes. For most nuclei the energy of the first excited state ranges from $\sim$20 kev to $\sim$2 Mev. Until recently, identification of various neutron groups from neutron-out reactions in the Mev energy range has been difficult. In the present work this identification was accomplished by determining the flight time for the scattered neutrons over known distances.

Neutrons of 1-Mev energy travel with a velocity of approximately 1.4 cm/mμsec. In order to measure the energy of 1-Mev neutrons to 2% with a flight path of 1 meter, the times of origin and detection must each be known to $\sim$1 mμsec. This may be accomplished by using millimicrosecond techniques for scintillation detection together with the short pulse of neutrons obtained from the natural phase bunching of particles accelerated in a fixed-frequency cyclotron.[1] These techniques were applied to the study of elastic and inelastic neutron scattering from iron and carbon.

Elastic and inelastic angular distribution measurements were made for Fe at 1.66, 1.58, and 1.48 Mev, and for C at 1.66 Mev. The elastic distribution exhibits an optical diffraction type pattern, and the inelastic distribution shows a slight asymmetry about 90 degrees.

EXPERIMENTAL PROCEDURE

For this work the 18-in. cyclotron at the Brookhaven National Laboratory produced a 2.45$\pm$0.03-Mev external proton beam. About 60 μa of the external beam were focused on a $\frac{1}{4}$-in. diameter target 12 ft from the cyclotron by using a pair of double-wedge magnets. The natural phase bunching[2] of the machine resulted in a 2-mμsec pulse of protons striking the target every 54.1 mμsec (rf=18.4 Mc/sec).

Neutrons were obtained from a thin target ($\sim$50 kev) of Zr-T on a water cooled copper backing. Three different primary neutron energies (1.66, 1.58, and 1.48 Mev) were obtained by positioning the scatterer at 0, 20, and 30 degrees with respect to the direction of the protons incident on the target. The target design (see Fig. 1) allowed for very little scattering material in the direction of the forward beam. The targets used showed only a negligible loss of tritium over a 50 000 μa-hr period.

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† Present address: Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

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