tion into the electron or muon pairs vanishes for all energies.

III. PROTON-ANTIPROTON ATOM

As an application of the above formulas we consider the two-photon lifetime of the $n\ ^1S_0$ state of the protonic analog of positronium.[4] Using the approximation of Wheeler[5] and Eq. (3), we find

$$\tau_{2\gamma}=1.8\times 10^{-15}n^3 \text{ sec.} \tag{8}$$

The effect of the Pauli moment here is to decrease the two-photon lifetime by the factor $X=38.5$. This will be reduced if the anomalous moment is spread over the pion Compton wavelength, rather than being an effective point moment as we have assumed.

It is a pleasure to acknowledge the assistance of Mr. Isadore Harris with these calculations.

[4] We have estimated the probability of slowing down in hydrogen or in heavy matter and of capture into atomic orbits in liquid hydrogen. We find that when Bevatron-produced antiprotons are slowed in heavy matter, then even if the high-energy cross section for annihilation in flight is twice geometric, at least a few percent of them will be captured into inner orbits so that the effects of selection rules will be observable. For slowing in liquid hydrogen we can make only a rough estimate. Using reasonable assumed annihilation and scattering cross sections, we find that at least thirty percent will probably be captured.

[5] J. A. Wheeler, Ann. N. Y. Acad. Sci. 48, 219 (1946). For $l\neq 0$, $\psi(0)$ vanishes and this method does not apply.

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Effect of π-π Interaction on High-Energy π-p Scattering

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The object of this paper is to examine the possibility of explaining the second maximum in (π-p) scattering in the framework of the Tamm-Dancoff approximation for (π-p) interaction, by postulating a resonant π-π interaction in the isotopic spin state $I=0$ of the π-π system. For this purpose the Tamm-Dancoff method in pseudoscalar theory, as formulated by Dyson and others, is first extended so as to include the effects of a π-π interaction. This is achieved by introducing a Green's function whose form is determined on the assumption of a zero-range π-π interaction in the state $I=0$ of the π-π system. The solution of the "modified" integral equation is greatly simplified by assuming the π-p interaction to be weak compared with the π-π interaction (the reasons for which are given). The total inelastic cross section is then derived by using the unitary property of the S matrix.

It is found that the Tamm-Dancoff approximation is unable to explain the (π-p) maximum on the basis of the proposed model. This is contrary to the result derived by other authors, who used the impulse approximation for the interaction of the incident meson with the meson cloud surrounding the nucleon.

1. INTRODUCTION

THE Brookhaven experiments on π-p scattering near 1 Bev have shown a definite maximum in the total π^--p cross section around this energy. There has been a natural tendency to explain this maximum as a resonant interaction. An attempt to explain it as a single-meson resonant state has so far given a negative result.[1] This perhaps may not be too surprising, for at this energy the inelastic processes play a very important part, so that their effects have to be taken into account more adequately than was possible in B. In fact the cross section for the production of an extra meson is larger than the elastic cross section above 1 Bev.

There was a suggestion by the Brookhaven theoretical group that a resonant $P_{\frac{3}{2},\frac{3}{2}}$ interaction of each of the two mesons with the nucleon in the final state might account for the observed maximum. However, such an explanation meets with the immediate difficulty that it gives rise to a nonzero contribution to the total $T=\frac{3}{2}$ state of the π-p system, as a result of which the π^+-p scattering would also show a maximum at the same energy. Since, on the other hand, such a maximum has not been observed, the strength of this argument is somewhat reduced. Dyson[2] has recently proposed another mechanism which successfully avoids the difficulty of enhancement in the $T=\frac{3}{2}$ state of the π-p system. He postulates a short-range π-π interaction in the state of isotopic spin $I=0$ of the meson-meson system.[3] Taking the impulse approximation for the interaction of the incident meson with the meson cloud surrounding the proton, he finds that the observed magnitude of the π^--p cross section can be explained on this basis. It is clear that this sort of interaction can affect only the state $T=\frac{1}{2}$ of the π-p system. Moreover, it involves no assumption about the relative angular momentum of the pion and the nucleon, so that a number of partial waves can be simultaneously affected.

A consequence of this hypothesis is that only neutrons

[1] A. N. Mitra, Phys. Rev. 99, 957 (1955); referred to as B.

[2] F. J. Dyson, Phys. Rev. 99, 1037 (1955).

[3] We are using the notation "I" for the total isotopic spin of the π-π system, as distinguished from "T" which stands for the total isotopic spin of the resultant π-p system.

should come out in the final state, as can be seen from considerations of charge conservation combined with the fact that the total isotopic spin $I=0$ is obtained only for the combinations (π^+-π^-) and (π^0-π^0). This prediction seems to be fairly well fulfilled according to the analysis of Walker's data[4] on the total π^--p cross section at 1 Bev, showing that neutrons come out in $\sim$70% of the cases.

An attempt has been made in this paper to examine whether the conclusion reached by Dyson on the basis of the impulse approximation is also maintained in the framework of the Tamm-Dancoff formalism. In other words, we calculate here the total (π^--p) cross section by incorporating a phenomenological (π-π) interaction of the resonant type in the Tamm-Dancoff equation for (π-p) scattering. In order to make the calculation reasonably simple we make the following assumptions:

(1) Strong interaction between the mesons exists only in the state $I=0$ of the 2-meson system; the interaction is neglected in the other isotopic spin states $I=1$, 2.

(2) The interaction between the nucleon and a meson is assumed to be weak.

The second assumption essentially amounts to postulating the Born approximation for the π-p interaction in the state $T=\frac{1}{2}$. The reason for taking this approximation may be summed up as follows:

(a) No single-meson resonant state of $T=\frac{1}{2}$ has been discovered so far. As discussed in B, the $D_{\frac{5}{2},\frac{1}{2}}$ state was as good a candidate for resonant behavior as might be expected *a priori*. However, the increase of phase shift over the Born approximation result was extremely small for $G^2/4\pi=16$. This result may be taken as a general index for other *single-meson* states of $T=\frac{1}{2}$ as well, at least in the framework of the lowest order Tamm-Dancoff formalism. It is quite likely, though, that the situation might change if intermediate states involving more than two mesons are also included in the Tamm-Dancoff (T.D.) formalism.

(b) The $P_{\frac{3}{2},\frac{3}{2}}$ state (which is known to be the only resonant state discovered so far) does not seem to have any influence on the present model which assumes a resonant π-π interaction in the state $I=0$ (not 1, 2).

(c) The Born approximation for π-p interaction appears to be a partial improvement over the impulse approximation in the sense that the latter completely neglects the interaction of the incident meson with the nucleon *core*.

(d) Lastly, the Born approximation simplifies the calculations enormously. Exact solution by partial wave analysis would mean solving at least four or five complicated integral equations corresponding to the same number of partial waves, on account of the high energies ($\sim$1 Bev) involved.

[4] W. D. Walker, *Proceedings of the Fifth Annual Rochester Conerence on High-Energy Physics, 1955* (Interscience Publishers, Inc., New York, 1955).

It is of course realized (in spite of these arguments) that the interaction of the meson field surrounding the nucleon core is too strong to be described by the Born approximation. The present Tamm-Dancoff method, on the other hand, does not seem to provide a very effective means of taking this interaction appropriately into account.

In Sec. 2 we indicate the derivation of the meson-nucleon integral equation including the effects of π-π interaction. Since the detailed derivation of this equation without the effects of π-π interaction has already been given by Dyson *et al.*,[5] it is enough for our purpose to outline the main steps leading to the modified equation for pion-nucleon scattering. The effect of the π-π interaction can be conveniently represented by means of a "Green's function" modifying the amplitudes of the two-meson intermediate states.

Section 3 is devoted to the derivation of the form of the Green's function on the assumption of a zero-range interaction between the two mesons. The Green's function involves the π-π scattering amplitude whose form is taken to be of the usual Breit-Wigner type.

In Sec. 4 the total cross section for *inelastic* pion-nucleon scattering is derived by using the unitary property of the S-matrix, so that one has essentially to calculate the amplitude for forward scattering. The details of various integrations involved in this section are given in the Appendix.

The fifth and final section is devoted to a brief discussion of the results.

2. MODIFIED π-p INTEGRAL EQUATION

In the notation of A, the equation connecting the amplitude $g(\mathbf{p},\mathbf{k})$ of one nucleon and one meson, to the other possible states is given by:

$$
\begin{aligned}
&(E-E_p-\omega_k)g_{u\alpha}(\mathbf{p},\mathbf{k})\\
&\quad=iG(16\pi^3)^{-\frac{1}{2}}\bigg[\sum_{u'\alpha'}\int d^3\mathbf{k}'(M^2/E_pE_{\mathbf{p}-\mathbf{k}'}\omega_{k'})^{\frac{1}{2}}\\
&\qquad\times\bar{u}(p)\gamma_5\tau_{\alpha'}u'(\mathbf{p}-\mathbf{k}')g_{u'\alpha\alpha'}(\mathbf{p}-\mathbf{k}';\mathbf{k},\mathbf{k}')\\
&\qquad+\sum_{u'}(M^2/E_{\mathbf{p}+\mathbf{k}}E_p\omega_k)^{\frac{1}{2}}\bar{u}(\mathbf{p})\gamma_5\tau_\alpha u'(\mathbf{p}+\mathbf{k})g_{u'}(\mathbf{p}+\mathbf{k})\\
&\qquad+\sum_{vu'}\int d^3\mathbf{p}'(M^2/E_{\mathbf{p}'-\mathbf{k}}E_{p'}\omega_k)^{\frac{1}{2}}\\
&\qquad\times\bar{v}(-\mathbf{p}'+\mathbf{k})\tau_\alpha\gamma_5u'(\mathbf{p}')g_{uu'v}(\mathbf{p},\mathbf{p}',-\mathbf{p}'+\mathbf{k})\\
&\qquad+\sum_{vu'\alpha'}\int d^3\mathbf{q}d^3\mathbf{k}'(M^2/E_{\mathbf{q}+\mathbf{k}'}E_q\omega_{k'})^{\frac{1}{2}}\\
&\qquad\times\bar{v}(\mathbf{q})\tau_{\alpha'}\gamma_5u'(-\mathbf{q}-\mathbf{k}')\\
&\qquad\qquad\times g_{uu'v\alpha\alpha'}(\mathbf{p},-\mathbf{q}-\mathbf{k}',\mathbf{q};\mathbf{k},\mathbf{k}')\bigg].\qquad(1)
\end{aligned}
$$

[5] Dyson, Ross, Salpeter, Schweber, Sundaresan, Visscher, and Bethe, Phys. Rev. **95**, 1644 (1954); referred to as A.

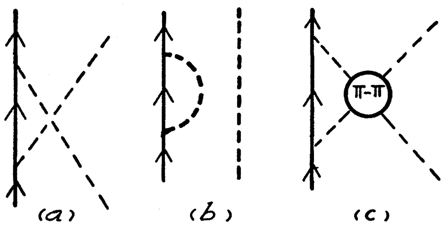

FIG. 1. Graphs for π-p scattering: (a) denotes the graph for meson scattering without π-π effects; (b) is a pure self-energy graph; (c) represents π-p scattering including π-π effects.

Here, the spinors for nucleon states are denoted by u, u', etc., and those for antinucleon states by v, v', etc.

The first and the last terms in the above expression contain wave functions for two mesons, so that the equations connecting them back to the (1,1) state will be modified by the π-π interaction. The other two connecting equations will remain unchanged. We shall now discuss the connecting equation from the (1,2) to the (1,1) state. The treatment for the other state (3,2) is very similar.

The connecting equation with the neglect of π-π effects is

$$\begin{aligned}(E-E_p-\omega_k-\omega_{k'})g_{u'\alpha\alpha'}(\mathbf{p}';\mathbf{k},\mathbf{k}') &= iG(16\pi^3)^{-\frac{1}{2}}[M(E_{p'}\omega_k E_{\mathbf{p}'+\mathbf{k}})^{-\frac{1}{2}} \\ &\times\sum_{u''}\bar{u}'(\mathbf{p}')\gamma_5\tau_\alpha u''(\mathbf{p}'+\mathbf{k})g_{u''\alpha'}(\mathbf{p}'+\mathbf{k},\,\mathbf{k}') \\ &+\{(\mathbf{k},\alpha)\rightleftarrows(\mathbf{k}',\alpha')\}]. \quad (2)\end{aligned}$$

The second term in (2) gives rise to self-energy effects only and was omitted in the treatment given in A. However, it is no longer permissible to omit this term when the π-π effects are taken into account. This can be seen immediately from an inspection of the graphs of Fig. 1.

As long as the π-π effects are neglected, one can always distinguish between the two graphs 1(a) and 1(b). However, with the inclusion of a π-π interaction, both 1(a) and 1(b) reduce to the form 1(c) and it is no longer possible to picture them separately.

Now Eq. (2) is of the form[6]:

$$g(\mathbf{k},\mathbf{k}')=K(\mathbf{k},\mathbf{k}')g(\mathbf{k}')+K(\mathbf{k}',\mathbf{k})g(\mathbf{k}). \quad (2a)$$

To represent the effects of a π-π interaction, one has now to add on the right-hand side of (2a) a term of the form:

$$\int I(E_0,\,\mathbf{k}'-\mathbf{k},\,\mathbf{k}'''-\mathbf{k}'')g(\mathbf{k}''',\mathbf{k}'')\times\delta^3(\mathbf{k}+\mathbf{k}'-\mathbf{k}''-\mathbf{k}''')d^3\mathbf{k}''d^3\mathbf{k}'''. \quad (3)$$

Here I denotes the interaction kernel for the meson-meson interaction and depends on the relative momenta of the particles before and after the interaction, and perhaps also on their total energy E_0 in the center-of-mass system. The δ function in (3) represents the overall conservation of momentum.

The Eq. (2a) for $g(\mathbf{k},\mathbf{k}')$ should therefore read

$$\begin{aligned}g(\mathbf{k},\mathbf{k}')&=K(\mathbf{k},\mathbf{k}')g(\mathbf{k}')+K(\mathbf{k}',\mathbf{k})g(\mathbf{k}) \\ &+\int I(E_0,\,\mathbf{k}'-\mathbf{k},\,\mathbf{k}+\mathbf{k}'-2\mathbf{k}'') \\ &\times g(\mathbf{k}+\mathbf{k}'-\mathbf{k}'',\,\mathbf{k}'')d^3\mathbf{k}''. \quad (4)\end{aligned}$$

To see the nature of the solution of (4) somewhat more clearly, it will be convenient to relabel the arguments of $g(\mathbf{k},\mathbf{k}')$ as follows:

$$\mathbf{k}+\mathbf{k}'=2\mathbf{k}_0,\quad \mathbf{k}'-\mathbf{k}=2\mathbf{s},$$

$$g(\mathbf{k},\mathbf{k}')\to g(\mathbf{k}_0,\mathbf{s}). \quad (5)$$

Then

$$g(\mathbf{k}+\mathbf{k}'-\mathbf{k}'',\,\mathbf{k}'')\to g(\mathbf{k}_0,\,\mathbf{k}''-\mathbf{k}_0).$$

Equation (4) now reads:

$$\begin{aligned}g(\mathbf{k}_0,\mathbf{s})&=K(\mathbf{k}_0-\mathbf{s},\,\mathbf{k}_0+\mathbf{s})g(\mathbf{k}_0+\mathbf{s}) \\ &+K(\mathbf{k}_0+\mathbf{s},\,\mathbf{k}_0-\mathbf{s})g(\mathbf{k}_0-\mathbf{s}) \\ &+\int I(E_0,\,2\mathbf{s},\,-2\mathbf{s}_1)g(\mathbf{k}_0,\mathbf{s}_1)d^3\mathbf{s}_1, \quad (6)\end{aligned}$$

where

$$\mathbf{s}_1=\mathbf{k}''-\mathbf{k}_0.$$

The solution of (6) depends on that of the integral equation:

$$G(\mathbf{s})=\delta^3(\mathbf{s}-\mathbf{s}_0)+\int I(E_0,\,2\mathbf{s},\,-2\mathbf{s}_1)G(\mathbf{s}_1)d^3\mathbf{s}_1,$$

where $\mathbf{s}_0$ is some arbitrary initial momentum. The solution of this equation will be denoted by $G(E_0,\mathbf{s},\mathbf{s}_0)$.

In terms of G, the solution of (6) can now be expressed as:

$$\begin{aligned}g(\mathbf{k},\mathbf{k}')=g(\mathbf{k}_0,\mathbf{s})&=\int d^3\mathbf{s}_0G(E_0,\mathbf{s},\mathbf{s}_0) \\ &\times[K(\mathbf{k}_0-\mathbf{s}_0,\,\mathbf{k}_0+\mathbf{s}_0)g(\mathbf{k}_0+\mathbf{s}_0) \\ &+K(\mathbf{k}_0+\mathbf{s}_0,\,\mathbf{k}_0-\mathbf{s}_0)g(\mathbf{k}_0-\mathbf{s}_0)]. \quad (7)\end{aligned}$$

The function G depends, among other things, on the charge coordinates of the interacting mesons. In the absence of interaction, we have simply:

$$G_{\alpha\alpha',\beta\beta'}(\mathbf{s},\mathbf{s}_0)=\delta^3(\mathbf{s}-\mathbf{s}_0)\delta_{\alpha\beta}\delta_{\alpha'\beta'}, \quad (8)$$

where β, β' and α, α' denote the charge coordinates of the mesons before and after the interaction.

The intermediate state (3,2) can be treated in an exactly similar manner.

The integral equation for $g(\mathbf{p},\mathbf{k})$ may now be obtained by substituting an equation of the type (7) for each of

[6] We are, for the time being, ignoring the nucleon momentum in the wave function $g(\mathbf{p},\mathbf{k})$.

the amplitudes of the (1,2) and (3,2) states in Eq. (1). This equation being rather lengthy, we shall write it out only after some essential simplifications have been carried out. The elimination of the various spinors u, u', etc., can be effected by introducing the projection operators $\Lambda_{\pm}$ defined in A, and the wave functions:

$$g_{\alpha}'(\mathbf{p},\mathbf{k})=\sum_{u} u(\mathbf{p})g_{u\alpha}(\mathbf{p},\mathbf{k}). \tag{9}$$

Next, the reduction of the wave function to large components can be achieved in the manner outlined in A. After these simplifications and some reduction, the integral equation takes the following form in the c.m. system:

$$\begin{aligned}(E-E_k-\omega_k)g_{\alpha}(\mathbf{k})&\\ =(G^2/32\pi^3)&\int d^3\mathbf{k}'(E_kE_{k'}\omega_k\omega_{k'})^{-\frac{1}{2}}L_{\alpha\alpha'}{}^{(0)}(\mathbf{k},\mathbf{k}')g_{\alpha'}(\mathbf{k}')\\ &+(G^2/64\pi^3)\int d^3\mathbf{k}'d^3\mathbf{q}L^{(1)}(\mathbf{k},\mathbf{k}',\mathbf{q})\\ &\times[\tau_{\alpha'}G_{\alpha\alpha',\beta\beta'}(E_0,\,\mathbf{k}'-\mathbf{k}-\mathbf{q},\,\mathbf{k}'-\mathbf{k}+\mathbf{q})\tau_{\beta}g_{\beta'}(\mathbf{k}')\\ &+\tau_{\alpha'}G_{\alpha\alpha',\beta\beta'}(E_0,\,\mathbf{k}'-\mathbf{k}-\mathbf{q},\,\mathbf{k}-\mathbf{k}'-\mathbf{q})\tau_{\beta'}g_{\beta}(\mathbf{k}')]\\ &-(G^2/64\pi^3)\int d^3\mathbf{k}'d^3\mathbf{q}L^{(2)}(\mathbf{k},\mathbf{k}',\mathbf{q})\\ &\times[G_{\alpha\alpha',\beta\beta'}(E_0,\,\mathbf{k}'-\mathbf{k}-\mathbf{q},\,\mathbf{k}'-\mathbf{k}+\mathbf{q})\tau_{\beta}\tau_{\alpha'}g_{\beta'}(\mathbf{k}')\\ &+G_{\alpha\alpha',\beta\beta'}(E_0,\,\mathbf{k}'-\mathbf{k}-\mathbf{q},\,\mathbf{k}-\mathbf{k}'-\mathbf{q})\tau_{\beta'}\tau_{\alpha'}g_{\beta}(\mathbf{k}')].\end{aligned} \tag{10}$$

Here the function G, regarded as a matrix in the charge coordinates, operates on the "initial" coordinates β, β' and changes them to α, α'. $L^{(0)}$ is the part of the kernel which is related to the (1,0) and (3,0) intermediate states and will not concern us in the subsequent calculations. The other quantities appearing in (10) are defined as follows:

$$\begin{aligned}E_{\mathbf{k}+\mathbf{k}'-\mathbf{q}}&(E-E_{\mathbf{k}+\mathbf{k}'-\mathbf{q}}-\omega_{k'}-\omega_{\mathbf{k}-\mathbf{q}})\\ &\times(E_kE_{k'}\omega_{\mathbf{k}-\mathbf{q}}\omega_{\mathbf{k}'-\mathbf{q}})^{\frac{1}{2}}L^{(1)}(\mathbf{k},\mathbf{k}',\mathbf{q})\\ &=(M+E_k)(E_{\mathbf{k}+\mathbf{k}'-\mathbf{q}}-M)-(\boldsymbol{\sigma}\cdot\mathbf{k})[\boldsymbol{\sigma}\cdot(\mathbf{k}+\mathbf{k}'-\mathbf{q})]\\ &\quad+(M+E_{k'})^{-1}[(E_{\mathbf{k}+\mathbf{k}'-\mathbf{q}}+M)(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{k}')\\ &\quad-(E_k+M)[\boldsymbol{\sigma}\cdot(\mathbf{k}+\mathbf{k}'-\mathbf{q})](\boldsymbol{\sigma}\cdot\mathbf{k}')];\end{aligned} \tag{11}$$

$$\begin{aligned}E_q&(E-E_k-E_k'-E_q-\omega_{k'}-\omega_{\mathbf{k}-\mathbf{q}})\\ &\times(E_kE_{k'}\omega_{\mathbf{k}-\mathbf{q}}\omega_{\mathbf{k}'-\mathbf{q}})^{\frac{1}{2}}L^{(2)}(\mathbf{k},\mathbf{k}',\mathbf{q})\\ &=(M+E_k)(M+E_q)+(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{q})\\ &\quad+(M+E_{k'})^{-1}[(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{k}')(E_q-M)\\ &\quad+(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{k}')(E_k+M)].\end{aligned} \tag{12}$$

3. π-π INTERACTION

If one calculates the lowest order π-π interaction with the help of the so-called "square diagram," one encounters the isotopic spin dependence of the "effective" Hamiltonian in the form:

$$\mathrm{Tr}(\tau_{\alpha}\tau_{\beta}\tau_{\alpha'}\tau_{\beta'})=2(\delta_{\alpha\beta}\delta_{\alpha'\beta'}-\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\delta_{\alpha\beta'}\delta_{\alpha'\beta}). \tag{13}$$

The right-hand side of (13) can be expressed in terms of three operators ξ, η, ζ defined as follows:

$$\begin{aligned}\xi_{\alpha\alpha',\beta\beta'}&=\delta_{\alpha\alpha'}\delta_{\beta\beta'},\\ \eta_{\alpha\alpha',\beta\beta'}&=\delta_{\alpha\beta'}\delta_{\alpha'\beta}-\delta_{\alpha\beta}\delta_{\alpha'\beta'},\\ \zeta_{\alpha\alpha',\beta\beta'}&=\delta_{\alpha\beta'}\delta_{\alpha'\beta}+\delta_{\alpha\beta}\delta_{\alpha'\beta'}.\end{aligned} \tag{14}$$

Thus ξ and ζ are *even* operators (i.e., symmetric in the interchange of α and α', β and β') and η is an *odd* operator.

Now even if one does not take the "square diagram" seriously as representative of the π-π interaction, it is clear that the isotopic-spin dependence of the interaction can be expressed as a *linear* function of the above three operators only. Further, since mesons obey Bose-Einstein statistics, it is easily seen that ξ and ζ will be associated only with states of even angular momenta (of the π-π system) and η with only those of odd angular momenta.

From the definitions (14) of these operators, one can easily find their eigenvalues and put them into proper correspondence with the three different states of total isotopic spin I (*viz.* 0, 1, and 2) that can be formed out of the two-meson system. This is discussed in Appendix I. Here we give only the results (with obvious notation):

$$\begin{aligned}&\xi_0=3, &&\xi_1=\xi_2=0;\\ &\eta_0=\eta_2=0, &&\eta_1=-2;\\ &\zeta_0=\zeta_2=2, &&\zeta_1=0.\end{aligned} \tag{15}$$

We now find that the Green's function $G(\mathbf{s},\mathbf{s}_0)$ can be written as

$$G(\mathbf{s},\mathbf{s}_0)=A\xi+B\eta+C\zeta, \tag{16}$$

where A and C are even functions of s and s_0 and B is an odd function. The functions A, B, and C, can be expressed in terms of the "eigenvalues" G_0, G_1, and G_2 of the operator G in the three states of $I=0$, 1, and 2, by means of (15). Thus,

$$G_0=3A+2C,\quad G_1=-2B,\quad G_2=2C.$$

Therefore, (16) gives

$$G=\tfrac{1}{3}G_0\xi-\tfrac{1}{2}G_1\eta+\tfrac{1}{2}G_2(\zeta-\tfrac{2}{3}\xi). \tag{17}$$

In order to find the forms of the functions G_i, we must make some plausible assumptions about the nature of the π-π interaction. From the very mechanism proposed, it is clear that the interaction should be of a strong, short-range type. The two mesons should there-

fore come close together in order that this final state interaction may be at all effective in increasing the total cross section for π-p scattering. This is possible if the relative energy of these particles is not large.[7]

We shall assume a zero-range interaction in the S-state of isotopic spin $I=0$ of the π-π system. The departure of the Green's function from (8) will then arise from the state $l=0$, $I=0$ only.

A partial wave analysis of $\delta^3(\mathbf{s}-\mathbf{s}_0)$ gives the following result:

$$\delta^3(\mathbf{s}-\mathbf{s}_0)=(4\pi^2 ss_0)^{-1}\delta(s-s_0)\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta_{ss_0}). \quad (18)$$

The integral equation for G_0 will be the same as that for the wave function of the two-meson system. Using normalizations similar to those used in A, one then finds from (18) that the equation for the S-state interaction with isotopic spin $I=0$ is of the form:

$$G_0(\mathbf{s},\mathbf{s}_0)=-\int\frac{d^3\mathbf{s}'}{\omega_s\omega_{s'}}V(\mathbf{s},\mathbf{s}')\times\left[-\frac{\delta(s'-s_0)}{4\pi^2 s's_0}+P\frac{1}{E_0-E'}G_0(\mathbf{s}',\mathbf{s}_0)\right], \quad (19)$$

where

$$V(\mathbf{s},\mathbf{s}')=\int V(\mathbf{r})\exp[i\mathbf{r}\cdot(\mathbf{s}-\mathbf{s}')]d^3\mathbf{r}. \quad (20)$$

Since the range of the interaction is assumed to be small, the exponential in (20) may be approximately replaced by unity, so that $V(\mathbf{s},\mathbf{s}')$ reduces only to a constant, say $4\pi^2\lambda$. The integral Eq. (19) is then seen to have an approximate solution of the form:

$$G_0(\mathbf{s},\mathbf{s}_0)=\alpha(\omega_s\omega_{s0})^{-1}, \quad (21)$$

where α is roughly independent[8] of E_0, s, and s_0. To relate "α" to the π-π scattering amplitude, we note that the "wave function" for the π-π system satisfies the integral equation[9]:

$$\psi(\mathbf{s},\mathbf{s}_0)=-\int\frac{d^3\mathbf{s}'}{\omega_s\omega_{s'}}V(\mathbf{s},\mathbf{s}')\times\left[-\delta(E'-E_0)+P\frac{1}{E_0-E'}\psi(\mathbf{s}',\mathbf{s}_0)\right]. \quad (19a)$$

The solution of (19a), analogously to (19), is

$$\psi(\mathbf{s},\mathbf{s}_0)=(2\pi^2\alpha s_0/\omega_s). \quad (21a)$$

[7] See K. M Watson, Phys. Rev. **88**, 1163 (1952).

[8] This can be easily seen by assuming a function of the form (21) and noting that the last term in (19) contributes mostly in the upper limit of integration, since the integral is evaluated as a principal value. Thus the approximate independence of G_0 from E_0 is established.

[9] Note that (19) and (19a) differ only in the inhomogeneous term on the right-hand side.

This function, evaluated at the "initial value" $s=s_0$ is proportional to $\tan\delta$ where δ is the phase shift. Thus α is proportional to $(s_0^{-1}\omega_{s0}\tan\delta)$. Now if we take outgoing waves only, $\tan\delta$ should be replaced by $\sin\delta\exp(i\delta)$. Thus we have finally:

$$G_0=\left(\frac{a}{s_0\omega_s}\right)(\sin\delta e^{i\delta}), \quad (22)$$

a being a constant with the dimensions of a length. Equation (22) gives G_0 as a function of the relative momenta $\mathbf{s}_0$ and $\mathbf{s}$ of the two pions before and after the interaction. However, the zero-range approximation cannot give the correct dependence of δ upon energy near resonance. To specify this dependence of δ we make the additional assumption that it is given by the usual Breit-Wigner form, *viz.*

$$\sin\delta e^{i\delta}=(i\Gamma/2)/[(\epsilon-\epsilon_r)-i\Gamma/2], \quad (22a)$$

where Γ is the width at half-maximum, ϵ is the energy of either meson in the center-of-mass system of the two mesons, and ϵ_r is the resonant value of ϵ.

The functions G_1 and G_2 which correspond to the states $I=1$ and $I=2$, vanish according to our assumptions. From (8), (14), (17), (21), and (22) we therefore obtain the Green's function as

$$G_{\alpha\alpha',\beta\beta'}(\mathbf{s},\mathbf{s}_0)=\delta^3(\mathbf{s}-\mathbf{s}_0)\delta_{\alpha\beta}\delta_{\alpha'\beta'}+(a/3s_0\omega_s)\sin\delta e^{i\delta}\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \quad (23)$$

The charge dependence of the terms in the right-hand side of (10) can now be simplified with the help of (23). Introducing the operators Q, Q' defined in A, we have the following results:

$$\begin{aligned}&\tau_{\alpha'}G(\mathbf{k}'-\mathbf{k}-\mathbf{q},\mathbf{k}'-\mathbf{k}+\mathbf{q})\tau_\beta g_{\beta'}(\mathbf{k}')\\&\quad+\tau_{\alpha'}G(\mathbf{k}'-\mathbf{k}-\mathbf{q},\mathbf{k}-\mathbf{k}'-\mathbf{q})\tau_{\beta'}g_\beta(\mathbf{k}')\\&=[\tfrac{2}{3}a|\mathbf{k}'-\mathbf{k}-\mathbf{q}|^{-1}\omega_{\mathbf{k}'-\mathbf{k}+\mathbf{q}}^{-1}(\sin\delta e^{i\delta})Q\\&\quad+\delta^3\mathbf{q}Q'+3\delta^3(\mathbf{k}-\mathbf{k}')]g(\mathbf{k}'); \end{aligned}\quad (24a)$$

$$\begin{aligned}&G(\mathbf{k}'-\mathbf{k}-\mathbf{q},\mathbf{k}'-\mathbf{k}+\mathbf{q})\tau_\beta\tau_{\alpha'}g_{\beta'}(\mathbf{k}')\\&\quad+G(\mathbf{k}'-\mathbf{k}-\mathbf{q},\mathbf{k}-\mathbf{k}'-\mathbf{q})\tau_{\beta'}\tau_{\alpha'}g_\beta(\mathbf{k}')\\&=[\tfrac{2}{3}a|\mathbf{k}'-\mathbf{k}-\mathbf{q}|^{-1}\omega_{\mathbf{k}'-\mathbf{k}+\mathbf{q}}^{-1}(\sin\delta e^{i\delta})Q'\\&\quad+\delta^3\mathbf{q}Q+3\delta^3(\mathbf{k}-\mathbf{k}')]g(\mathbf{k}'). \end{aligned}\quad (24b)$$

The last terms in each of the equations (24) are pure self-energy effects and will henceforth be neglected. The terms involving $\delta^3\mathbf{q}$ are the contributions obtained by neglecting the π-π interaction. The first terms are the specific contributions from meson-meson effects.

4. INELASTIC CROSS SECTION

We can now obtain the scattering matrix from the integral equation (10) for $g(\mathbf{p},\mathbf{k})$ following the pro-

cedure of Møller.[10] This equation is the same as that satisfied by the "wave matrix" defined by Møller. Since we are interested only in the Born approximation for the π-p interaction, we can substitute a plane wave for the initial wave function in the right-hand side of (10). We then obtain the submatrix $\mathbf{T}$ of the scattering operator which is defined for conservation of energy and momentum, in the form:

$$\mathbf{T}(\mathbf{k},\mathbf{k}')=(2\pi)^3\mathbf{M}(\mathbf{k},\mathbf{k}'), \tag{25}$$

where $\mathbf{M}$ is defined by the equation:

$$(E-E_k-\omega_k)g(\mathbf{k})=\int \mathbf{M}(\mathbf{k},\mathbf{k}')g(\mathbf{k}')d^3\mathbf{k}'. \tag{25a}$$

This equation, in conjunction with (10), (11), and (12), helps to identify $\mathbf{M}$ immediately in terms of known quantities.

The contribution of $\mathbf{T}$ to the S matrix is in the usual manner, given by

$$-i\mathbf{T}(2\pi)^4\delta^4(p_\mu+k_\mu-p_\mu'-k_\mu').$$

Now using the unitary character of the S matrix and following the procedure of Lippmann and Schwinger,[11] we obtain the total transition probability from the initial state in the form:

$$\textstyle\sum_b \omega_{ba}=\sum_b|T_{ba}|^2=-2\,\mathrm{Im}\mathbf{T}_{aa}=-2\,\mathrm{Im}\mathbf{T}(\mathbf{k},\mathbf{k}).$$

The total inelastic π-p cross section is then given by:

$$\sigma_{\mathrm{in}}=\frac{\text{transition probability}}{\text{incident flux}}=\frac{E_k\omega_k}{k(E_k+\omega_k)}[-2\,\mathrm{Im}\mathbf{T}(\mathbf{k},\mathbf{k})]. \tag{26}$$

Our problem thus reduces to that of calculating $\mathrm{Im}\mathbf{T}(\mathbf{k},\mathbf{k})$, i.e., the matrix element for *forward scattering*.

The expression for $\mathbf{T}(\mathbf{k},\mathbf{k}')$ is greatly simplified when we put $\mathbf{k}=\mathbf{k}'$. Further from an inspection of (24) it appears that the contribution to $\mathrm{Im}\mathbf{T}$ comes only from the term involving $\sin\delta\exp(i\delta)$. We have thus to take the imaginary part of (22a), which is given by

$$\sin^2\delta=(\Gamma^2/4)/[(\epsilon-\epsilon_r)^2+(\Gamma^2/4)]. \tag{27}$$

As regards the dependence of ϵ on the momenta of the two mesons, we note from (10), (11), and (12) that the meson momenta (before π-π scattering) are $\mathbf{k}$ and $\mathbf{k}'-\mathbf{q}$, i.e., $\mathbf{k}$ and $\mathbf{k}-\mathbf{q}$ (since we have put $\mathbf{k}'=\mathbf{k}$). The c.m. energy of each meson is then given by

$$4\epsilon^2=(\omega_k+\omega_{\mathbf{k}-\mathbf{q}})^2-(2\mathbf{k}-\mathbf{q})^2. \tag{28}$$

Collecting all our results contained in (10), (11), (12), and (24)–(27) and simplifying, we have finally:

$$\sigma_{\mathrm{in}}=-2[E_k\omega_k/k(E_k+\omega_k)](a\pi/3)(G^2/4\pi)\times[Q_{\frac{1}{2}}J_1(k,\epsilon_r)+Q_{\frac{1}{2}}'J_2(k,\epsilon_r)], \tag{29}$$

where

$$J_1=\int d^3\mathbf{q}\frac{\sin^2\delta}{q\omega_q}\frac{2(E_kE_{2\mathbf{k}-\mathbf{q}}-M^2)+2\mathbf{k}\cdot\mathbf{q}-4k^2}{E_k\omega_{\mathbf{k}-\mathbf{q}}E_{2\mathbf{k}-\mathbf{q}}(E_k-E_{2\mathbf{k}-\mathbf{q}}-\omega_{\mathbf{k}-\mathbf{q}})}, \tag{30}$$

and

$$J_2=\int d^3\mathbf{q}\frac{\sin^2\delta}{q\omega_q}\frac{2(E_kE_q+M^2)+2\mathbf{k}\cdot\mathbf{q}}{E_kE_q\omega_{\mathbf{k}-\mathbf{q}}(E_k+E_q+\omega_{\mathbf{k}-\mathbf{q}})}. \tag{31}$$

$Q_{\frac{1}{2}}$ and $Q_{\frac{1}{2}}'$ are the eigenvalues of these operators in the total isotopic spin state $T=\frac{1}{2}$, i.e.,

$$Q_{\frac{1}{2}}=3,\quad Q_{\frac{1}{2}}'=-1.\quad\text{(See A.)} \tag{32}$$

The evaluation of the integrals J_1 and J_2 can be considerably simplified by noting that the quantity $\sin^2\delta$ which appears in each of them can, according to (27), be approximately represented as

$$\sin^2\delta\approx(\pi\Gamma/2)\delta(\epsilon-\epsilon_r), \tag{33}$$

since the width Γ of the π-π resonance may be expected to be small compared with the "Doppler width" due to the momentum distribution of the mesons. The actual evaluation of these integrals is shown in Appendix II. Here we quote only the results $[\epsilon_r=(\mu^2+r^2)^{\frac{1}{2}}=\omega_r]$:

$$J_1\approx-\left(\frac{4\pi^2\Gamma\omega_r}{kE_k\omega_{2r}}\right)\left[-\left(\frac{\pi}{2}+\tan^{-1}\frac{\omega_k-\mu}{\omega_{2r}}\right)+\left(\frac{E_k+\omega_k}{E_1}\right)\left(\frac{\pi}{2}+\sin^{-1}\frac{\omega_k-\mu}{E_{2k}}\right)+\left(\frac{\omega_{2r}}{E_1}\right)\ln\frac{(E_2+E_1)}{\{(\omega_k-\mu)^2+\omega_{2r}^2\}^{\frac{1}{2}}}\right], \tag{34}$$

where

$$E_1^2=E_{2k}^2-8r^2-\mu^2,\qquad E_2^2=E_1^2+(\omega_k-\mu)^2+\omega_{2r}^2. \tag{35}$$

$$J_2\approx\left(\frac{4\pi^2\Gamma\omega_r}{kE_k\omega_{2r}}\right)\left[\left(\frac{E_k-\omega_k+M}{M}\right)\left(\frac{\pi}{2}+\tan^{-1}\frac{\omega_k-\mu}{\omega_{2r}}\right)-\left(\frac{\omega_{2r}}{M}\right)\ln\frac{(E_3+M)}{\{(\omega_k-\mu)^2+\omega_{2r}^2\}^{\frac{1}{2}}}\right], \tag{36}$$

where

$$E_3^2=M^2+(\omega_k-\mu)^2+\omega_{2r}^2.$$

5. DISCUSSION

Equations (29), (34), and (36) express the total inelastic (π-p) cross section as a function of the c.m. momentum k of the meson and the nucleon, and the parameter r which represents the resonance momentum of either of the two mesons in their own c.m. frame. The

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relevant energy region of the incident meson (lab system), i.e., 700–1200 Mev corresponds to k ranging between 0.5 M and 0.8 M. Thus we may calculate σ_{in} in this energy region for different values of the parameter r and try to find the "optimum" value of r for which σ_{in} has a maximum at $k=0.62\,M$ which corresponds to $\sim$900 Mev (lab). Now a necessary condition on r is that it should not be too large, since the mesons have to come fairly close together for the interaction to be effective. Reasonable value of r should vary between 100 and 200 Mev (see reference 2 for a typical value of r). We have tried to determine the "optimum" value of r from the equation

$$(d\sigma/dk)=0 \quad \text{for} \quad k=0.62\,M.$$

This equation is quite complicated, but after making some straightforward approximations based on the assumed smallness of (r/M), it reduces to the following:

$$0.92\left(\frac{\pi}{2}+\tan^{-1}\frac{1}{\rho}\right)-0.45\left(\frac{\rho}{1+\rho^2}\right)+0.5\rho$$
$$-\frac{7.9+1.65\rho}{(2.54-0.77\rho^2)^{\frac{3}{2}}}-\frac{4.7+1.65\rho}{(2.54-0.77\rho^2)^{\frac{1}{2}}}$$
$$+\frac{0.9(1-1.9\rho^2)}{(2.54-0.77\rho^2)^2(1+\rho^2)}$$
$$-(0.1\rho)[(1.4+0.4\rho^2)^{\frac{1}{2}}+1.4+0.4\rho^2]^{-1}=0, \quad (37)$$

where

$$\rho=(\omega_{2r}/\omega_k), \quad k=0.62M. \quad (38)$$

It is easy to see from (37) that the expression on the left is overwhelmingly negative and does not change sign for any value of ρ in the range $0<\rho\leqslant 1$ (the limit $\rho=1$ is certainly high from what has been said about r). It must therefore be concluded that Eq. (37) does not have a solution for any reasonable value of r. Hence the cross section $\sigma(k,r)$, as a function of k, does not have a maximum at the relevant energy region. More explicitly, $\sigma(k)$ is a monotonically decreasing function of the meson energy throughout the relevant energy region.

The negative nature of this result forces the conclusion that a resonant π-π interaction in the state $I=0$ is hardly effective in producing a maximum in the final (π-p) cross section. This result, it may be noted, has been derived in the framework of the lowest order Tamm-Dancoff approximation for the (π-p) interaction. It is quite possible that a more adequate treatment of the meson-nucleon interaction (e.g., inclusion of 3-meson intermediate states, etc.) might give a result of a more encouraging nature. However, the object of this investigation was only to examine the working of the lowest order Tamm-Dancoff formalism in producing the desired result. Probably this approximation is enough for the interaction of the incident meson with the nucleon *core*, but the interaction of the latter with its surrounding meson cloud is apparently too strong to be described by the Tamm-Dancoff approximation. The impulse approximation, on the other hand, takes account of this latter interaction in a more realistic (though phenomenological) manner, in that the meson cloud is regarded as tightly bound to the "nucleore." A field-theoretical description of this important feature in a more effective manner than provided by the Tamm-Dancoff approximation does not seem to be available at present.

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APPENDIX I

Let the charge functions be denoted by $g_{\alpha\alpha'}$. Then from (14) we have

$$(\xi g)_{\alpha\alpha'}=\xi_{\alpha\alpha',\beta\beta'}g_{\beta\beta'} \quad \text{(summation convention)}$$
$$=\delta_{\alpha\alpha'}\delta_{\beta\beta'}g_{\beta\beta'}=\delta_{\alpha\alpha'}g_{\gamma\gamma};$$
$$(\xi^2 g)_{\alpha\alpha'}=\{\xi(\xi g)\}_{\alpha\alpha'}=\delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\beta\beta'}g_{\gamma\gamma}$$
$$=3\delta_{\alpha\alpha'}g_{\gamma\gamma}=3(\xi g)_{\alpha\alpha'}.$$

This equation shows that the operator ξ has eigenvalues ξ_I given by the equation

$$\xi_I^2-3\xi_I=0,$$

i.e.,

$$\xi_I=3,\ 0. \quad (\text{I.1})$$

In order to associate these values with the correct isotopic spin states ($I=0$, 1, 2) of the π-π system, we proceed as follows:

Since the "weight" of a state of isotopic spin I is $(2I+1)$, we have the well-known relation

$$\sum_I (2I+1)\xi_I=\text{Tr}\xi=\delta_{\alpha\alpha'}\delta_{\alpha\alpha'},$$

i.e.,

$$1\cdot\xi_0+3\xi_1+5\xi_2=3.$$

Since the ξ's are restricted to the values 3 and 0, the only possible solution of the last equation is

$$\xi_0=3, \quad \xi_1=\xi_2=0. \quad (\text{I.2})$$

In an exactly similar manner one can obtain the eigenvalues of the operators η and ξ which are as given in Eq. (15) of the text.

APPENDIX II

Here we give an approximate derivation of the results (34) and (36) for the integrals J_1 and J_2 defined by (30) and (31), respectively. We examine J_2 first.

From (31) and (33), one finds

$$J_2=\int\left(\frac{\pi\Gamma}{2}\right)\frac{d^3\mathbf{q}}{q\omega_q}\delta(p-r)\times\left(\frac{\omega_p}{p}\right)\frac{2(E_kE_q+M^2)+2\mathbf{k}\cdot\mathbf{q}}{E_kE_q\omega_{\mathbf{k-q}}(E_k+E_q+\omega_{\mathbf{k-q}})},\quad\text{(II-1)}$$

where

$$\epsilon=\omega_p=(p^2+\mu^2)^{\frac{1}{2}},\quad \epsilon_r=\omega_r.\quad\text{(II-2)}$$

We note that the integral J_2 does not diverge for $\mu\to 0$. We shall therefore neglect μ compared with M, since this involves an error of $O(\mu/M)^2$ at most. We shall, however, not neglect μ compared with ω_r since "r" is also of the same order of magnitude as μ. Equation (II-1) can then be simplified by noting the following:

$$2(E_kE_q+M^2)+2\mathbf{k}\cdot\mathbf{q}=(E_k+E_q+\omega_{\mathbf{k-q}})(E_k+E_q-\omega_{\mathbf{k-q}}).$$

Putting

$$\omega_{\mathbf{k-q}}=x,\quad\text{(II-3)}$$

one now obtains

$$J_2=\int_0^\infty(\pi^2\Gamma/kE_k)(dq/E_q\omega_q)\times\int dx\delta(p-r)(\omega_r/r)(E_k+E_q-x).\quad\text{(II-1)}$$

For integration over x, it is convenient to change the variable to p according to (28), which gives

$$x=\omega_k\pm(q^2-4p^2)^{\frac{1}{2}}.\quad\text{(II-4)}$$

On account of the appearance of the δ function, non-vanishing contribution will arise only from $x=x_0$, where

$$x_0=\omega_k\pm(q^2-4r^2)^{\frac{1}{2}}.\quad\text{(II-5)}$$

Since x_0 has to be real, q must be $\geqslant 2r$. Further, since x_0 has to be positive and $\geqslant\mu$, two values of x_0 are admissible for

$$q^2\leqslant 4r^2+(\omega_k-\mu)^2,$$

and only one if the opposite inequality is satisfied. Thus the two regions of q integration are

$$\begin{aligned}&(1)\qquad 2r\leqslant q\leqslant[4r^2+(\omega_k-\mu)^2]^{\frac{1}{2}};\\&(2)\qquad [4r^2+(\omega_k-\mu)^2]^{\frac{1}{2}}\leqslant q\leqslant\infty.\end{aligned}\quad\text{(II-6)}$$

Now most of the contribution to the integral comes from moderate values of $q(\sim k)$, so that one may approximate by using the multiplicative factor (q/ω_q) in the integrand. This results in a considerable simplification in the integration procedure. We now use the new variable z defined by

$$z^2=q^2-4r^2,\quad\text{(II-7)}$$

and obtain after integration over x,

$$J_2=\int_0^{\omega_k-\mu}dz\left(\frac{4\pi^2\Gamma\omega_r}{kE_k}\right)\frac{2E_k+2E_q-2\omega_k}{E_q(\omega_{2r}^2+z^2)}+\int_{\omega_k-\mu}^\infty dz\left(\frac{4\pi^2\Gamma\omega_r}{kE_k}\right)\frac{E_k+E_q-\omega_k-z}{E_q(\omega_{2r}^2+z^2)}.$$

Integrating over z and neglecting some terms of $O(r,\mu/M)$, one obtains Eq. (36) of the text.

The integration over J_1 proceeds in a very analogous manner since all the considerations used in the evaluation of J_2 are needed for J_1 as well. A slight additional complication arises from the appearance of the quantity $E_{2\mathbf{q-k}}$ in J_1. Putting $x=x_0$ [defined in (II-5)] in this expression, one obtains

$$\begin{aligned}E_{2\mathbf{q-k}}^2&=M^2+4q^2+k^2+2(x^2-q^2-k^2-\mu^2)\\&=M^2+4q^2+k^2-8r^2\pm4\omega_k(q^2-4r^2)^{\frac{1}{2}}\\&=(E_k^2+4q^2-8r^2)\pm4\omega_k(q^2-4r^2)^{\frac{1}{2}}.\end{aligned}\quad\text{(II-8)}$$

The second term in (II-8) is small compared with the first throughout the range of q integration. The desired approximation consists in neglecting this term.

In the case of J_1 also, one has the simplification

$$2\mathbf{k}\cdot\mathbf{q}-4k^2+2(E_kE_{2\mathbf{k-q}}-M^2)=(E_{2\mathbf{k-q}}+x-E_k)(x+E_k-E_{2\mathbf{k-q}}).$$

After these simplifications the integration is now exactly analogous to that of J_2. Neglecting certain terms of $O(r,\mu/M)$ one finally gets Eq. (34) of the text.