Theory of Pion Photoproduction*†

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The photoproduction of pions from nucleons is calculated in the Tamm-Dancoff approximation to the relativistic pseudoscalar theory. The matrix elements are explicitly separated, in each partial wave, into production into plane waves and production associated with final-state scattering. These matrix elements are exhibited and compared in the energy region up to 500 Mev. The agreement between theory and experiment is fairly good if one uses the same coupling constant used to calculate scattering in this approximation. The enhancement of the $P\frac{3}{2}$, $T=\frac{3}{2}$ ($M1$) matrix element is about 2.5. A correction (applicable near threshold) for the nonconservation of isotopic spin due to the $\pi^+-\pi^0$ mass difference is evaluated.

PHOTOPRODUCTION of pions from nucleons was first observed at Berkeley in 1949. Since that time, extensive measurements of differential cross sections for π^0 and π^+ production from hydrogen have been made at Berkeley, Cornell, Massachusetts Institute of Technology, California Institute of Technology, and Illinois, from the threshold region up to 500-Mev γ-ray energy.[1,2] Production of π^0 and π^- mesons from neutrons has been inferred from photoproduction from deuterons.

The purpose of this work is twofold: (1) to test a particular approximate meson theory and (2) assuming that the theory is fairly good, to provide a theoretical partial-wave analysis of the photoproduction matrix elements which may in turn provide some guidance for empirical analyses. Before turning in Sec. II to our particular theoretical study of photoproduction, we would like in Sec. I to make a survey of predictions, setting them into those of a more or less general nature and those which are properties of particular phenomenological models or approximate meson theories. In Sec. III, the results will be discussed in connection with the experimental data.

I. SURVEY OF PHOTOPRODUCTION THEORIES

A. General Properties

It is possible to make some statements about pion photoproduction without recourse to any detailed theory or model. Several authors[3-6] have demonstrated or employed the result that, aside from an arbitrary common phase factor, photoproduction matrix elements are real numbers times the same phase factors that characterize the corresponding scattering states. The essential reality of the photoproduction matrix elements when the scattering phase shift is abstracted depends on the assumption that in the scattering, real processes into other channels do not occur, for this would alter the phases of the wave function at close distances. Thus one must not extend this convenient result to very high energy, i.e., of the order of 1 Bev or up.

This result is especially useful since the differential scattering cross sections have been measured in an energy region corresponding to that for photoproduction (Note: The laboratory energy in $n+\pi^+$ scattering plus 151 Mev is the equivalent laboratory energy of the gamma ray for production of π^+, while for scattering of π^0 by p the relevant energy is 145 Mev.)

It has also been shown by Watson[7] that the assumption of charge independence yields a relation between the four different processes that occur in photoproduction:

$$\sigma^+:\quad \gamma+p\to n+\pi^+, \tag{1}$$

$$\sigma_p{}^0:\quad \gamma+p\to p+\pi^0, \tag{2}$$

$$\sigma^-:\quad \gamma+n\to p+\pi^-, \tag{3}$$

$$\sigma_n{}^0:\quad \gamma+n\to n+\pi^0. \tag{4}$$

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† A preliminary discussion of this work appears in H. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, 1955), Vol. 2.

[1] Charged pion production from hydrogen and deuterium, experimental work: J. Steinberger and A. S. Bishop, Phys. Rev. **86**, 174 (1952); Feld, Frisch, Lebow, Osborne, and Clark, Phys. Rev. **85**, 680 (1952); J. M. Keck and R. Littauer, Phys. Rev. **88**, 139 (1952); White, Jakobson, and Schulz, Phys. Rev. **88**, 836 (1952); Jarmie, Repp, and White, Phys. Rev. **91**, 1023 (1953); G. S. Janes and W. L. Kraushaar, Phys. Rev. **93**, 900 (1954); G. Bernardini and E. L. Goldwasser, Phys. Rev. **94**, 729 (1954); Jenkins, Luckey, Palfrey, and Wilson, Phys. Rev. **95**, 179 (1954); Sands, Teasdale, and Walker, Phys. Rev. **95**, 595 (1954); Leiss, Robinson, and Penner, Phys. Rev. **98**, 201 (1955); G. W. Repp, University of California Radiation Laboratory, UCRL-2953, 1955 (unpublished); Walker, Teasdale, Peterson, and Vette, Phys. Rev. **99**, 210 (1955); Tollestrup, Keck, and Worlock, Phys. Rev. **99**, 210 (1955); Beneventano, Bernardini, Goldwasser, Lee, and Stoppini (to be published); G. Bernardini (private communication).

[2] Neutral pion production from hydrogen and deuterium: G. Cocconi and A. Silverman, Phys. Rev. **88**, 1230 (1952); Wolfe, Silverman, and De Wire, Phys. Rev. **99**, 268 (1955); Goldschmidt-Clermont, Osborne, and Scott, Phys. Rev. **57**, 188 (1955); D. C. Oakley and R. L. Walker, Phys. Rev. **97**, 1283 (1955); Walker, Oakley, and Tollestrup, Phys. Rev. **97**, 1279 (1955); F. E. Mills and L. J. Koester, Phys. Rev. **98**, 210 (1955); L. J. Koester, Phys. Rev. **98**, 211 (1955); L. J. Koester (private communication).

[3] K. Aizu, *Proceedings of the International Conference on Theoretical Physics, Tokyo, September 1953* (Science Council of Japan, Tokyo, 1954).

[4] M. Ross, Phys. Rev. **94**, 454 (1954).

[5] K. Watson, Phys. Rev. **95**, 228 (1954).

[6] M. Kawaguchi and S. Minami, Progr. Theoret. Phys. Japan **12**, 789 (1954).

[7] K. Watson, Phys. Rev. **85**, 852 (1952).

B. Phenomenological Theories

To clarify the following discussions, let us express the photoproduction matrix element as a sum of Born-approximation and final-state scattering terms (see Sec. I.D; compare reference 4):

$$\sum_\Gamma(\mathbf{B}_\Gamma \cos\delta_\Gamma+\mathbf{A}_\Gamma \sin\delta_\Gamma)e^{i\delta_\Gamma}\phi_\Gamma. \quad (5)$$

Here the "Born approximation" or nonenhancement $\mathbf{B}_\Gamma$ refers to all production into the Γth partial wave of the plane-wave part of the final state (i.e., $\mathbf{B}$ is the sum of all processes not involving a proper one-pion scattering in the final state). A phenomenological theory generally takes the form of (1) a model which prescribes the form of the energy dependence of some matrix elements $\mathbf{A}_\Gamma$ and $\mathbf{B}_\Gamma$, and (2) an assumption that other matrix elements are negligible. The magnitudes of the matrix elements at some energy and some features of any rapid energy dependence are the parameters. In fact, beyond the threshold region one expects that these quantities $\mathbf{A}_\Gamma$ and $\mathbf{B}_\Gamma$, defined as in (5), will be slowly varying with energy.

In the threshold region the energy dependence of the cross section will be given by the square of the matrix element (5) times the final meson momentum k. In this region the energy dependence of the Born terms $\mathbf{B}_\Gamma$ can be reliably taken as the product of the centrifugal barrier dependence on final meson momentum, k^l, and an unknown function of the slowly varying quantities, photon energy q and total meson energy ω. This latter function has generally been taken from the indications of meson field theory. For meson momenta small compared to the photon momenta, and small compared to momenta characterizing the size of the physical nucleon, this unknown function plays little role. The energy dependence at threshold may be taken in this case as

$$d\sigma/d\Omega\sim(k/q)|\sum\alpha_\Gamma(k/q)^l\phi_\Gamma|^2.$$

The α's are real parameters. This momentum dependence is in fairly good agreement with experiment. If meson scattering is strong but short-ranged at threshold, so that $\delta/(ka)^{2l+1}>1$, where a is the range of the interaction, Watson[8] has shown that the k dependence of the $\mathbf{A}\sin\delta$, or enhancement, term in the matrix element goes as $\sin\delta/k^{l+1}$ at threshold. The cross section due to such a term behaves again as k^{2l+1}. The detailed calculation to be presented here substantiates this statement roughly where it applies, i.e., in the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state. It is indicated, however, that the $\mathbf{A}\sin\delta$ term increases somewhat more rapidly with k than is deduced in this strong-scattering limit. Use of the foregoing relatively reliable results in the threshold region has been discussed by Watson and Gell-Mann,[8,9] and Kawaguchi and Minami.[6]

It is not possible to predict fully the magnitude of the photoproduction when final-state scattering plays an important role, nor to untangle the situation at high energies, without an assumption of detailed knowledge of meson nucleon systems; i.e., more than knowledge of the scattering phase shifts alone. This detailed knowledge, for example, of the form of the meson-nucleon system at close distances, can come from a meson field theory or, in the absence of such a detailed theory or of faith therein, it may be parametrized with the aid of a phenomenological model.[10,11] In theories of photoproduction a basic aim will be of course to establish connections with scattering results. The photoproduction cannot be considered independently of scattering.

C. Meson Field Theories

The more basic field theoretical approach to photoproduction has by now yielded extensive and interesting results. Without detailed calculations, theorems applicable to photoproduction at threshold have been derived[12] from a rigorous treatment which shows that in the low energy, zero-meson-mass limit, the cross section can be expressed in terms of matrix elements of particularly simple form which are in fact the same as the matrix elements which define the strength of the basic interactions (i.e., they define the observed, or renormalized coupling constants e^2 and G_r^2). The Kroll-Ruderman theorem presents the matrix element for S-wave photoproduction at threshold in the form of an expansion in ascending power of μ/M. The leading term in this expansion is essentially the lowest order perturbation theory result for charged mesons involving the renormalized coupling constant while the corresponding term vanishes for neutrals. Similar theorems for P- and higher-wave photoproduction at threshold have been recently developed by Low[13] and by Klein.[14] Klein has

[8] K. Watson, Phys. Rev. **88**, 1163 (1953).

[9] M. Gell-Mann and K. Watson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 219.

[10] Treatments of higher energies based on particular physical assumptions have been made by Brueckner and Watson [K. Brueckner and K. Watson, Phys. Rev. **86**, 923 (1952)], and by Watson and Gell-Mann (reference 9). These discussions generally involve the one-level resonance formalism. This one-level resonance model parameterizes the cross sections associated with $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ scattering. In terms of the model and these parameters the energy dependence of the photoproduction term $\mathbf{A}$ is defined (i.e., the term representing enhancement of the photoproduction into the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state due to the attractive interaction). One can add to this resonant matrix element smaller slowly varying terms representing other matrix elements. There is a large number of such terms however, which fact leads to some ambiguity with the present state of the data. See also Watson, Keck, Tollestrup, and Walker, Phys. Rev. **101**, 1159 (1956).

[11] A phenomenological approach has been developed by R. G. Sachs, Phys. Rev. **87**, 1100 (1952); **95**, 1065 (1954), in order to describe the properties of the physical nucleon and pion scattering. It would seem that his treatment of the scattering is in a convenient representation in case the intermediate coupling theory is a good approximation. Possibilities exist for extending this work to photoproduction. To do so will again, however, require in effect some detailed assumption about pion nucleon states at close distances.

[12] N. Kroll and M. Ruderman, Phys. Rev. **93**, 233 (1954).

[13] F. Low, *Proceedings of the Fifth Annual Rochester Conference on High Energy Nuclear Physics* (Interscience Press, New York, 1955).

[14] A. Klein, Phys. Rev. **99**, 998 (1955).

shown that the outgoing meson current term is correctly given at threshold to corrections of relative order $(\mu/M)^2$ by lowest order perturbation theory in all partial waves (at low energies this term will be the only one for which waves with $l>1$ are important). The remaining portion of the P-state matrix element is given, to corrections of relative order μ/M, by lowest order perturbation theory (including the observed anomalous moment in the photon nucleon interaction). These theorems may enable one to determine the coupling constant and then to check the basic theory. Application of these theorems depends basically on the suggestion that the succeeding terms in the μ/M expansion do not have large coefficients. It is well to keep in mind that this suggestion will generally be expected to be valid in the pseudoscalar theory, where high energy, $E\gg\mu c^2$, intermediate states play an important role so that the cross sections are not sensitive to letting $\mu\rightarrow 0$. On the other hand, this suggestion will generally be invalid if (1) the scattering is large in the final state, (2) meson-meson interaction plays an important role. The second possibility is of a highly speculative nature, but the former is of immediate interest. It is clearly possible for the $\mathbf{A}_\Gamma$ terms associated with final state scattering to be comparable with the $\mathbf{B}_\Gamma$ terms at threshold, if strong scattering is observed at threshold. It is found, however, in the detailed calculation of matrix elements to follow, that the enhancement is particularly small for charged mesons in the S state as the scattering in the two isotopic spin states interferes destructively. In the $P\frac{3}{2}$, $T=\frac{3}{2}$ state the enhancement term does contribute substantially compared to the Born term at threshold. These two results indicate that the leading terms of the S-wave threshold theorem (Kroll-Ruderman) should be a good approximation, but that care must be taken in a comparison with data, because the data always come from above threshold and the production via other processes and into other waves must be properly subtracted. These questions are fully discussed in Sec. III A.

Beyond the scope of the threshold theorems, i.e., in order to treat cases of strong final-state scattering, and at higher energy, detailed calculations must be made. Low-order perturbation theory,[15,16] fairly successful at threshold as we have mentioned, does not agree with experiments at higher energy (say above 200-Mev gamma-ray energy). Inclusion of the observed magnetic moments of the nucleon in the interaction (as a Pauli term), as suggested above, yields considerable improvement in the π^0 predictions.[16] These predictions are still far from satisfactory, however.

More ambitious approximations to meson field theories have been recently considered by Chew,[17] Dyson *et al.*, (D),[18] Friedman, Lee, and Christian,[19] and Low and Chew.[13,20] These theories have been applied with some success to some aspects of pion scattering and to some properties of the nucleon and two nucleon systems. This paper contains an extension of the work of (D) to photoproduction.

The theories of Chew[17] and of Dyson, although the basic philosophy is quite different, are in one sense very similar in their physical content where the same situations can be considered. These are both Tamm-Dancoff (TD) approximations limiting the number of mesons present at one time to the minimum nontrivial value of two. The Chew theory uses gradient meson-nucleon coupling with a cutoff, or finite fixed source. It is a two-parameter theory (assuming one makes no use of detailed features of the source such as its shape). The Dyson theory uses the relativistic pseudoscalar coupling and includes the possibility of nucleon pairs. This is a one-parameter theory. One observes in this theory that there are important contributions in meson scattering processes from terms involving large energy denominators which, it should be kept in mind, imply that the approximation is probably not good (by chance the coupling constant fitting the scattering agrees with that determined by use of the threshold theorems discussed above). A comparable physical situation exists in the Chew theory. Some physical quantities calculated therein are diverging at the cutoff and large contributions are made right in the cutoff region. Adjustment of the cutoff allows one to bring in large contributions effectively equivalent to the large relativistic contributions of the other theory. Countering this disadvantage, as a result of the cutoff, Chew's TD theory can be applied to situations (i.e., properties of the physical nucleon) where the relativistic version has not been tractable. The cutoff theory is also easier to work with. All static models do, however, have the quality of being gauge variant and so essentially recoil corrections should be made in calculations of electromagnetic processes.[21] It is this gauge variant quality which leads to serious ambiguities in, for example, anomalous magnetic moment and electron neutron interaction calculations.[15,22]

The intermediate-coupling theory and the approximation proposed by Low (distinct from his threshold theorems already mentioned) are basically different from the TD approximation. None of these approximate theories includes the other, i.e., they all have

[15] Benoist-Gueutal, Prentki, and Ratier, Compt. rend. **230**, 1146 (1950); G. Araki, Progr. Theoret. Phys. Japan **5**, 507 (1950); K. Brueckner, Phys. Rev. **79**, 641 (1950).

[16] M. Kaplon, Phys. Rev. **83**, 712 (1951).

[17] G. Chew, Phys. Rev. **95**, 1669 (1954); K. Sawada, Progr. Theoret. Phys. Japan **9**, 53 (1953); Fukuda, Goto, Okuba, and Sawada, Progr. Theoret. Phys. Japan **12**, 79 (1954); Chiba, Yamazaki, and Fukuda, Progr. Theoret. Phys. Japan **12**, 767 (1954).

[18] F. Dyson *et al.*, Phys. Rev. **95**, 1644 (1954), to be referred to as (D); M. Kalos and R. Dalitz, Phys. Rev. **100**, 1515 (1955).

[19] G. Takeda, Phys. Rev. **95**, 1078 (1954); Friedman, Lee, and Christian, Phys. Rev. **100**, 1494 (1950).

[20] F. Low, Phys. Rev **97**, 1392 (1955). G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[21] R. Capps, Phys. Rev. **99**, 926 (1955); N. Fukuda (private communication).

[22] R. Capps and W. Holladay, Phys. Rev. **99**, 931 (1955); M. H. Friedman, Phys. Rev. **97**, 1123 (1955); G. Salzman, **99**, 973 (1955).

different physical content. At the present time one cannot demonstrate that any of these approximations yields results faithful to a particular basic theory.

D. Cross-Section Formula

In this section, the expression for the photoproduction matrix element used in the subsequent calculations is derived. Some details of the separation of the matrix elements into two parts as given by Eq. (5) are shown and discussed.

The interaction of the electromagnetic field with the mesons and nucleons, H_e, is treated as a perturbation. The initial and final wave functions Ψ_I, Ψ_F, are each considered as a sum of terms $\Psi^{(m,n;l)}$ involving m nucleons, n antinucleons, and l mesons. In the final state, the free-particle center-of-mass momenta will always be specified by $\mathbf{k}$, $-\mathbf{k}$ for the meson and nucleon, respectively. The final state wave function depends on all the angle and charge variables:

$$\Omega \equiv (\hat{k},\sigma,\tau,i),$$

in addition to the energy E specified by k. The meson charge is denoted by i; $\hat{k}$ denotes the unit vector $\mathbf{k}/k$. Let ξ denote all these eigenvalues of the one-nucleon, one-meson component of the final state $\Psi_F{}^{(1,0;1)}$. The photoproduction matrix element can be written

$$\mathfrak{M}^M(\xi)\delta(\mathbf{I}-\mathbf{F}) = (\Psi_F(\xi), H_e\Psi_I), \tag{6}$$

where $\mathbf{I}$, $\mathbf{F}$ are the initial and final total momenta, respectively, and M is the total magnetic quantum number. The differential cross section is then, in the center-of-mass system:

$$\frac{d\sigma}{d\Omega}(\xi) = \tfrac{1}{2}\sum_M\sum_\sigma |\mathfrak{M}^M|^2(2\pi)^4\left[\frac{dq}{dE_I}\frac{dk}{dE_F}\right]k^2. \tag{7}$$

We take $\hbar = M = c = 1$ (here M is the nucleon mass) and $E_I = q+(q^2+\mu^2)^{\frac{1}{2}}$, $E_F = E(k)+\omega(k) = (1+k^2)^{\frac{1}{2}}+(\mu^2+k^2)^{\frac{1}{2}}$, where μ is the meson mass. Let

$$\Psi_F{}^{(1,0;1)}(\xi) = \int d\xi'\psi(\xi,\xi')a^*(\xi')c^*(\xi')\Phi_0, \tag{8}$$

where a^* and c^* are creation operators for the nucleon and meson fields. In the Born approximation of photoproduction, $\Psi_F{}^{(1,0;1)}$ is the plane wave given by

$$\begin{aligned}\psi_B(\xi,\xi') &= \delta(\xi-\xi')\\ &= (dE/dk)(1/k^2)\delta(E-E')\sum\phi_\Gamma{}^*(\Omega)\phi_\Gamma(\Omega'),\end{aligned} \tag{9}$$

where the ϕ_Γ form a complete orthonormal set. In this calculation we will use, instead of the plane-wave final state ψ_B, the wave function ψ expressed in the Goldberger formalism,[23] which is the stationary-state solution to the meson-nucleon scattering problem‡:

$$\psi(\xi,\xi') = \frac{dE}{dk}\frac{1}{k^2}\sum_\Gamma \phi_\Gamma{}^*(\Omega)\phi_\Gamma(\Omega') \times\cos\delta_\Gamma e^{i\delta_\Gamma}\left[\delta(E-E')+P\frac{f_\Gamma(E')}{E-E'}\right], \tag{10}$$

where δ_Γ is the scattering phase shift in the state Γ and $f_\Gamma(E) = (1/\pi)\tan\delta$. The δ-function singularity in the momentum-space wave function (10) corresponds to the incident plane wave. The additional term $f(E')$ represents the effects of scattering in the Γth state (the principal value of this term is to be taken at $E=E'$). The term in f corresponds to standing waves at large distances. These standing waves form a complete set; the perturbation theory yielding the transition to this final state selects a linear combination corresponding to outgoing waves.[24]

We shall again introduce the concepts of Born approximation and enhanced portions of the matrix element as discussed in connection with Eq. (5) by writing the matrix element:

$$\mathfrak{M}^M(\xi) = \int d\xi'\psi(\xi,\xi')\mathbf{B}^M(\xi'). \tag{11}$$

This implies that we express the final state Ψ_F in the form of an operator times $\Psi_F{}^{(1,0;1)}$. Let

$$\mathbf{B}_\Gamma{}^M(k') = \frac{1}{(4\pi)^{\frac{1}{2}}}\int d\Omega'\phi_\Gamma{}^*(\Omega')\mathbf{B}^M(\xi'); \tag{12}$$

then we have the final form for $\mathfrak{M}$:

$$\mathfrak{M}^M(\xi) = \sum_\Gamma (4\pi)^{\frac{1}{2}}\phi_\Gamma(\Omega) \times\cos\delta_\Gamma e^{i\delta_\Gamma}[\mathbf{B}_\Gamma{}^M(k)+\mathbf{A}_\Gamma{}^M(k)\tan\delta_\Gamma], \tag{5}$$

which was previously stated, where the $\mathbf{A}_\Gamma$ are written:

$$P\left\{\frac{dE_F}{dk}\frac{1}{k^2}\int k'^2dk'\mathbf{B}_\Gamma(k')\frac{1}{E-E'}f_\Gamma(k')\frac{1}{\tan\delta_\Gamma}\right\}. \tag{13}$$

The fruitfulness of expressing the matrix element in this way depends on the $\mathbf{A}$ and $\mathbf{B}$ being insensitive to variations in the phase shift. We find this lack of sensitivity to obtain with the interaction with which we will deal.

Matrix elements to states of high angular momentum are less strongly suppressed by the centrifugal barrier in the case of photoproduction than in scattering. It will be seen later for example, that D and F states can play a moderately important role in the photoproduction differential cross section in the terms $\mathbf{B}_\Gamma$ of Eq. (5) in an energy region where they are not important in scattering or in the $\mathbf{A}_\Gamma$ terms. The proper way to calcu-

[23] M. Goldberger, Phys. Rev. 84, 929 (1951).

‡ *Note added in proof.*—We could equivalently remark that $\psi_\Gamma{}^+ = \cos\delta_\Gamma e^{i\delta_\Gamma}\psi_\Gamma{}^{(1)}$ where ψ^+, $\psi^{(1)}$ are the Γth partial waves of outgoing and standing wave solutions, respectively.

[24] The author would like to thank Dr. N. Austern for an interesting discussion of these questions and for early reference on them, such as N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1933), first edition, p. 258.

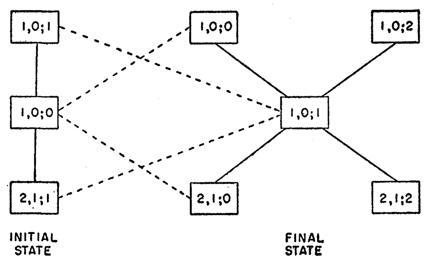

FIG. 1. Configurations considered in this calculation. The solid lines represent connection via the meson-nucleon interaction, the dotted lines connection via the interaction with the electromagnetic field.

late the cross section is then to include all partial waves in $\mathbf{B}_\Gamma$. Both experiment and theory indicate, however, that only S and P states are involved in meson-nucleon scattering up to an energy of about 200 Mev. Thus it will be correct to write the component $\Psi_F{}^{(1,0;1)}$ of the final state in photoproduction as a plane wave corrected to include effects of the meson-nucleon final state interaction only in those S and P states where they are important (actual calculation of the $\mathbf{A}_\Gamma$'s shows that these terms are negligible when the scattering is small). The $\mathbf{A}\sin\delta$ terms will, for simplicity, be considered explicitly only in the S states and in the $P\frac{3}{2}$, $T=\frac{3}{2}$ state.

II. CALCULATION OF THE MATRIX ELEMENT

We wish to employ an approximate meson theory which provides the detailed results necessary to evaluate the enhancement or final state scattering contributions to photoproduction and which is consistent with the threshold theorems, i.e., the $\mathbf{B}_\Gamma$ terms agree with lowest order perturbation theory (including the anomalous moment interaction) at threshold within corrections of order μ/M or $(\mu/M)^2$. We shall use the lowest order Tamm-Dancoff approximation to the symmetric pseudoscalar theory. We shall include the anomalous magnetic moments in the electromagnetic interaction (as a Pauli term).[25] The general philosophy of the approach can be expressed in terms of a two step approximation: (1) We approximate the scattering or photoproduction matrix element (using perturbation theory language) limiting the contributing diagrams considered to a relatively simple set. But in each diagram considered, all vertices and propagation functions are renormalized. The renormalized operators correspond approximately to the unrenormalized ones for processes involving little momentum transfer. Our second step then is (2) to approximate each diagram by replacing some or all of these operators by the unrenormalized forms (with renormalized coupling constant), where the convergence of momentum integrals suggest that this step is not ridiculous (some time orderings are also dropped). The results of this TD approximation already applied to the meson scattering problem in (D) will be taken over here to describe the final state. Only a brief discussion will be given here of the scattering wave functions. A basic problem is to treat the initial state in a consistent manner. A fairly convincing method is proposed which relates the photoproduction matrix elements to the renormalized coupling constant and the final state functions, assuming that the latter are correct. Such a method exists by virtue of the gauge invariance of the electromagnetic interaction.

A. Final State

The treatment of the scattering problem in which the meson nucleon wave function and associated phase shift were determined is very successful in predicting the $P\frac{3}{2}$, $T=\frac{3}{2}$ phase shift through 300 Mev,[18] the coupling constant being adjusted to make a best fit to this data. The other predicted phase shifts are qualitatively in agreement with experiment. Scattering in the $S\frac{1}{2}$ and $P\frac{1}{2}$ states with $T=\frac{1}{2}$ was not handled because of special renormalization problems (i.e., in the spirit of the discussion of the above paragraph).[26] The explicit TD approximation made in order to determine the wave function involves two assumptions: (1) The physical system of a meson and nucleon at low energy can be limited to the simplest configurations (i.e., in addition to the basic nucleon-meson configuration we consider only those configurations which are connected to them through one power of interaction). (2) Self-energy diagrams have a vanishing contribution. Thus in the block diagram showing the processes taken into account (Fig. 1) the final state is represented as a combination of the four configurations: (1,0;0), (1,0;2), (2,1;0), and (2,1;2) in addition to the basic (1,0;1) configuration, where $(m,n;l)$ denotes a configuration of m nucleons, n antinucleons, and l mesons. In effect, the proper TD approximation is used in only a portion of the final state configuration space in order to avoid

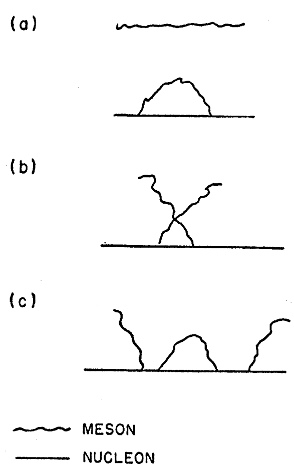

FIG. 2. Some possible scattering processes in the final state. See text, Sec. II.

[25] We do not attempt to calculate the anomalous moments themselves because the theory does not seem tractable in this respect. So it can be said of the photoproduction that we are "reducing the problem to a previously unsolved problem."

[26] See also R. H. Dalitz and F. J. Dyson, Phys. Rev. 99, 301 (1955).

self-energy process. Self-energy diagrams are found to occur in two different ways. One type is found, for example, in the process $(1,0;1)\rightleftarrows(1,0;2)$ which contains the diagram of Fig. 2(a) as well as that of Fig. 2(b) and others. Self-energy contributions of the type (2a) occur separated from other contributions in the Tamm-Dancoff integral equations and are readily dropped.[27] The second type of self-energy contributions occurs only by virtue of the repetition of certain processes by the integral equation; thus $(1,0;1)\rightleftarrows(1,0;0)$ contains the diagram of Fig. 2(c). This latter type of effect exists only when $J=T=\frac{1}{2}$. It is hard to separate out this effect. Among the simpler possibilities for handling the two states involved here, the most consistent is to use first-order perturbation theory for the entire states, with the noninteracting (1,0;1) configuration as zero order wave function.

B. Initial State

One gets into difficulty if one applies the same approximations (1), (2) of the preceding paragraph to the initial state and then calculates the photoproduction. [In this case, first-order perturbation theory would be used to determine the wave function of the initial state from the zero order (1,0;0) configuration.] In this scheme, a typical photoproduction process involves photon absorption immediately "after" the first meson nucleon vertex followed by a meson nucleon scattering process as shown in Fig. 3(a). Meanwhile a process such as shown in Fig. 3(b) is not included. This procedure is not gauge-invariant. The fact is that although we have a wave function of possible merit (which we would like to test) in one configuration of the final state, photoproduction is a transition between two states; and the initial physical system of the nucleon is not represented in a comparable fashion by the wave function

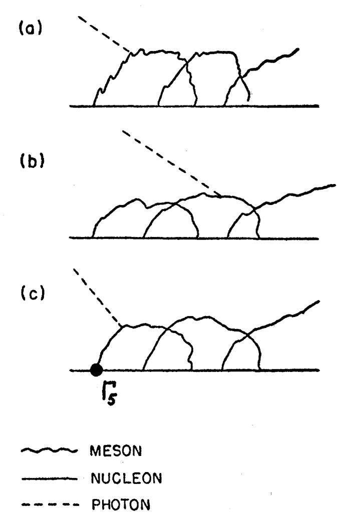

FIG. 3. Some possible photoproduction processes. See text, Sec. II.

calculated in lowest order perturbation theory. The procedure we can follow instead is to construct a gauge-invariant expression at first and then approximate this as well as possible. In order to do this we shall make the assumption that the final wave function $\Psi_F^{(1,0;1)}$ is good. Then it is necessary, for a consistent gauge-invariant theory, to insert a renormalized vertex operator Γ_5 for the first meson-nucleon vertex as indicated in Fig. 3(c). In order to evaluate matrix elements involving this operator, we can calculate the matrix element of the equation of continuity:

$$\left\langle\frac{\partial}{\partial x_\mu}J_\mu\right\rangle_{IF}=0, \tag{14}$$

in addition to the photoproduction matrix element:

$$\langle A_\mu J_\mu\rangle_{IF}.$$

We can assume that $\Gamma_5\rightarrow\gamma_5$ is a good approximation for momentum or free-particle energy exchanges, at our time-ordered vertices, which are small compared to the nucleon mass. This is just to say that only the high-momentum intermediate states are troublesome. The high-momentum portion of the matrix element (14) can be brought into correspondence with the same portion of the photoproduction matrix element so that the necessary integrals involving the Γ_5 operator are evaluated for us in (14).

C. Enhanced Matrix Elements Using γ_5

Let us evaluate all the matrix elements using γ_5 and insert the correction discussed above at the end. Let H_0 be the free-particle energy. Let

$$H_{\pi n}=iG\int\bar{\psi}\gamma_5\tau_i\phi_i\psi d^3x,$$

$$\begin{aligned}H_e=&ie\int\bar{\psi}\gamma_\mu A_\mu\tfrac{1}{2}(1+\tau_3)\psi d^3x\\&-ie\int\left(\frac{\partial\phi^*}{\partial x_\mu}\phi-\frac{\partial\phi}{\partial x_\mu}\phi^*\right)A_\mu d^3x\\&+\tfrac{1}{2}e\int\bar{\psi}\gamma_\mu A_\mu\gamma_\nu q_\nu[\tfrac{1}{2}(1+\tau_3)\mu_p+\tfrac{1}{2}(1-\tau_3)\mu_n]\psi d^3x,\end{aligned}$$

where ψ, A, and ϕ are the nucleon, electromagnetic, and meson fields; H_e is the sum of the electromagnetic interactions with the nucleon charge, meson charge, and anomalous moments μ_p and μ_n of proton and neutron (the latter term in H_e is correctly stated for photon absorption).

The photoproduction matrix element $B(\xi')$ is given in our scheme by second-order perturbation theory terms of the form:

$$\left(a^*(\xi')c^*(\xi')\Phi_0,H_e\frac{1}{E-H_0}H_{\pi n}a_{\sigma'\tau'}{}^*(-\mathbf{q})b_\nu{}^*(\mathbf{q})\Phi_0\right), \tag{15}$$

[27] If the reader does not want to drop these terms, see W. M. Visscher, Phys. Rev. 96, 788 (1954).

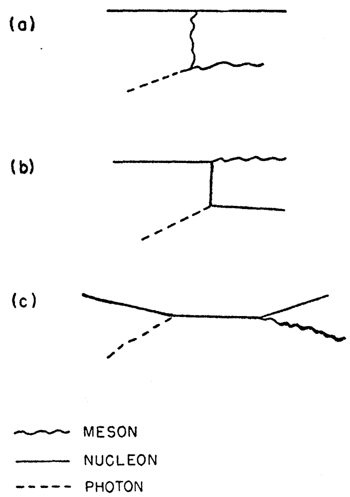

FIG. 4. The three lowest order perturbation-theory processes for photoproduction.

and those with H_e and $H_{\pi n}$ interchanged. The three types of processes involved in (15) are shown in Fig. 4.

Let us fix the polarization as $\nu=(x+iy)/\sqrt{2}$ so that the $\mathbf{q}$ component of total angular momentum is $M=1+\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}$. The final isotopic spin states of charge one and zero, respectively, in terms of proton and neutron functions φ^+, φ^- can be defined as follows:

$$T=\tfrac{1}{2}:\quad (1/\sqrt{3})\tau_i\varphi^{\pm},$$
$$T=\tfrac{3}{2}:\quad (\tfrac{3}{2})^{\frac{1}{2}}(\delta_{i3}-\tfrac{1}{3})\tau_i\varphi^{\pm}.$$

We define the S and P space-spin functions as follows:

$$\begin{aligned}
&(4\pi)^{\frac{1}{2}}\phi_{S\frac{1}{2}}{}^{M=\frac{1}{2}}=\chi^+,\\
&(4\pi)^{\frac{1}{2}}\phi_{P\frac{1}{2}}{}^{M=\frac{1}{2}}=-(\cos\theta\,\chi^+ +\sin\theta\, e^{i\phi}\chi^-),\\
&(4\pi)^{\frac{1}{2}}\phi_{P\frac{3}{2}}{}^{M=\frac{1}{2}}=\left(\sqrt{2}\cos\theta\,\chi^+ -\frac{1}{\sqrt{2}}\sin\theta\, e^{i\phi}\chi^-\right),\\
&(4\pi)^{\frac{1}{2}}\phi_{P\frac{3}{2}}{}^{M=\frac{3}{2}}=(\tfrac{3}{2})^{\frac{1}{2}}\sin\theta\, e^{i\phi}\chi^+.
\end{aligned}\tag{16}$$

The matrix elements $B_\Gamma(k')$ and the corresponding $C_\Gamma(k')$ of the continuity equation, of Eq. (12) to the S and P states, with $M=\frac{1}{2}$ unless stated, are then:

$$\begin{aligned}
-\alpha B_{S\frac{1}{2}}=&-\tfrac{2}{3}\mathfrak{f}'(I_0-I_2)T^{(a)}+[(1/\mathfrak{f}')K_0\\
&+\mathfrak{q}(J_1+K_1)+\tfrac{1}{3}\mathfrak{f}'(J_0+2J_2)]T^{(b)}\\
&+(\bar{\mu}q/2)[(1/\mathfrak{f}')K_0+K_1-J_1\\
&-\tfrac{1}{3}\mathfrak{f}'(J_0+2J_2)](1-\mathfrak{q})T^{(bA)}+[1/(E+1)]T^{(c)}\\
&\qquad-[1/(E+1)](\bar{\mu}q/2)(1+\mathfrak{q})T^{(cA)},
\end{aligned}$$

$$\begin{aligned}
-\alpha B_{P\frac{1}{2}}=&\tfrac{2}{3}\mathfrak{q}(I_0-I_2)T^{(a)}-[\tfrac{1}{3}\mathfrak{q}(J_0+2J_2)\\
&+\mathfrak{q}K_0+(1/\mathfrak{f}')K_1+\mathfrak{f}'J_1]T^{(b)}\\
&+(\bar{\mu}q/2)[\tfrac{1}{3}(J_0+2J_2)-K_0-(1/\mathfrak{f}')K_1\\
&+\mathfrak{f}'J_1](1-\mathfrak{q})T^{(bA)}-[1/(E-1)]\mathfrak{q}\mathfrak{f}'T^{(c)}\\
&\qquad-[1/(E-1)](\bar{\mu}q/2)(1+\mathfrak{q})T^{(cA)},
\end{aligned}$$

$$\begin{aligned}
-\alpha B_{P\frac{3}{2}}{}^{M=\frac{1}{2}}=&(1/\sqrt{2})[\tfrac{2}{3}\mathfrak{q}(I_0-I_2)-(6/5)\mathfrak{f}'(I_1-I_3)]T^{(a)}\\
&+(1/\sqrt{2})[\tfrac{2}{3}\mathfrak{q}(J_0+2J_2)+(2/\mathfrak{f}')K_1\\
&+(2/5)\mathfrak{f}'(2J_1+3J_3)+2\mathfrak{q}K_2]T^{(b)}\\
&+(1/\sqrt{2})(\bar{\mu}q/2)[-\tfrac{2}{3}(J_0+2J_2)+(2/\mathfrak{f}')K_1\\
&\quad+2K_2-(2/5)\mathfrak{f}'(2J_1+3J_3)](1-\mathfrak{q})T^{(bA)},
\end{aligned}$$

$$\begin{aligned}
-\alpha B_{P\frac{3}{2}}{}^{M=\frac{3}{2}}=&(3/2)^{\frac{1}{2}}[\tfrac{2}{3}\mathfrak{q}(I_0-I_2)-(2/5)\mathfrak{f}'(I_1-I_3)]T^{(a)}\\
&+(3/2)^{\frac{1}{2}}[\tfrac{2}{3}\mathfrak{q}(J_0-J_2)-(2/5)\mathfrak{f}'(J_1-J_3)]T^{(b)}\\
&+(3/2)^{\frac{1}{2}}(\bar{\mu}q/2)[\tfrac{2}{3}(J_0-J_2)\\
&\qquad+(2/5)\mathfrak{f}'(J_1-J_3)](1+\mathfrak{q})T^{(bA)},
\end{aligned}$$

$$\begin{aligned}
\alpha C_{P\frac{3}{2}}=&(1/\sqrt{2})\left\{\tfrac{1}{3}\mathfrak{q}(I_0+2I_2)-\tfrac{1}{5}\mathfrak{f}'(2I_1+3I_3)\right.\\
&-[\omega(k')/k'](\mathfrak{q}I_1+\mathfrak{f}'I_2)+\tfrac{1}{2}[E(k')+\omega(k')-E]\\
&\left.\times\frac{1}{2}\int_{-1}^{1}dx(\mathfrak{q}P_1+\mathfrak{f}'P_2)D(a)\right\}T^{(a)}\\
&+(1/\sqrt{2})\left\{\tfrac{1}{3}\mathfrak{q}J_0-\tfrac{2}{5}\mathfrak{f}'J_1+\tfrac{1}{2}[E(k')+\omega(k')-1]\right.\\
&\qquad\left.\times\frac{1}{2}\int_{-1}^{1}dxP_1D(b)\right\}T^{(b)},
\end{aligned}$$

where $T^{(a)}$ is the isotopic spin matrix element for process (a) which, in terms of the amplitudes to the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states, respectively, is:

$$\begin{aligned}
T^{(a)}&=\{-\sqrt{\tfrac{2}{3}},\sqrt{\tfrac{1}{3}}\}, &&\text{for } \gamma+p,\\
&=\{\sqrt{\tfrac{2}{3}},-\sqrt{\tfrac{1}{3}}\}, &&\text{for } \gamma+n,
\end{aligned}$$

and

$$\begin{aligned}
T^{(b)}&=(1/\sqrt{2})\{\sqrt{\tfrac{1}{3}},\sqrt{\tfrac{2}{3}}\}, &&\text{for } \gamma+p,\\
&=(1/\sqrt{2})\{2\sqrt{\tfrac{1}{3}},-\sqrt{\tfrac{2}{3}}\}, &&\text{for } \gamma+n,\\
T^{(bA)}&=(1/\sqrt{2})\{-1/\sqrt{3},2\sqrt{\tfrac{2}{3}}\}, &&\text{for } \gamma+p,\\
&=(1/\sqrt{2})\{1/\sqrt{3},-2\sqrt{\tfrac{2}{3}}\}, &&\text{for } \gamma+n,\\
T^{(c)}&=\{\sqrt{6},0\}, &&\text{for } \gamma+p,\\
&=\{0,0\}, &&\text{for } \gamma+n,\\
T^{(cA)}&=\{\sqrt{6},0\}, &&\text{for } \gamma+p,\\
&=\{-\sqrt{6},0\}, &&\text{for } \gamma+n,
\end{aligned}$$

$$\mathfrak{f}=\mathbf{k}/[E(k)+1],\quad \mathfrak{q}=\mathbf{q}/[E(q)+1],$$

$$\begin{aligned}
D^{-1}(a)&=\omega(\mathbf{k}'-\mathbf{q})[-E(q)+E(k')+\omega(\mathbf{k}'-\mathbf{q})],\\
D^{-1}(a')&=\omega(\mathbf{k}'-\mathbf{q})[-q+\omega(k')+\omega(\mathbf{k}'-\mathbf{q})],\\
D^{-1}(b)&=E(\mathbf{k}'+\mathbf{q})[-E(q)+\omega(k')+E(\mathbf{k}'+\mathbf{q})],\\
D^{-1}(b')&=E(\mathbf{k}'+\mathbf{q})[-q+E(k')+E(\mathbf{k}'+\mathbf{q})],
\end{aligned}$$

and the common factor is:

$$\alpha^{-1}=-\frac{ieG}{16\pi^3}\frac{1}{2}\left[\frac{[E(k')+1][E(q)+1]}{E(k')\omega(k')E(q)q}\right]^{\frac{1}{2}};\tag{17}$$

also,

$$I_n=\frac{k'}{2}\int_{-1}^{1}dxP_n(x)[D(a)+D(a')],$$

$$J_n=\frac{k'}{2}\int_{-1}^{1}dxP_n[D(b)+D(b')],$$

$$K_n=\frac{k'}{2}\int_{-1}^{1}dxP_n\{\omega(k')D(b)+[E-E(k')]D(b')\}.$$

Some small terms were dropped from $C_{P\frac{3}{2}}$ for simplicity. Also we used $\bar{\mu}=\frac{1}{2}(\mu_p-\mu_n)$ and we neglected $\mu_p+\mu_n$.

We could just as well have considered the $P\frac{3}{2}$ contribution in terms of magnetic dipole ($M1$) and electric quadrupole ($E2$) matrix elements instead of $M=\frac{1}{2}$ and $M=\frac{3}{2}$ matrix elements. If we define the matrix elements so that one obtains the same total cross section for a given magnitude of either matrix element, then

$$\begin{aligned}B_{M1}&=\tfrac{1}{2}\sqrt{3}B_{P\frac{3}{2}}{}^{M=\frac{3}{2}}-\tfrac{1}{2}B_{P\frac{3}{2}}{}^{M=\frac{1}{2}},\\ B_{E2}&=\tfrac{1}{2}B_{P\frac{3}{2}}{}^{M=\frac{3}{2}}+\tfrac{1}{2}\sqrt{3}B_{P\frac{3}{2}}{}^{M=\frac{1}{2}}.\end{aligned}\tag{18}$$

Production into the $P\frac{1}{2}$ state must proceed by $M1$ absorption but the above matrix element and this process are independent.

D. Gauge Invariance and Modification of the Enhancement

Now let us consider the effect of the modified vertex operator Γ_5. The vertex γ_5 in the time-ordered processes should be replaced by $f((\mathbf{k}'-\mathbf{q})^2,k'^2)\gamma_5$, where f also depends on the process involved, as discussed in II B. Consider the functions f to be included appropriately in the integrands of I_n, J_n, etc. We see that the $E2$ (electric quadrupole) matrix element [Eq. (18)] and the matrix element $C_{P\frac{3}{2}}$ have almost the same form in the two leading terms (i.e., leading powers of k'). Thus the $E2$ matrix element should be well approximated by the combination (for $T=\frac{3}{2}$):

$$B_{E2}=B_{E2}'+2\sqrt{3}C_{P\frac{3}{2}}'(1+\tfrac{1}{2}\bar{\mu}q),$$

where the prime indicates matrix elements calculated with the unmodified vertex operator. In other words, according to the discussion of Sec. II B, we evaluate the terms in Γ_5 about which we know the least, using Eq. (14). A simple check on this procedure is to examine the contribution of the integrand of the matrix element (13) as a function of k'. This is done in Fig. 5. The curve B_{E2}' before subtraction of $C_{P\frac{3}{2}}'$ is preserved for comparison. It is seen that momenta k' in the region of M and above contribute very little. The assumption is of course that for the small momenta the unmodified γ_5 is a good approximation. Similarly the contribution of the pair process (b') to B_{E2} becomes very small.

The procedure is not as satisfactory for the $M1$ matrix element [Eq. (18)] where the forms of the photoproduction and equation of continuity matrix elements are not as similar. Here again, however, the leading term at high momentum is the same. We note that the leading term in modification of the coefficient of γ_5 (i.e., in f) will be a function of k' and independent of the process. But it appears that terms which are important for $k'\approx M$ do not occur in the same way in the two matrix elements B' and C'. On carrying through the calculation of the leading term we have:

$$B_{M1}=B_{M1}'+2C_{P\frac{3}{2}}\left[1+\left(\frac{2}{q}+1\right)\frac{\bar{\mu}q}{2}\right].$$

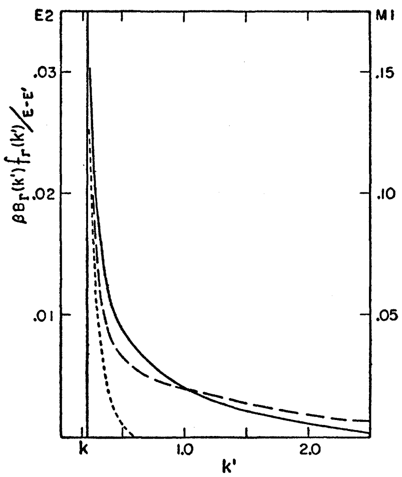

FIG. 5. The integrand of the enhanced matrix element, i.e., the integrand of Eq. (13) showing the relative importance of high momentum intermediate states. The dashed curve shows the $E2$ matrix element before the gauge invariance correction is applied. The dotted and solid curves show the corrected $E2$ and $M1$ integrands respectively. The vertical line just above k is chosen so that the integral from $k'=0$ to this point is approximately zero in all three cases.

The integrand of (13) is again plotted in Fig. 5. The situation is not as satisfactory, the contribution for $k'>M$ being 30% of the total matrix element. Thus the $M1$ matrix element is probably only a fair approximation to the gauge invariant expression.

For the S state this procedure is not of practical interest as the contribution of high momenta to the matrix elements is small and C' is very small. This result arises from the form of the final state wave function.

It should be noted that we evaluated the photoproduction and continuity matrix elements as if we did not need to consider, in addition to processes of the type of Fig. 3(b), where the diagram is taken as time-ordered, processes where the photon vertex is moved in time so that three or more mesons are present. Such processes should have been considered in both matrix elements. If a rough comparison[28] is made of the most

[28] One needs to make strong assumptions such as that the scattering process off the energy shell goes to lowest order perturbation theory [K. Brueckner and K. Watson, Phys. Rev. **92**, 1023 (1953)], in order to handle these terms.

TABLE I. Isotopic spin 3/2 enhancement.

$E_{\mathrm{lab}}\gamma$	$\beta\mathbf{A}_S$	$\beta\mathbf{A}_{P\frac{3}{2}}{}^{M=\frac{1}{2}}$	$\beta\mathbf{A}_{P\frac{3}{2}}{}^{M=\frac{3}{2}}$
163	1.02	−0.72	1.68
264	0.41	−0.32	0.73
520	0.07	−0.22	0.55

important terms of this kind to the terms already calculated as a function of the intermediate momentum k', one finds of course that the ratio is small for $k'<M$ but grows to order of $\frac{1}{2}$ for $k'>M$. Thus one can estimate that the contribution of these high energy intermediate state terms will be considerably less than $\frac{1}{2}$ the matrix elements already calculated, and they will tend to cancel when the vertex operator is modified as above. Roughly then the additional contributions would be $<\frac{1}{4}$ the $M1$ matrix element as already calculated, and negligible for $E2$ and S-state matrix elements. These contributions will be neglected in the following.

The enhanced matrix elements as defined by Eq. (13) were integrated numerically using the $B(k')$ discussed above and wave function $f(E')$ numerically obtained by Kalos and Dalitz[18] from the Tamm-Dancoff integral equations. Results are given in Table I for the quantity

$$\beta\mathbf{A}_\Gamma(k)\equiv(4\pi/ieG\sqrt{2})(2\pi)^2 \times[(dq/dE_I)(dk/dE_F)]^{\frac{1}{2}}k\mathbf{A}_\Gamma(k). \tag{19}$$

From the differential cross section, Eq. (7), it is seen that the tabulated quantity will yield microbarns/steradian when squared and multiplied by

$$445(e^2/4\pi)(G^2/4\pi),$$

since

$$10^{-30}/(\hbar/Mc)^2=445. \tag{20}$$

It is convenient to list approximate analytic forms for these results ($T=\frac{3}{2}$):

$$\begin{aligned} \beta\mathbf{A}_S&=[0.11/(kq)^{\frac{1}{2}}](1-1.2q),\\ \beta\mathbf{A}_{P\frac{3}{2}}{}^{M=\frac{1}{2}}&=-0.31q/kE, \\ \beta\mathbf{A}_{P\frac{3}{2}}{}^{M=\frac{3}{2}}&=0.73q/kE. \end{aligned} \tag{21}$$

E. Born Approximation

To conclude, it is worthwhile to discuss the plots of the matrix elements $\mathbf{B}_\Gamma(k)$ on the energy shell. The integrals in the expressions above simplify:

$$I_n\to\tfrac{1}{2}k\int_{-1}^{1}dxP_n(x)/[q\omega(k)-\mathbf{q}\cdot\mathbf{k}],$$

$$J_n\to\tfrac{1}{2}k\int_{-1}^{1}dxP_n(x)/[qE(k)+\mathbf{q}\cdot\mathbf{k}],$$

$$K_n\to\omega(k)J_n/[E(k)+1].$$

The integral J_n in particular can be well approximated by expanding the denominator in $k/E(k)$. This was partially done in reference 4, where in order to obtain the matrix elements $-\alpha B_\Gamma(k)$ it is necessary to divide by the factor

$$-qC\equiv-\left[\frac{k[E(k)+1][E(q)+1]}{8qE^2}\right]^{\frac{1}{2}}.$$

In addition the anomalous moment contributions must be added. This will not be done here, however. We merely plot the $\mathbf{B}_\Gamma(k)$ in Fig. 6. The quantities plotted are actually $\pm\beta\mathbf{B}_\Gamma$ [see Eq. (19)]. Simple analytic forms for these quantities near threshold are given in Eqs. (27) and (32).

The production into higher waves, $l>1$, in Born approximation is shown graphically in Fig. 7. The quantities plotted are defined by the relation:

$$\sum_\sigma K_\sigma{}^M(\theta)\chi^\sigma=\beta\sum_{j,l>1}\mathbf{B}_\Gamma{}^M\phi_\Gamma{}^M(\theta). \tag{22}$$

In addition, the appropriate combination of isotopic spins is taken for π^+ production. Only two of the four functions for π^+ production are important and none of those for π^0 production is important. The latter result is expected, as high partial waves arise only from the outgoing meson current terms [such as process (a) of Fig. 4]. For negative-meson production $K(\pi^-)=-K(\pi^+)$.

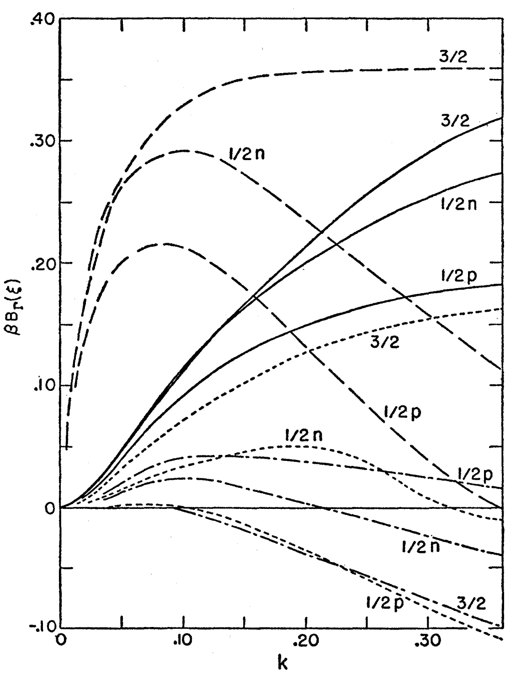

FIG. 6. Born approximation matrix elements to S and P states. Both photoproduction from protons and neutrons is considered, as indicated by p and n. The isotopic spin character of the meson-nucleon system is indicated as $\frac{1}{2}$ or $\frac{3}{2}$. In the $\frac{1}{2}$ p case the negative of the matrix element is plotted. In the $T=\frac{3}{2}$ case, the matrix element for production from protons is shown, from neutrons the matrix element is equal but of opposite sign. The dashed curve is for the S state, the solid for the $P\frac{3}{2}$ state with $M=\frac{3}{2}$, the dotted for the $P\frac{1}{2}$ and the dot-dashed for the $P\frac{3}{2}$ with $M=\frac{1}{2}$.

III. COMPARISON WITH EXPERIMENT

Using Eqs. (4) and (7), one obtains the differential cross section in μbarns/sterad for producing a meson of given charge from the calculated $\mathbf{A}_\Gamma$ and $\mathbf{B}_\Gamma$ as:

$$d\sigma/d\Omega=445(e^2/4\pi)(G^2/4\pi)\sum_M\sum_\sigma|X_J{}^M\phi_J{}^M|^2,$$
$$\beta^{-1}X_J{}^M\equiv\sum_{T=\frac{1}{2},\frac{3}{2}}a_T(B_\Gamma\cos\delta_\Gamma+A_\Gamma\sin\delta_\Gamma)\exp(i\delta_\Gamma), \tag{23}$$

where the ϕ_J are the angle spin functions given for S and P states in Eq. (17), and the factor β is defined by Eq. (19). Certain combinations of $T=\frac{3}{2}$ and $T=\frac{1}{2}$ matrix elements are taken to produce a meson of given charge. The required coefficients a_T yielding the various processes are:

$$\sigma^+,\sigma^-:\quad -(\tfrac{2}{3})^{\frac{1}{2}}(T=\tfrac{1}{2})+(\tfrac{1}{3})^{\frac{1}{2}}(T=\tfrac{3}{2}),$$
$$\sigma_p{}^0,\sigma_n{}^0:\quad (\tfrac{1}{3})^{\frac{1}{2}}(T=\tfrac{1}{2})+(\tfrac{2}{3})^{\frac{1}{2}}(T=\tfrac{3}{2}). \tag{24}$$

Writing out Eq. (23) explicitly for S and P states and grouping the higher waves as in Eq. (22), we have the expression most convenient for direct evaluation of the cross section in μb/sterad:

$$\begin{aligned}d\sigma/d\Omega=445(e^2/4\pi)(G^2/4\pi)|X_S+K_+{}^{\frac{1}{2}}(\theta)\\+(-X_{P\frac{1}{2}}+\sqrt{2}X_{P\frac{3}{2}}{}^{\frac{1}{2}})\cos\theta|^2\\+|X_{P\frac{1}{2}}+(1/\sqrt{2})X_{P\frac{3}{2}}{}^{\frac{1}{2}}|^2\sin^2\theta\\+\tfrac{3}{2}|X_{P\frac{3}{2}}{}^{\frac{3}{2}}|^2\sin^2\theta+|K_-{}^{\frac{3}{2}}(\theta)|^2.\end{aligned} \tag{25}$$

The predictions and experimental results are generally expressed in the form

$$d\sigma^+/d\Omega=A_0{}^++A_1{}^+\cos\theta+A_2{}^+\cos^2\theta, \tag{26}$$

for π^+ production from protons, for example. One assumes in this expansion that D and higher waves are unimportant (except for S-D interference). Outside the extreme threshold region this is not true for charged meson production. It is seen in Fig. 11 that strong deviation from the form of Eq. (26) is predicted at extreme forward angles, $\theta\lesssim 20°$. There are no extensive measurements that penetrate this region, so that in the following we will present the π^+ predictions in the form of the A's of Eq. (26), using a least squares analysis of the predicted cross sections at 25°, 45°, 90°, 135°, and 180°. The lower angle is omitted below 230 Mev.

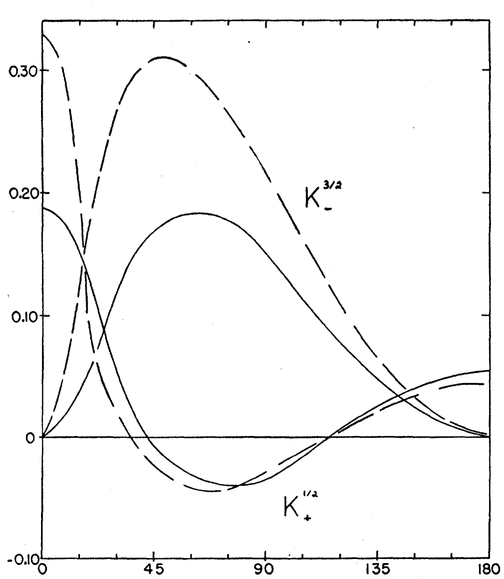

FIG. 7. Born approximation matrix elements summed for $l>1$. For 265 Mev and 520 Mev $E_{\mathrm{lab}}{}^\gamma$. See Eq. (22).

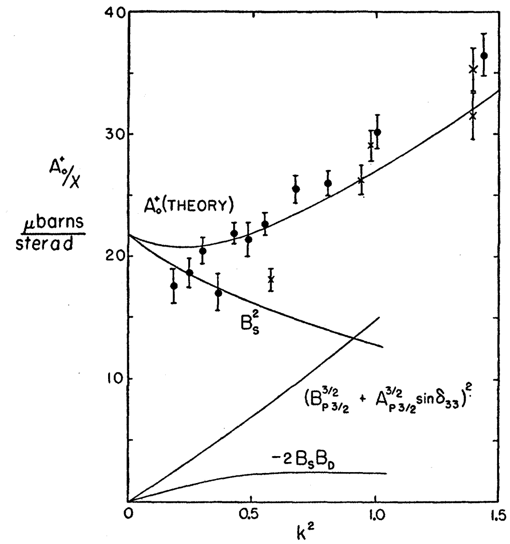

FIG. 8. The isotropic term $A_0{}^+$ in photoproduction of positive mesons from protons near threshold, *vs* the square of the center-of-mass meson momentum in units of μc. See Sec. III A. The correction due to the $\pi^+-\pi^0$ mass difference taken from Table II would increase the cross section about 1 μb/sterad except at extremely low energy. The circles are weighted averages prepared by G. Bernardini from work at Illinois and Cornell. The crosses are points from California Institute of Technology of Walker *et al.* and Tollestrup *et al.* (see reference 1).

A. Charged Pions at Threshold

Charged meson production near threshold is particularly interesting because of the possibility of application of the Kroll-Kuderman theorem. To get a feeling for the situation, let us exhibit the S and P Born approximation matrix elements in rough analytic expansions:

$$\begin{aligned}\beta\mathbf{B}_S&=(1/\sqrt{2}E)(k/q)^{\frac{1}{2}}[1-q/2+O(k^2/q^2)+O(q^2)],\\\beta\mathbf{B}_{P\frac{1}{2}}&=(1/4\sqrt{2}E)(k/q)^{\frac{3}{2}}[1-(4\bar{\mu}q/3)],\\\beta\mathbf{B}_{P\frac{3}{2}}{}^{M=\frac{1}{2}}&=(1/4E)(k/q)^{\frac{3}{2}}\{1-[3+(2\bar{\mu}/3)q]\},\\\beta\mathbf{B}_{P\frac{3}{2}}{}^{M=\frac{3}{2}}&=(1/2\sqrt{3})(k/q)^{\frac{3}{2}}[1-(\tfrac{1}{2}-\bar{\mu})q].\end{aligned} \tag{27}$$

(The nucleon mass, it is recalled, is set equal to unity.) The small role played by anomalous moment type processes is evident.

The calculated $(d\sigma/d\Omega)$ (90°) near threshold, dividing

out the kinematic factor

$$\chi = (1/E^2)(k/q), \tag{28}$$

as suggested by Bernardini in a similar analysis,[29] is plotted in Fig. 8 along with experimental data, with $G^2/4\pi = 15.7$. The large contributions are Born-approximation S wave, and also $P\frac{3}{2}$ wave production. The Born S and Born S-D interference and the full $P\frac{3}{2}$ term for $M=\frac{3}{2}$ are plotted (for $P\frac{3}{2}$, $M=\frac{1}{2}$ the matrix element is very small). The latter has the approximate analytic form:

$$X_{P\frac{3}{2}}{}^{\frac{3}{2}} = [(0.31k/q)+0.74(\tfrac{1}{3})^{\frac{1}{2}}(\delta_{33}/k^3)(kq)^{\frac{3}{2}}], \tag{29}$$

with $\delta_{33}/k^3 \approx 0.25$ in units of $1/\mu^3$. We neglect the imaginary part of the matrix element. The second term on the rhs, the enhancement term, is approximately equal to the Born term at $E_{\text{lab}}{}^{\gamma} = 265$ Mev, but vanishes more rapidly at threshold. It seems accidental that the $P\frac{3}{2}$ contribution is just proportional to k^2. It is seen that the S state contribution plotted this way is strongly energy dependent and is not a linear function of k^2 [at threshold the cross section reduces to $445(e^2/4\pi)(G^2/4\pi)\chi\times 2/(2M+\mu)^2$]. It is important to note that the S-state contribution is not overwhelming at energies sufficiently high to have plenty of statistics. Thus a reliable estimate of the P-wave production is necessary.

S-state enhancement contributions $\mathbf{A}_S$ were neglected in the calculations here or at higher energy for positive mesons. It is fairly well established experimentally that the S-wave phase shifts are:

$$\delta_3 = -0.11k/\mu,$$
$$\delta_1 = 0.16k/\mu,$$

at least for $E_{\text{lab}}{}^{\pi} \lesssim 100$ Mev.[30] Meanwhile the TD theory of scattering yields

$$\delta_3 = -0.38k/\mu,$$

and is inadequate to make a prediction for δ_1. To get a rough idea of the enhancement in this situation, the quantity $\mathbf{A}$ was taken from the TD S-state calculation for $T=\frac{3}{2}$ (Table I) and the ratio $\mathbf{A}/\mathbf{B}$ assumed to be the same in both isotopic spin states. Then the experimental phase shifts of Eq. (27) were used. The resulting enhancement term is small: $X_S \approx 1.05\beta\mathbf{B}_S$.

Corrections have been calculated to account for violation of conservation of isotopic spin resulting from the mass difference between charged and neutral mesons.[31] This correction has not been included in the curve, Fig. 8, or in the cross sections, Eqs. (27) and (29). The calculation of this correction is given in the appendix. It is based on the assumption that for sufficiently small r, $\psi^{(1,0;1)}(r)$ does not depend on small changes in the meson masses. The corrected cross sections are expressed as functions of $\kappa = 2(k_0-k_+)/(k_0+k_+)$ and are proportional to the cross sections calculated assuming equal masses. In the S state the corrections to the cross section are of order $(\delta\kappa)^2$ and can be neglected [see Eq. (A1)]. In the P states the corrections are of order κ and are important. Let $\mu_0{}^e$, $\mu_+{}^e$ be the experimental π^0, π^+ masses, respectively. Indicating which masses are employed for π^0 and π^+ in the calculation by the notation $\sigma(\mu_0,\mu_+)$, the corrected P state cross sections are [from Eq. (A3)]:

$$\begin{aligned}\sigma^+(\mu_0{}^e,\mu_+{}^e) &= \sigma^+(\mu_0{}^e,\mu_0{}^e)\left(1-\frac{2e^{i\delta_+}-1}{ikr_0}\kappa\right)\frac{k_+}{k_0},\\ \sigma^0(\mu_0{}^e,\mu_+{}^e) &= \sigma^0(\mu_+{}^e,\mu_+{}^e)\left(1+\frac{2e^{i\delta_0}-1}{ikr_0}\kappa\right)\frac{k_0}{k_+},\end{aligned} \tag{30}$$

where δ_+ and δ_0 are defined in Eq. (A2). Using the calculated cross sections to make the transition $\sigma(\mu_+{}^e,\mu_+{}^e)\rightarrow\sigma(\mu_0{}^e,\mu_0{}^e)$ we find for example the production into the $P\frac{3}{2}$ state to be (assuming $\delta_{33}\ll 1$, $\delta_{13}=0$):

$$\begin{aligned}\sigma^+(\mu_0{}^e,\mu_+{}^e) &= \{1+[2-(4\delta_{33}/3kr_0)]\kappa\}\sigma^+(\mu_+{}^e,\mu_+{}^e),\\ \sigma^0(\mu_0{}^e,\mu_+{}^e) &= \{1-[3-(8\delta_{33}/3kr_0)]\kappa\}\sigma^0(\mu_0{}^e,\mu_0{}^e),\end{aligned} \tag{31}$$

where the cross sections of the right-hand side are those we calculated before considering the mass difference. It is seen that the radius, r_0, dividing the free-particle region from that of strong interaction, enters explicitly (it does not for the S waves). This would not be a satisfactory situation if we wanted more than a rough idea of these corrections. We will take $r_0=\frac{1}{2}\mu$. The corrections to the $P\frac{3}{2}$ cross sections are given numerically in Table II.

The associated predictions for $A_1{}^+$ and $A_2{}^+$ are shown in Fig. 10. Extensive experimental data below 230 Mev from the same experiment yielding the results for $A_0{}^+$ (Fig. 9) are not shown. The errors on these data are about the same as those shown as higher energy. The predicted $A_2{}^+$ curve agrees well with these data, while the $A_1{}^+$ approaches closer to zero than experiment, the upper ends of the experimental bars roughly coinciding with the predicted curve. The quantity $A_2{}^+$ is principally due to the contribution of the $P\frac{3}{2}$, $M=\frac{3}{2}$ term

TABLE II. Relative corrections to be added to $P\frac{3}{2}$ cross sections due to the $\pi^+ - \pi^0$ mass difference.

$E_\gamma{}^{\text{lab}}$ (Mev)	κ	π^+ production $[2-(8\mu\delta_{33}/3k)]\kappa$	π^0 production $-[3-(16\mu\delta_{33}/3k)]\kappa$
160	0.26	0.50	−0.74
170	0.14	0.26	−0.37
180	0.10	0.17	−0.25
190	0.08	0.13	−0.18
200	0.06	0.10	−0.14
225	0.04	0.06	−0.08
250	0.03	0.03	−0.04

[29] G. Bernardini and E. L. Goldwasser, Phys. Rev. **95**, 857 (1954).

[30] J. Orear, Phys. Rev. **96**, 178 (1954).

[31] The author would like to thank Dr. L. J. Koester for arousing his interest in this problem. H. P. Noyes, Phys. Rev. **101**, 320 (1956) has investigated the effect of nonconservation of isotopic spin on low-energy pion scattering. Basically, his considerations differ from the present ones in that he assumes pion-nucleon interactions which may be long-tailed, while we employ a theory where this does not occur.

given in Eq. (29), while A_1^+ is due to the S wave interference with the difference between contribution to the $P\frac{1}{2}$ and $P\frac{3}{2}$, $M=\frac{1}{2}$ waves.

In addition to these π^+ predictions the predicted π^- cross sections may be compared with experiment. The σ^-/σ^+ ratio at $k=0$ for production from deuterium is found in the recent work of Beneventano *et al.*[2] to be 1.80 ± 0.15. Coulomb corrections will reduce this result about 10%. The predicted value for σ^-/σ^+ is the same here as in perturbation theory, i.e., $[(M+\mu)/M]^2=1.32$. This result is insensitive to enhancement of the S state matrix element. Such enhancement contributes about equally to the two processes. There is, however, considerable question as to the significance of this comparison of single-nucleon calculations with deuteron data.[32] The data actually contain a rather sharp increase in σ^-/χ as threshold is approached (in the region up to 50 Mev above threshold). The theory under discussion offers no hint as to the explanation of this rise.

In light of the above comparisons, we note two points relative to the application of the Kroll-Ruderman and similar theorems. The enhancement contribution in the S state is estimated above to be small only because the experimental phase shifts are roughly equal and opposite. We see that, if one considered instead the *a priori* reasonable phase shifts $\delta_1=\delta_3\approx k/M$, one would obtain a considerable enhancement term for the charged photoproduction matrix elements, a term approximately of order (μ/M) compared to Born approximation. Now the Kroll-Ruderman theorem predicts such terms to be $(\mu/M)^2$, and the theorem is sufficiently general to describe more theories than the true theory yielding the phase shifts (24). This is probably not a disagreement with the theorem which results from our approximations but is perhaps an example of the possibility of terms formally of order $(\mu/M)^2$, say, to appear as terms of order $\ln^n(M/\mu)(\mu/M)^2$. The second point is the example of how the coefficients of various terms of relative order μ/M, $(\mu/M)^2$, etc., may become large due to resonance processes, so that one may observe large contributions for other than leading terms in the series. One, of course, observes this substantial deviation from Born approximation in the $P\frac{3}{2}$ production.

There are apparent two immediate weaknesses in the theory as used to predict charged pion photoproduction near threshold. One was choice of the coupling constant (call it G_γ) to be equal to that used in the scattering problem (i.e., equal to the G_s which was fitted to the $\frac{3}{2}$, $\frac{3}{2}$ scattering data). We used a partial and approximate renormalization procedure to make the photoproduction calculation consistent with that for scattering. The more correct the final state wave function used in this calculation the closer should be the agreement between G_γ and the correct renormalized constant. The coupling constant, G_s, appearing in the scattering integral equations (the latter are finite but high-momentum mesons make large contributions) is also an approximation to the renormalized constant. The relation between G_γ and G_s is otherwise obscure. A second weakness in the predictions lies in handling the S state enhancement. The estimate of this contribution as small is probably correct, but it may not be so small as to exclude quantitatively significant effects, considering the accuracy of the data now becoming available.

[32] In these experiments, $E_{\mathrm{lab}}{}^{\gamma}$ is calculated from the observed meson energies, assuming that the meson was produced from a free nucleon at rest. Within 50 Mev of threshold, the relative energy of the two final nucleons is a fraction of a Mev. There has been no published treatment of the problem taking the strong interactions of these two nucleons into account at these energies. The question is whether or not there might be sufficient difference between the n-n and p-p systems to explain the different behavior of π^+ and π^- cross sections. Without extensive investigation it can be observed that correcting for this effect is in the direction of reducing σ^-/σ^+. If there is a strong very-low-energy nucleon attraction in the final state, the energy of the incident photon is lower than one would otherwise calculate from the observed pion energy. Thus there are relatively more events near threshold for σ^+ than one calculates using single-nucleon kinematics. Another way to look at this question is to ask: How reliable is the extrapolation to threshold considering the lack of knowledge of $E_{\mathrm{lab}}{}^{\gamma}$?

B. Neutral Pions at Threshold

Neutral meson production proceeds qualitatively through an $M1$ transition to the $P\frac{3}{2}$ state without the large Born approximation S wave production that occurs for charged pions. Near threshold, however, we are in an advantageous position to observe the S and $P\frac{1}{2}$ π^0 production.

The Born approximation matrix elements near threshold are approximately:

$$
\begin{aligned}
\beta\mathbf{B}_S&=(1/2E)(kq)^{\frac{1}{2}}(1+\bar{\mu}),\\
\beta\mathbf{B}_{P\frac{1}{2}}&=(1/3E)(k/q)^{\frac{3}{2}}q(1+\bar{\mu}),\\
\beta\mathbf{B}_{P\frac{1}{2}}{}^{\frac{1}{2}}&=-(1/6\sqrt{2}E)(k/q)^{\frac{3}{2}}q(1+\bar{\mu}),\\
\beta\mathbf{B}_{P\frac{3}{2}}{}^{\frac{3}{2}}&=(1/2\sqrt{6}E)(k/q)^{\frac{3}{2}}q(1+\bar{\mu}).
\end{aligned}
\tag{32}
$$

The simple and important way in which the anomalous moment $\bar{\mu}=\frac{1}{2}(\mu_p-\mu_n)$ enters is of interest. The corresponding π^+ matrix elements were given in Eq. (27).

The S-state enhancement will now be important as the production into the S state arises from the difference between large $T=\frac{3}{2}$ and $\frac{1}{2}$ matrix elements. This enhancement is estimated, as above, by using the $\mathbf{A}_S$ of the theory in conjunction with the empirical phase shifts of Eq. (28). The full S and $P\frac{3}{2}$ matrix elements near threshold are then approximately:

$$
\begin{aligned}
X_S&=(1/2E)(k/q)^{\frac{1}{2}}[(0.42q/\mu)-0.24],\\
X_{P\frac{3}{2}}{}^{M=\frac{1}{2}}&=-(1/E)(k/q)^{\frac{1}{2}}[(0.050k/\mu)\\
&\qquad+0.25(\delta_{33}/k^3)(kg)^{\frac{3}{2}}],\\
X_{P\frac{3}{2}}{}^{M=\frac{3}{2}}&=(1/E)(k/q)^{\frac{1}{2}}[(0.086k/\mu)\\
&\qquad+0.60(\delta_{33}/k^3)(kq)^{\frac{3}{2}}]
\end{aligned}
\tag{33}
$$

where the Born and enhancement terms are exhibited separately in that order and $(\delta_{33}/k^3)\approx 0.25$ in units of $1/\mu^3$. The effect of the $\pi^0-\pi^+$ mass difference has not been included in Eqs. (32) and (33).

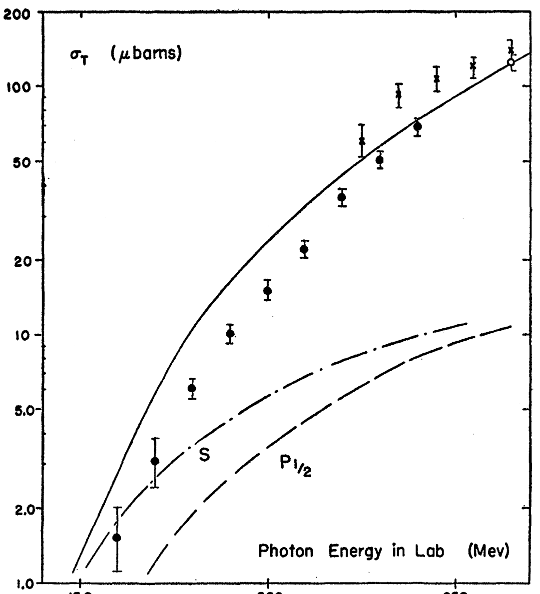

FIG. 9. Neutral pion photoproduction from protons. Semilog plot of total cross section near threshold. The filled-in circles were taken from Mills and Koester (see reference 34). The crosses are points from Goldschmidt-Clermont *et al.* (M.I.T. group), and the open circle from Oakley and Walker. See reference 2. The partial contributions of S and $P\frac{1}{2}$ wave production are shown. The correction due to $\pi^+-\pi^0$ mass difference as given in Table II is included.

It is seen that the S wave production becomes small, as is observed experimentally, only through cancellation by the enhancement term (this term is but crudely estimated here). Note that complete cancellation is not possible as we have neglected the imaginary part of the matrix element.[33] Thus an S term with one higher power of k than that shown in Eq. (33) must at least be present.

The $P\frac{3}{2}$ matrix elements come almost entirely from magnetic dipole contribution. In the region where $k^{\frac{1}{2}}q^{\frac{3}{2}}/\mu^2\approx 1$, the ratio of $M1$ to $E2$ matrix elements, defined analogously to the $\mathbf{B}$'s of Eq. (18), is

$$X_{M1}/X_{E2}=15.$$

Meanwhile the ratio of the Born term to enhancement term in $X_{(M1)}$ is 1/1.6 and not zero as has been assumed in other theoretical approaches. Both of these ratios become smaller at higher energies.

An interesting aspect of the calculation is that the $P\frac{1}{2}$ contribution [Eq. (32)] is comparable with the $P\frac{3}{2}$. The ratio is

$$X_{P\frac{1}{2}}/X_{M1}=0.55,$$

taking $k^{\frac{1}{2}}q^{\frac{3}{2}}/\mu^2\approx 1$. The effect of this $P\frac{1}{2}$ contribution is to increase the magnitude of the coefficient of the $\cos\theta$ term in the matrix element, thus increasing $-A_1$ and A_2.

[33] Watson (reference 5) has derived, as a consequence, an inequality for X_S in terms of the π^+ matrix element which for small phase shifts is

$$|X_S|\gtrsim\tfrac{1}{3}(\delta_1-\delta_3)\beta\mathbf{B}_S(\pi^+)\approx 0.09(k/\mu)(1/\sqrt{2}E)(k/q)^{\frac{1}{2}}.$$

In Fig. 9, we have plotted the total cross section including the correction for the $\pi^+-\pi^0$ mass difference.[34] The predicted cross section is somewhat high near threshold. This difficulty may be associated with the poorly evaluated S wave production. The angular distributions are shown in Figs. 13 and 14. It is seen that both $A_0{}^0$ and $A_2{}^0$ are too small in magnitude§ in the threshold region even where the total cross section predictions are fairly good. There are sizeable experimental errors and discrepancies (some of the latter are probably just a question of energy resolution which, for example, might shift some M.I.T. points 5 or 10 Mev upward). As a result one has difficulty in pinning down which partial wave contributions are incorrectly predicted. It seems clear, however, that consistency requires that $A_2{}^0$ should be much more negative than predicted. This would allow a slightly larger $A_0{}^0$ and leave room for adjustment of σ_T [where, of course, $\sigma_T=4\pi(A_0+\frac{1}{3}A_2)$]. This is a deficiency in the theory which is independent of the S-wave production. There are two possibilities: (a) a strong reduction in the $P\frac{1}{2}$ contribution will accomplish a decrease in A_2 as would (b) a strongly

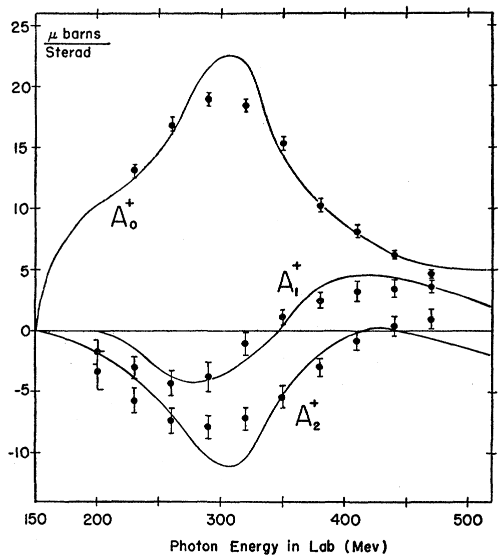

FIG. 10. Differential cross section for positive pions as $d\sigma/d\Omega=A_0{}^++A_1{}^+\cos\theta+A_2{}^+\cos^2\theta$. The experimental points are taken from the consensus of California Institute of Technology, Cornell, and Illinois work, prepared by Walker *et al.* See reference 1.

[34] Note that the threshold points of Mills and Koester (reference 2) and L. J. Koester (private communication) involve assumption of the ratio $A_2{}^0/A_0{}^0\approx -0.6$. But they are insensitive to this assumed ratio. For example, their σ_T would be increased 5% at 180 Mev and 10% at 140 Mev by assuming $A_2{}^0/A_0{}^0=0$.

§ *Note added in proof.*—It has been suggested to the author that it would also be interesting to divide the matrix elements into the different contributing physical processes. Let us divide the processes according to whether the photon interaction is with: (a) meson current (not including anomalous moment type processes), (b) nucleon current (not including anomalous moment), (bA) anomalous moment. The approximate ratios of the contributions of (a), (b), (bA) in that order to the following $T=3/2$ matrix elements, are: M1 (Born) 4:2:7, M1 (enhancement) 2:1:7, E2 (Born or enhancement) 9:1:0, S (Born) -1:6:2. These numbers apply at 265 Mev.

increased E_2 contribution, i.e., a more positive $P_{\frac{3}{2}}{}^{M=\frac{1}{2}}$ matrix element. *Empirically* the possibilities (a) and (b) might be distinguished if we assume that the energy dependence of the discrepancy in A_2 must have a resonance behavior, if it proceeds through the $P\frac{3}{2}$ state. With the present data and theory, we cannot be sure whether this effect shows a resonance. We illustrate this situation by plotting the differential cross-section predictions after setting the $P\frac{1}{2}$ matrix elements equal to zero. There is sufficient improvement so that remaining differences could arise from a variety of relatively unimportant theoretical or experimental sources.

C. Higher Energies

Comparison of experiment with prediction for $d\sigma^+/d\Omega$ as a function of energy is shown in Fig. 10, while $d\sigma^+/d\Omega$ at 265 and 455 Mev is shown in Figs. 11 and 12. To give some feeling for these predictions, let us examine a simple approximate expression for the differential cross section. We use the approximation

$$|X_{P\frac{3}{2}}|^2 = |a_{\frac{1}{2}}B_{\frac{1}{2}}+a_{\frac{3}{2}}(B_{\frac{3}{2}}\cos\delta_{33}+A_{\frac{3}{2}}\sin\delta_{33})e^{i\delta_{33}}|^2$$
$$\approx (B\cos\delta_{33}+A\sin\delta_{33})^2=(A^2+B^2)\sin(\delta_{33}+\alpha), \quad (34)$$

where

$$\alpha=\tan^{-1}(B/A).$$

The strongly enhanced $P\frac{3}{2}$ contributions then have the form of peaks shifted from the scattering peak as a result of the production into plane waves. In the region 250–400 Mev the principal contributions to σ^+ are then approximately (in μb/sterad):

$$A_0{}^+=[6+16.5\sin^2(\delta_{33}+36°(q/1.8\mu))](\mu/k),$$

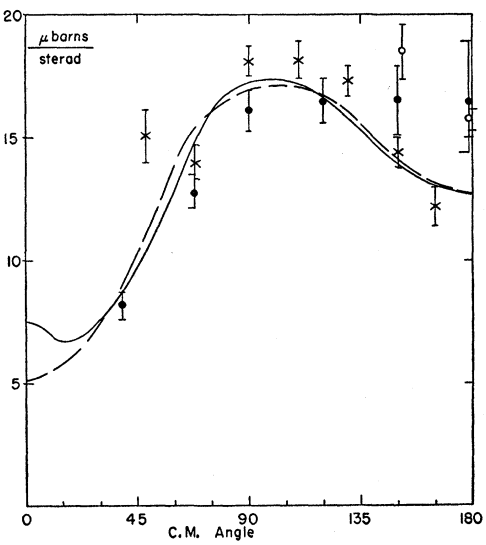

FIG. 11. Differential cross section for positive pions at 265 Mev. The dashed curve is the least-squares fit to the solid curve at 25°, 45°, 90°, 135°, and 180°, assuming no higher terms than $\cos^2\theta$. The filled-in circles are from Walker *et al.* at 260 Mev, the crosses from Tollestrup *et al.* at 260 Mev. The open circles are from the data of Repp at 267 Mev. See reference 1.

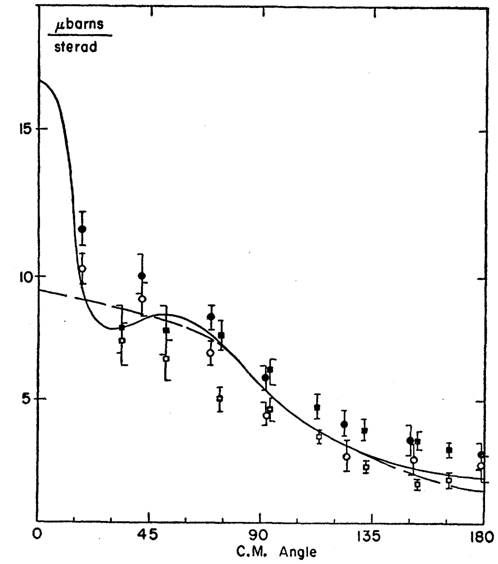

FIG. 12. Differential cross section for positive pions at 455 Mev. The dashed curve is the least-squares fit to the solid curve at 25°, 45°, 90°, 135°, and 180° assuming no higher terms than $\cos^2\theta$. The circles are the points of Walker *et al.*, the squares of Tollestrup *et al.* The filled-in points are at 440 Mev, the open points at 470 Mev.

where the constant term is due to S and $l>1$ waves.

$$A_1{}^+=-9\sin\delta_{33}\cos\delta_{33}(\mu/k),$$

where this contribution is the S interference with enhancement of the $P_{\frac{3}{2}}{}^{M=\frac{1}{2}}$ wave.

$$A_2{}^+=-\{13\sin^2[\delta_{33}+36°(q/1.8\mu)]-2\}(\mu/k),$$

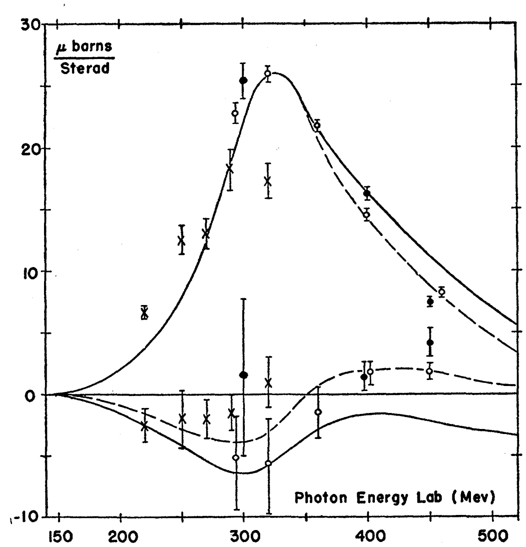

FIG. 13. Differential cross section for neutral pions from protons as $d\sigma/d\Omega=A_0{}^0+A_1{}^0\cos\theta+A_2{}^0\cos^2\theta$. The upper curves and points refer to $A_0{}^0$, the lower to $A_1{}^0$. See Fig. 14 for $A_2{}^0$. See text for explanation of dashed curve, Sec. III B and D. The crosses are points of Goldschmidt-Clermont *et al.*, the filled-in circles of Walker *et al.*, the open circles of Oakley and Walker. See reference 2. See also reference 35.

TABLE III. Properties of $P\frac{3}{2}$ matrix elements.

$E_{\rm lab}{}^{\gamma}$	δ_{33}	$\lvert X_{M1}{}^{+}/X_{E2}{}^{+}\rvert$	$\lvert X^{+}/\mathbf{B}^{+}\rvert_{M1}$	$\lvert X^{+}/\mathbf{B}^{+}\rvert_{E2}$	$\lvert X_{M1}{}^{0}/X_{E2}{}^{0}\rvert$	$\lvert X^{0}/\mathbf{B}^{0}\rvert_{M1}$
265	27	2.66	1.93	1.08	9.0	3.8
310	62	3.65	2.14	0.98	5.9	4.8
390	115	3.01	0.93	0.62	5.8	3.0
455	130	1.45	0.41	0.62	4.2	1.6

where the constant term in the bracket is due essentially to S-D interference. The photon momentum $q=1.8\mu$ occurs as $E_{\rm lab}{}^{\gamma}=330$ Mev. The phase shift, δ_{33}, used in the figures is that of Kalos and Dalitz[18] reduced slightly at the highest energies to conform with experiment[35] (typical values of δ_{33} used are given in Table III). It is seen that although the pion scattering resonance occurs at say 195 Mev in the laboratory, corresponding to 345-Mev photons, the peak will be shifted down about 40 Mev (near $\delta_{33}+36°=90°$). We dissect the $P\frac{3}{2}$ contribution to $d\sigma^{+}/d\Omega$ in another manner by exhibiting in Table III the ratio $\lvert X_{M1}/X_{E2}\rvert$ [using definitions analogous to Eq. (18)]. These matrix elements X are then expressed as sums of two terms: the Born and enhancement contributions. The ratio of the absolute value of X to the Born contribution, *abbreviated* $\lvert X/\mathbf{B}\rvert_{M1}$, for example, is tabulated.

The importance of the Born contributions is obvious. One remaining feature may be pointed out again: the peak at extreme forward angles due to D- and F-wave interference with S and P waves. The relevant matrix element ($K_{+}{}^{\frac{1}{2}}$ of Fig. 7) is increasing with energy roughly as $(k/q)^{\frac{5}{2}}$. Experiments to find this will have to be done inside 15° c.m. and preferably at the cross section resonance or higher energies (to avoid the cancellation of the S- and P-matrix elements in the forward direction at lower energies).

As a function of energy, $d\sigma_P{}^{0}/d\Omega$ is exhibited in Figs. 13[36] and 14. In Fig. 15, $d\sigma_P{}^{0}/d\Omega$ is plotted at 260 Mev. The motivation for the latter plot is merely to contrast the more difficult π^0 experiments and their lack of points in the forward direction with the π^+ experiments. The approximate analytic expressions for $\sigma_P{}^{0}$, valid say from 250–400 Mev, are:

$$A_0{}^{0}=22(1.8\mu/q)\sin^2(\delta_{33}+11°(q/1.8\mu)^2)+2\sin^2(\delta_{33}-35°),$$

where the first term is the $P_{\frac{3}{2}}{}^{M=\frac{3}{2}}$ (essentially $M1$) production and the second term arises from $P\frac{1}{2}$ and $P_{\frac{3}{2}}{}^{M=\frac{1}{2}}$ contributions.

$$A_1{}^{0}=-2.8(q/1.8\mu)(1+2.3\sin\delta_{33}\cos\delta_{33}),$$

where the first term in the parentheses gives the Born approximation S-P interference and the second the enhanced $P\frac{3}{2}$ wave interference with the S wave. This is the only place where S waves play a role.

$$A_2{}^{0}=-A_0{}^{0}+10\sin^2[\delta_{33}+26°(q/1.8\mu)],$$

where the new term is a sum of further contributions of $P\frac{1}{2}$ and the enhancement of the $P_{\frac{3}{2}}{}^{M=\frac{1}{2}}$ waves. In Table III we again dissect further the $P\frac{3}{2}$ contributions to $\sigma_P{}^{0}$ as discussed in the above paragraph (note that B_{E2} for π^0's is essentially zero). We have an important discrepancy between theory and experiment in that the theoretical A_2 is too positive at the peak and below.‖ This may be explained by (a) a reduced $P\frac{1}{2}$ matrix element or (b) an increased enhancement of the $P\frac{3}{2}$ $E2$ contribution as discussed in the previous section. Either (a) or (b) would tend to eliminate the term added to $-A_0$ in the expression for A_2 above.

D. Conclusion

This calculation consists of an evaluation of the enhancement of a production process due to final-state scattering, and proper combination of this matrix element with that associated with production into plane waves.

Let us first discuss the magnitude of enhancement. Does this provide a test of the meson theory independent of the two well-established experimental tests, i.e., of the coupling constant, through the threshold limits of photoproduction and p-wave scattering (particularly σ^+ as discussed in the Kroll-Ruderman theorem), and the $P\frac{3}{2}$ resonance energy? Technically the enhancement

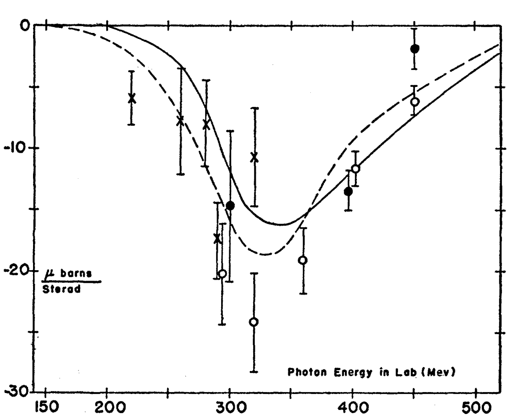

FIG. 14. Differential cross section for neutral pions from protons. See caption of Fig. 13 for explanation.

[35] S. Lindenbaum and L. Yuan, Phys. Rev. **100**, 306 (1955).

[36] Data on $d\sigma/d\Omega$ (135° c.m.) of Koester (reference 2) are not included because we don't have sufficient information to calculate the A^0's from these data. The quantity is best a measure of $A_1{}^{0}$ when combined with σ_T and one finds, for example, assuming that $A_2{}^{0}/A_0{}^{0}=-0.6$ that $A_1{}^{0}=-0.9\pm0.2$ at 200 Mev. The result is, however, fairly sensitive to the assumed $A_2{}^{0}/A_0{}^{0}$.

‖ *Note added in proof.*—More recent data reported by Corson (Proceedings of the Sixth Annual Rochester Conference on High Energy Physics) leads to a considerable improvement of the theory with experiment.

must be a further test of the theory since it involves the magnitude of the final state wave function at small distances, which is not a quantity simply related to the phase shifts, but depends on the details of the interaction. It may not provide a sensitive test (see below). At least the calculations using a static source and this relativistic calculation do not lead to quantitatively different enhancement effects. It should be repeated, however, that it is a test. Calculations of the magnitude of enhancement of other pion production processes such as γ-d, p-p, π-p, have been consistently made without a satisfactory investigation of the behavior of the final state function at small distances. These calculations involve models or approximations which fit the asymptotic wave function but seem arbitrary at small distances (i.e., in their off the energy shell components). The magnitude of the enhancement is correspondingly arbitrary.

The form of the matrix element, Eq. (5), as a sum of the scattering and plane-wave (Born) terms with the phase shifts explicitly exhibited is borne out as convenient by this calculation. The terms **A** and **B** are indeed slowly varying with energy except at threshold where the rapid behavior is predictable. In addition, corresponding terms are of the same order of magnitude in adjacent spin-isotopic spin states. To further examine this point the terms **A** for the $P\frac{1}{2}$ and $P\frac{3}{2}$, $T=\frac{3}{2}$ states were calculated and found to be similar to those exhibited in Table I (they are somewhat smaller than the **A** associated with the attractive $P\frac{3}{2}$, $T=\frac{3}{2}$ state). Thus, for example we know that there is little enhancement in states with little scattering. The simple result of Eq. (34) also follows, giving the height and shift of the resonance peak which provides a very simple understanding of the magnitudes and positions of the π^+ and π^0 production peaks.[37] A specific feature which is interesting is that the enhancement to Born ratio in the $M1$ matrix element to the $P\frac{3}{2}$, $T=\frac{3}{2}$ state is not much greater than one (i.e., $\mathbf{A}/\mathbf{B}\approx 2.5$ at the resonance energy). Thus it would generally be dangerous to examine more than qualitative features of a production process if it is assumed to proceed entirely through this final state scattering mechanism. The production models mentioned above often depend on a larger value for the enhancement. The π^0 P-wave production is, however, fairly amenable to a discussion based on enhancement alone. There, with isotopic spin cancellation in the Born term, the enhanced matrix element is more than 5 times the Born.

In repeating some of the pitfalls of this approximate calculation, we shall end on a note of caution. The obvious weakness of the final-state wave function in its emphasis of very high momentum components may not play a very important role in the photoproduction. This may be true to the extent that the photoproduction

[37] These ideas were used without much justification in reference 4.

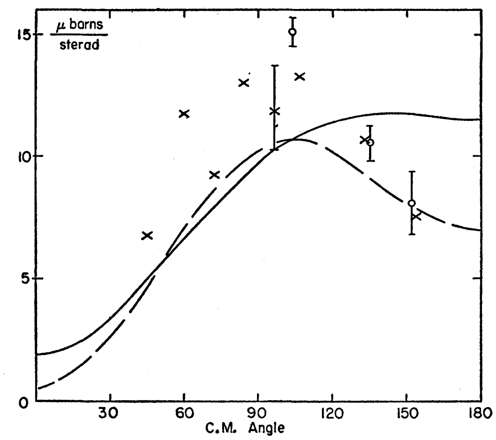

FIG. 15. Differential cross section for neutral pions from protons at 260 Mev. The crosses are points of Goldschmidt-Clermont *et al.* averaged from 240 to 280 Mev. A typical experimental error bar is shown on one point. The circles are points of Oakley and Walker at 270 Mev. See reference 2. See text for explanation of dashed curve, Sec. III B and D.

matrix elements (Fig. 5) do not extensively involve high-momentum intermediate states, coupled with the fact that the final state wave function is quite flat through a wide momentum range. But, taking the final-state function for granted, let us consider two other difficulties. The approximation to the gauge-invariant $M1$ matrix element is probably not very accurate because high momentum components still play a role (Fig. 5). At the same time the explicit production coupling constant, G_γ, was chosen equal to G_s, that used in the scattering integral equation. The resulting π^0 peak seems to agree with experiment. But the relative size of the $M1$ matrix element and the size of the appropriate coupling constant are probably both in error and this particular agreement is a coincidence. Further features of the comparison of theory with experiment show significant deviations. For example, the fact that the $E2$ π^0 production must be larger or the $P\frac{1}{2}$ smaller to bring some important theoretical features in line with experiment has been pointed out. Although it seems reasonable that the enhanced π^0 production should be $M1$, the deficiencies of the theory just mentioned are such that the former possibility seems the easier to accept. The latter possibility would not indicate an enhancement effect but a difficulty with the Born term, thus indicating that the threshold theorem in the $P\frac{1}{2}$ state does not yield a rapidly converging series.

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APPENDIX

We want to consider the effect of the $\pi^+-\pi^0$ mass difference on the production processes

$$\gamma+p\to\pi^++n,$$
$$\to\pi^0+p.$$

We assume that the basic effect is the difference in momentum that now characterizes the two events. At distances close to the nucleon, say $r<r_i$, there is a region throughout which the final state wave function $\psi^{(1,0;1)}(r)$ is characterized by relatively high momenta. More particularly, in this region the curvature of $\psi^{(1,3;1)}(r)$ is much more rapid than that of a free-particle function characterized by $k\lesssim\mu$ (analogous to a potential deep compared to k^2/μ) and much more rapid than that associated with the mass difference $\Delta\mu$ which will be characterized by the quantity $(\mu\Delta\mu)^{\frac{1}{2}}$. At large distances, say $r>r_0$, $\psi^{(1,3;1)}(r)$ takes on the free-particle form. The thickness of the boundary between the inside and outside region is small compared to the effect of the mass difference. In fact

$$r_0\ll 1/(\mu\Delta\mu)^{\frac{1}{2}}.$$

This follows from the nature of the meson-nucleon wave functions that are being considered here.[31] Once the outside and inside regions have been defined, the basic assumption can be made that isotopic spin remains a good quantum number in the inside region and the shapes of the various inside wave functions are unaffected by considering the pion mass difference (i.e., a characteristic virtual momentum p will differ relatively little from p', where $(\mu^2+p^2)^{\frac{1}{2}}=[(\mu+\Delta\mu)^2+p'^2]^{\frac{1}{2}}$).

In the outside region, we have the asymptotic radial functions

$$\psi_\alpha{}^{(1,0;1)}(r)\sim A(e^{-ik_\alpha r}/k_\alpha r)+(e^{ik_\alpha r}/k_\alpha r),$$

$$\psi_\beta{}^{(1,0;1)}(r)\sim Be^{-ik_\beta r}/k_\beta r,$$

where we are considering the production of mesons of type π^α, and ψ_α is the projection of the wave function which is associated with the meson π^α, etc. The coefficient A is given by an appropriate combination of phase factors determined by the scattering phase shifts, and $k_\alpha=k_\beta$, if $\mu_\alpha=\mu_\beta$. Assume we have calculated the production matrix element setting $\mu_\alpha=\mu_\beta=\mu_\beta{}^e$, where $\mu_\beta{}^e$ is the experimental or correct value for the mass of π_β. In making the correction $\mu_\alpha\to\mu_\alpha{}^e$, the logarithmic derivative,

$$\psi_\beta{}^{(1,0;1)\prime}(r_0)/\psi_\beta{}^{(1,0;1)}(r_0),$$

will remain the same. Then, since the inside wave function in each isotopic spin state remains the same, the combination of isotopic spin states inside required for π^α production, remains the same. Thus the logarithmic derivative,

$$\psi_\alpha{}^{(1,0;1)\prime}(r_0)/\psi_\alpha{}^{(1,0;1)}(r_0),$$

will not change. This condition on ψ_α immediately determines the new value of the coefficient A and thus the new value of $\psi_\alpha{}^{(1,0;1)}(r_0)$. Let us use the notation $\psi_\alpha(\mu_\alpha,\mu_\beta,r)$ and $\mathfrak{M}_\alpha(\mu_\alpha,\mu_\beta)$ to indicate the final-state function and matrix element as functions of the meson masses employed for π^α and π^β. Then the correct matrix element is given by the ratio:

$$R_\alpha=\frac{\mathfrak{M}_\alpha(\mu_\alpha{}^e,\mu_\beta{}^e)}{\mathfrak{M}_\alpha(\mu_\beta{}^e,\mu_\beta{}^e)}=\frac{\psi_\alpha(\mu_\alpha{}^e,\mu_\beta{}^e,r_0)}{\psi_\alpha(\mu_\beta{}^e,\mu_\beta{}^e,r_0)}.$$

The result for S waves is approximately

$$R_\alpha=1+i\delta_\alpha(k_\alpha-k_\beta)/k_\beta, \tag{A1}$$

where δ_α, $k_\alpha r_0$, and $k_\beta r_0$ were assumed to be small compared to one. The quantity δ_α is given for neutral and positive mesons in terms of the appropriate partial-wave phase shifts δ_3 and δ_1 (for isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively) as

$$\begin{aligned}e^{-2i\delta_0}&=\tfrac{2}{3}e^{-2i\delta_3}+\tfrac{1}{3}e^{-2i\delta_1},\\ e^{-2i\delta_+}&=\tfrac{1}{3}e^{-2i\delta_3}+\tfrac{2}{3}e^{-2i\delta_1}.\end{aligned} \tag{A2}$$

The result for P waves is approximately

$$R_\alpha=1\pm[(e^{2i\delta_\alpha}-1)/ikr_0]\kappa, \tag{A3}$$

where these are the first terms in an expansion in κ, with $\kappa=(k_0-k_+)/k$, $k=\frac{1}{2}(k_0+k_+)$. The $\pm$ signs apply to π^0 and π^+ production, respectively. These results are used to correct calculated cross sections in Secs. III A and B.

These results can also be applied to pion nucleon scattering. It is indicated that in $\pi^\pm$ or π^0 S state processes the mass difference has no important effect (i.e., $|R_\alpha|^2\approx 1$). One should just employ the appropriate momentum in the expression for initial flux and density of final states. This is in disagreement with the result of Noyes.[31]