PHYSICAL REVIEW VOLUME 103, NUMBER 3 AUGUST 1, 1956

On Einstein's λ Transformations*

PETER G. BERGMANN
Department of Physics, Syracuse University, Syracuse, New York
(Received April 16, 1956)

The invariance of the curvature with respect to λ transformations is given a geometric significance that permits its immediate extension to more general curvatures.

In the fifth edition of *The Meaning of Relativity* as well as in his last joint paper with Kaufman,[1,2] Einstein introduced a transformation that he called "λ-transformation" and that affects the components of the affine connection as follows:

$$\Gamma^{*\beta}{}_{\alpha\rho}=\Gamma^{\beta}{}_{\alpha\rho}+\delta^{\beta}{}_{\alpha}\lambda_{,\rho}. \tag{1}$$

He showed, by straightforward computation, that the curvature tensor $R_{\iota\kappa\lambda}{}^{\nu}$ is λ invariant. In this note we shall give this result an intuitive significance which permits its immediate extension to more general parallel displacement operations.

Given a vector at one point in a manifold, $V^{\rho}(x)$, we define a law of parallel displacement which yields a vector at the neighboring point $(x+dx)$,

$$dV^{\rho}=-\Gamma^{\rho}{}_{\sigma\kappa}dx^{\kappa}V^{\sigma}. \tag{2}$$

If we displace V along a curve given in the parametric representation $x(\theta)$, then the result of the displacement from θ_1 to θ_2 may be conveniently given the form

$$V(\theta_2)=P\left[\exp\left(-\int_{\theta_1}^{\theta_2}\Gamma_{\rho}dx^{\rho}\right)\right]V(\theta_1), \tag{3}$$

where the symbol $P[\ ]$ denotes the "chronological product," in which in every product the factors are ordered right to left according to increasing values of the parameter θ.[3]

If we vary the path from x_1 to x_2 by an infinitesimal displacement vector $\delta x(\theta)$, leaving the end points fixed, then the variation in $V(\theta_2)$ $[V(\theta_1)$ being given$]$ depends on the curvature. Using the same symbolic notation as in Eq. (3), i.e., suppressing the vector indices in V and the corresponding indices in the components of parallel displacement and in the curvature tensor, we find:

$$\delta V(\theta_2)=P\left[\int_{\theta_1}^{\theta_2}R_{\iota\kappa}\delta x^{\iota}dx^{\kappa}\exp\left(-\int_{\theta_1}^{\theta_2}\Gamma_{\rho}dx^{\rho}\right)\right]V(\theta_1). \tag{4}$$

Clearly, Eqs. (3) and (4) now appear in a form in which it is quite immaterial that V is a vector field. If we were to write down the law of parallel displacement for a spinor field, then Γ_{ρ} would represent the spin-affine connection and $R_{\iota\kappa}$ the spin-curvature tensor.

Regardless of its specific significance, the curvature $R_{\iota\kappa}$ is given by the expression

$$R_{\iota\kappa}=\Gamma_{\iota,\kappa}-\Gamma_{\kappa,\iota}+[\Gamma_{\kappa},\Gamma_{\iota}]. \tag{5}$$

That it is a commutator can be shown by virtue of the fact (which will not be demonstrated here) that the parallel displacement about an arbitrary closed curve may be represented by a surface integral of an expression somewhat similar to the chronological product appearing in Eq. (4). The two subscripts of the curvature are then multiplied by the two coordinate directions on the surface of integration. The whole equality resembles Stokes' theorem, except that we deal here not with a vector field but with an operator field having one "visible" vector index.

The transformation (1) may now be written in the symbolic form

$$\Gamma_{\rho}{}^{*}=\Gamma_{\rho}+I\lambda_{,\rho}, \tag{6}$$

where I stands for the identity operator. The additional term on the right "commutes" with all other operators. Accordingly we can integrate the relationship (1) in closed form. We find that $V_2{}^{*}$ differs from V_2 by a factor that depends only on the end points but not on the connecting path of parallel displacement:

$$V_2{}^{*}=\exp[\lambda(\theta_1)-\lambda(\theta_2)]V_2. \tag{7}$$

Accordingly, a variation of path with fixed end points, as it appears in Eq. (4), is unaffected by a λ transformation.

The generality of the result suggests that λ transformations may occur in various unified field theories involving any kind of nonholonomic parallel displacement of geometric objects. In fact, the λ transformation appears to be closely related in its conception to Weyl's original gauge transformations.[4] Weyl had introduced the gauge transformation as a local change in gauge (i.e., scale) of the metric tensor. The details were, of course, different. Nevertheless, the similarity in geometric significance and in the structure of the group is suggestive. Moreover, Eq. (6) differs from the gauge transformation law of de Broglie waves of charged particles only in that here the exponent in the trans-

* This work was supported by a grant from the National Science Foundation.

[1] Albert Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1955), fifth edition, p. 148.

[2] A. Einstein and B. Kaufman, Ann. Math. **62**, 128 (1955).

[3] F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[4] H. Weyl, Preuss. Akad. Wiss. Ber., p. 465 (1918); Ann. Physik **59**, 101 (1919).

formation law is real, whereas in de Broglie waves it is imaginary.

It appears likely that λ-type transformations will be encountered in any geometries with affine connections whenever they are not excluded by the type of symmetry condition characteristic of Christoffel symbols, and further, that λ invariance is likely to be intimately related to ordinary gauge invariance in any theory that purports to contain references to the electromagnetic field.

ACKNOWLEDGMENTS

The author wishes to acknowledge helpful discussions with Dr. Bruria Kaufman.

PHYSICAL REVIEW VOLUME 103, NUMBER 3 AUGUST 1, 1956

Vacuum Polarization and Proton-Proton Scattering in the ^3P State*

ERIK ERIKSEN, *Institute for Theoretical Physics, University of Oslo, Blindern, Norway*

AND

LESLIE L. FOLDY AND WILLIAM RARITA,† *Case Institute of Technology, Cleveland, Ohio*

(Received April 16, 1956)

The contribution of the vacuum polarization potential to the ^3P phase shift in proton-proton scattering has been calculated and employed to correct observed phase shifts to obtain the phase shifts associated with the specifically nuclear ^3P potential between two protons. In the energy range from 2 to 5 Mev, about half the observed phase shift can be attributed to vacuum polarization. From the results it is shown that if the potential between the two protons in both the ^1S and ^3P states is represented by a square well of range 2.6×10^{-13} cm, then the ^3P potential is opposite in sign (repulsive) and approximately 13% in magnitude relative to the ^1S potential.

IN two previous papers[1,2] the effect of vacuum polarization on proton-proton scattering in the ^1S state at low energies was examined. On the basis of experimental data, some evidence for the reality of vacuum polarization effects was obtained, and by then introducing a correction for these effects, new values of the constants characterizing the specifically nuclear interaction between two protons in the ^1S state were derived. The changes in these constants—the zero-energy scattering length and the effective range—were small but not negligible.

It was noted in the second paper cited that the correction for vacuum polarization would be relatively much more important in deriving the properties of the specifically nuclear interaction between two protons in the ^3P state for the following two reasons: the vacuum polarization potential has a relatively long range, and the nuclear interaction in this state is much weaker than in the ^1S state. Calculations have now been performed of the contribution of the vacuum polarization potential to the ^3P state phase shift in an appropriate approximation; the results and their implications are discussed in the present paper. In brief summary, the present calculations indicate that approximately one-half of the experimentally observed ^3P phase shift in the energy region from 1 to 5 Mev arises from the vacuum polarization potential, so that the correction for this effect is very important. After correction, the ^3P nuclear interaction is found to be approximately 13% as strong as the ^1S nuclear interaction. The rather anomalous behavior of the ^3P phase shift at higher energies as observed in several measurements is not explained by the vacuum polarization effect, however.

CALCULATION OF THE VACUUM POLARIZATION EFFECT

We shall ignore for the present the possibility of a tensor nuclear interaction between two protons and assume that the nuclear interaction, like the Coulomb and the vacuum polarization potentials, is purely central. One can then easily show that the P-wave phase shift δ_1 is given by the following formula:

$$\sin\delta_1=-\frac{M}{\hbar^2k}\int_0^\infty Vuv\,dr. \tag{1}$$

Here M is the proton mass, $k=(ME_L/2\hbar^2)^{\frac{1}{2}}$, where E_L is the energy in the laboratory system, V is the sum of the specifically nuclear potential V_n and the vacuum polarization potential V_{vp}, but not including the Coulomb potential e^2/r, v is the radial P-wave function in the presence of the potential V, while u is the radial P-wave function in the absence of this potential (that is, for a pure Coulomb potential), both u and v being normalized to unit amplitude for large r. Since both the

* This work has been supported by the U. S. Atomic Energy Commission, The National Science Foundation, and the Norwegian Research Council for Science and the Humanities.

† Visiting Professor during 1955; present address: Brooklyn College, Brooklyn, New York.

[1] L. L. Foldy and E. Eriksen, Phys. Rev. **95**, 1048 (1954).

[2] L. L. Foldy and E. Eriksen, Phys. Rev. **98**, 775 (1955).