Phase Shifts for Pion-Nucleon Scattering at 40 Mev*

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Approximate phase shifts for the pion-nucleon system are obtained from the Chew-Low-Wick equations. At 40 Mev, the set of phases: $\delta_1=6.26°$, $\delta_3=-4.2°$, $\delta_{11}=-2.05°$, $\delta_{13}=\delta_{31}=-0.82°$, and $\delta_{33}=4.1°$, has the properties: (i) it satisfies the first iteration and the effective-range approximations of the Chew-Low-Wick equations, (ii) it satisfies the dispersion relations, (iii) it gives good over-all agreement with the experimental data within the experimental errors. Iteration of the effective-range approximation, for energies up to 200 Mev, gives good agreement for the δ_{13} and δ_{33} phases but not for the δ_{11} phase shifts.

INTRODUCTION

LOW, Chew, and Wick[1] have recently given a very interesting new formulation of the pion-nucleon scattering problem. The main advantage of this theory is that one deals with a finite renormalized theory. On the other hand, in going to the nonrelativistic limit one neglects nucleon recoil and pair formation and, in order to reduce the problem to manageable size, one must use the so-called one-meson approximation. The validity of these approximations is not clear, since the solution of the Chew-Low-Wick equations, even for nonrelativistic energies, involves energy integrals over all energies. For these reasons a knowledge of the solutions of these equations, and comparison with the experimental data, is of great interest.

Unfortunately, the Chew-Low-Wick theory gives a system of coupled nonlinear integral equations for the p-waves which is very difficult to solve. A numerical solution is at present being investigated by Salzman.[2] Meanwhile, it seemed very desirable to obtain a preliminary orientation as to the character of these solutions and to compare them with experiment. For this purpose we have obtained two approximate solutions of the equations. The two available parameters of the theory were fitted by means of the experimentally determined δ_{33} phases, and to carry out a comparison with experimental data, we also needed the s-phases. For these the linear extrapolations of Orear[3] were used. Comparison of the cross sections so obtained in the energy region of 40 Mev, the energy chosen for the comparison, shows a quite surprising agreement with the available experimental data. It should of course be noted that this agreement may no longer persist when accurate solutions of the Chew-Low-Wick equations or better experimental data become available

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[1] F. E. Low, Phys. Rev. **97**, 1392 (1955); G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[2] G. Salzman (private communication).
[3] J. Orear, Phys. Rev. **96**, 176 (1954).

P-WAVE PHASES

We shall use the subscripts 1, 2, 3 to denote the $T=J=\frac{1}{2}$ state, the $T=\frac{1}{2}$, $J=\frac{3}{2}$ or $T=\frac{3}{2}$, $J=\frac{1}{2}$ states, and the $T=J=\frac{3}{2}$ state of the pion-nucleon system, respectively. The Chew-Low-Wick[1] equations are then given by

$$g_\alpha(z)=1-\frac{z}{\pi}\int_1^\infty \frac{p^3v^2(p)}{\omega_p{}^2}d\omega_p\left\{\frac{\lambda_\alpha}{\omega_p-(z+i\epsilon)}+\frac{H_\alpha(\omega_p)}{\omega_p+z}\right\}, \quad (1)$$

where

$$\lambda_\alpha=\tfrac{2}{3}f^2\begin{bmatrix}-4\\-1\\+2\end{bmatrix}, \quad (2)$$

$$H_\alpha(\omega_p)=\sum_{\beta=1}^{3}C_{\alpha\beta}\left|\frac{q_\alpha(-\omega_p-i\epsilon)}{g_\beta(\omega_p+i\epsilon)}\right|^2, \quad (3)$$

and $C_{\alpha\beta}$ is the matrix

$$(2/27)f^2\begin{bmatrix}-4 & 2 & -16\\ 32 & -7 & -16\\ 32 & 2 & 2\end{bmatrix}. \quad (4)$$

Here f is the renormalized unrationalized $ps(pv)$ coupling constant, $v(p)$ is the cutoff factor, to be specified below, for a meson of momentum p and energy ω_p, ϵ is a positive infinitesimal to define the integral in (1), and $g_\alpha(\omega_p)$ is related to the scattering phase-shift $\delta_\alpha(\omega_p)$ by

$$\frac{1}{\lambda_\alpha}\mathrm{Re}[g_\alpha(\omega_p)]=\frac{p^3v^2(p)}{\omega_p}\cot\delta_\alpha(\omega_p), \quad (5)$$

for $\omega_p\geqslant 1$.

Two approximate solutions of the Chew-Low-Wick equations were obtained:

(i) the first iteration solution, obtained by putting the unknown functions in the integrands in Eq. (1) equal to unity; i.e., for H_α we use, instead of Eq. (3),

$$H_\alpha=\sum_{\beta=1}^{3}C_{\alpha\beta}. \quad (6)$$

(ii) the effective range approximation (ERA) in which, in addition to the above approximation [i.e.,

TABLE I. Pion-nucleon scattering phase-shifts (in degrees) at 40 Mev. The s-wave phases of sets (A) to (C) are those of Orear.

Set	δ_1	δ_3	δ_{11}	δ_{13}	δ_{31}	δ_{33}	D_1	D_3
(A) ERA (effective range approx)	6.26	−4.2	−2.05	−0.82	−0.82	4.1	0.09	0.11
(B) First iteration solution	6.26	−4.2	−1.83	−0.76	−0.76	4.51	0.10	0.14
(C) ERA with $\omega_p{}^*$	6.26	−4.2	−1.96	−0.77	−0.77	3.73	0.095	0.087
(D) Bethe-deHoffmann	5.4	−5.0	−1.7	−1.1	0.3	4.5	0.051	0.13

defining H_α by Eq. (6)], the dependence on energy, i.e., on z, of the integrals in Eq. (1) is neglected, giving

$$\frac{1}{\lambda_\alpha}\mathrm{Re}[g_\alpha(\omega_p)]=\frac{1}{\lambda_\alpha}\left\{1-\omega_p A\begin{pmatrix}-1\\0\\+1\end{pmatrix}\right\}, \tag{7}$$

where

$$A=\frac{3}{\pi}\lambda_3\int^\infty\frac{d\omega_k}{\omega_k{}^3}k^3v^2(k). \tag{8}$$

In both cases, we took for the cutoff function (which seems to be arbitrarily at our disposal, apart from some mild restrictions) $v(p)=\beta^2/(p^2+\beta^2)$. β and the coupling constant f were determined by fitting our solutions to the δ_{33} phases obtained from phase-shift analyses of pion-nucleon scattering experiments.[4] In this way we found the values $\beta=8.56$ (all momenta and energies in units of μc and μc^2; $\mu=$meson mass) and $f^2=0.081$. The value of f^2 is that proposed by Chew and Low.[1] It agrees tolerably well with the experimental value of $f^2=0.066$, of Bernardini and Goldwasser.[5]

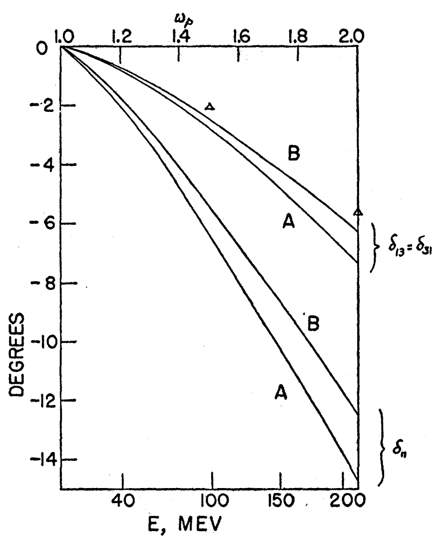

FIG. 1. δ_{11} and $\delta_{13}=\delta_{31}$ *vs* energy. $E=$meson kinetic energy in laboratory system, $\omega_p=$meson total energy in c.m. system (rest mass=1). Curves A and B are the ERA and first-iteration solutions respectively. (Δ)=first iteration to ERA for δ_{11}.

[4] H. A. Bethe and F. deHoffmann, *Mesons and Fields* (Row, Peterson, & Company, Evanston, 1955), Vol. 2. This book contains a comprehensive review of experimental data and references to their sources. Unless otherwise stated, experimental data quoted by us are taken from this book.

[5] G. Bernardini and E. L. Goldwasser, Phys. Rev. **95**, 857 (1954).

TABLE II. Cross sections given by set (A) (Table I) of phase-shifts for pion-proton scattering at 40 Mev. Total (differential) cross sections are expressed in mb (mb per unit solid angle). All angles are measured in the center-of-mass system.

	(A) π^+*scattering*		
1.	Differential cross section at 38°		0.18
2.	Differential cross section at 100°		0.76
3. (i)	Total cross section (30° to 180°)		8.96
(ii)	Total cross section (60°−180°)		8.52
	(B) π^- *scattering*		
4.	Differential cross section at 38°		0.48
5. (i)	Total cross section (30° to 180°)		2.86
(ii)	Total cross section (60° to 180°)		1.95
	(C) *Charge-exchange scattering*		
6.	Angular distribution $A+B\cos\theta+C\cos^2\theta$	A	0.38
		B	−0.71
		C	0.33
7.	Total cross section (0−180°)		5.72
8.	π^- scattering, total cross section (50°−180°) given by Set (D)		1.2

It has been proposed[6] to replace the linear ERA by a more general empirical relation, of the form

$$\frac{p^3v^2(p)}{\omega_p}\cot\delta_\alpha=\frac{1}{\omega_p}+F_\alpha(\omega_p), \tag{9}$$

where $F_\alpha(\omega_p)$ are unknown functions, with $F_\alpha(0)=1/\lambda_\alpha$. On account of the experimental errors, the linear ERA for δ_{33} seems to fit the experimentally derived phases as well as is to be expected. Hence from this viewpoint the more general relation (9) is not needed. Nor does it appear to us that there is any basic reason for separating out a $1/\omega_p$ term. Equation (5) is true only for $\omega_p\geqslant 1$, the general relation being

$$\frac{p^3v^2(p)}{\omega_p}\cot\delta_\alpha(\omega_p)=F_\alpha(0)g_\alpha(\omega_p)+i\frac{p^3v^2(p)}{\omega_p}. \tag{10}$$

Since $g_\alpha(0)=1$, and we can take $v^2(i)=1$ (for $\omega_p=0$, $p=i$), Eq. (10) indeed gives a $1/\omega_p$ term which dominates at $\omega_p=0$. However, to determine the coupling constant f by fitting the experimental δ_{33}-phases, we have certain experimental points in the range $\omega_p\geqslant 1$ and we want to determine the parameters associated with the curve *in this range.* If, for example, the curve in this range is a straight line, then the correct value of f^2 is obtained by extrapolating this straight line back to zero, without subtracting a $1/\omega_p$ term. Furthermore,

[6] Friedman, Lee, and Christian, Phys. Rev. **100**, 1494 (1955); S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100**, 306 (1955).

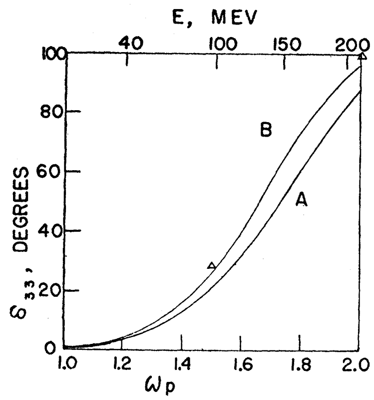

FIG. 2. δ_{33} *vs* energy. E=meson kinetic energy in laboratory system, ω_p =meson total energy in c.m. system (rest mass = 1). Curves A and B are the ERA and the first-iteration solutions, respectively. (Δ)=first iteration to ERA.

except near $\omega_p=0$, a $1/\omega_p$ term is not necessarily dominating.[7]

Chew and Low[1] have proposed taking kinematical effects of the nucleon recoil into account in the ERA by replacing ω_p by

$$\omega_p{}^*=\omega_p+(p^2/2M). \tag{11}$$

This seems to give a somewhat better fit for δ_{33} above the resonance energy, but on account of the experimental errors it is not clear whether this is significant.

The p-wave phase shifts as a function of energy are given in Figs. 1 and 2. Curves A and B show the ERA and first-iteration solutions. They are seen to agree well and their differences presumably represent a lower limit to the accuracy which can be ascribed to these phases. We also calculated the first iteration of the ERA. These are shown for $\delta_{13}=\delta_{31}$ and δ_{33} by the triangles (Δ) in Figs. 1 and 2; these also agree. For δ_{11}, however, there is violent disagreement ($\delta_{11}=+4°$ at $\omega_p=1.5$ and $\delta_{11}=+8.8°$ at $\omega_p=2.0$). Of course one does not expect the ERA to hold at all energies. If this iteration had agreed well, up to, say, 200 Mev, with the ERA, then one could have argued that it is a good approximation in this energy range and that the contributions from higher energies do not introduce errors,

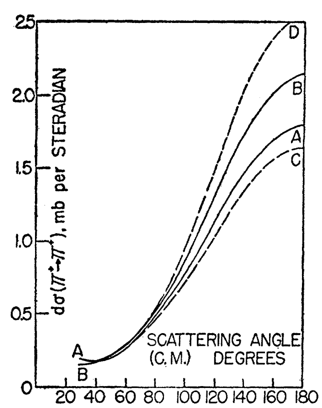

FIG. 3. Differential cross section (c.m.) for π^+-p scattering at 40 Mev in mb per steradian. (A) ERA; (B) first-iteration solution; (C) ERA using $\omega_p{}^*$ instead of ω_p; (D) Bethe-deHoffmann phases.

[7] The same conclusion has been reached independently by M. Friedman (private communication).

because of accidental cancellations and the cutoff factor. On account of the coupling of the phases, no conclusions can be drawn from the close agreement in the case of δ_{13} and δ_{33}.

The phases at 40 Mev are shown in Table I. Set (A) represents the ERA; set (B), the first iteration solution of the integral equations; set (C) the ERA, using $\omega_p{}^*$ instead of ω_p. The s-phases in sets (A) to (C) are those of Orear[3] and should be fairly reliable at 40 Mev. Set (D), in Table I, is the set of phases preferred by Bethe *et al.*[8] It does not satisfy $\delta_{13}=\delta_{31}$, but is "qualitatively similar" to the other sets.

Some indication of the consistency of our approximate phases with the Chew-Low-Wick equations is obtained

TABLE III. Experimentally observed pion-proton scattering cross sections (mb or mb per sterad) in the neighborhood of 40 Mev. Angles in laboratory and center-of-mass frames are labeled (lab) and (c.m.) respectively. The lines are numbered as in Table II.

	(A) π^+ *scattering*	
1.	Differential cross section at 43±4 Mev at 38±7.5° (c.m.)[a]	0.36±0.04
2.	Differential cross section at 43±4 Mev at 100±10° (c.m.)[a]	0.71±0.05
3. (i)	Total cross section at 40±3 Mev, 53°−180° (c.m.); obtained from integration of differential cross section of Perry and Angell[b]	10.9±3
(ii)	Total cross section at 37±6 Mev, 50°−180° (lab); transmission measurement by Angell[b]	11.8±1
	(B) π^- *scattering*	
4.	Differential cross section at 43±4 Mev at 38±7.5° (c.m.)[a]	0.61±0.08
5. (i)	Total cross section at 37 Mev, as deduced by Bethe *et al.*[c] from all available data, for the range 50°−180° (lab)	3 to 9
(ii)	Total cross section at 65 Mev, 42° to 180° (c.m.); (Bodansky *et al.*[b])	2.6±0.3
	(C) *Charge-exchange scattering*	
6.	Angular distribution $A+B\cos\theta+C\cos^2\theta$ at 40 Mev; (Roberts and Tinlot[b])[d] (A)	0.45±0.07
	(B)	−0.98±0.13
	(C)	0.54±0.21
7. (i)	Total cross section (0°−180°) at 37 Mev; (Spry[b])	6.4±1.0
(ii)	Total cross section at 40 Mev; (Roberts and Tinlot[b])[d]	7.9±1.8

[a] S. W. Barnes (private communication). The values given are those available at the time of writing of this paper, but they do not yet represent the final values.
[b] See reference 4.
[c] See reference 8.
[d] These measurements probably give very good relative values, i.e., the ratios $A:B:C$, but not the absolute values.

by seeing whether they satisfy the dispersion relations.[9] At 40 Mev, these read:

$$D_1\equiv\sin2\delta_1+2\sin2\delta_{13}+\sin2\delta_{11}=0.05\pm0.05,$$
$$D_3\equiv\sin2\delta_3+2\sin2\delta_{33}+\sin2\delta_{31}=0.15\pm0.05. \tag{12}$$

The right-hand sides of these equations are certain energy integrals over total cross sections, calculated by Anderson *et al.*[9] and the ±0.05 represents our rough

[8] See reference 4, p. 96.
[9] Anderson, Davidon, and Kruse, Phys. Rev. **100**, 339 (1955).

estimate of the errors. In the last column of Table I the values of D_1 and D_3 for those sets of phases are given, showing that all the sets [with the possible exception of (C)] are consistent with the dispersion relations.

COMPARISON WITH EXPERIMENT

In Table II, the pion-proton cross sections at 40 Mev, predicted by set (A) in Table I, are given. On account of the errors in the experimental data it did not seem worth while to consider the other sets. In addition, line 8 gives the π^- total cross section predicted by Set (D). Comparison with lines 5(i) and (ii) shows that "qualitatively similar" sets of phases can lead to very different results, even for total cross sections.

To compare the different sets of phases, we have plotted in Figs. 3 and 4 the differential cross sections for sets (A) to (D) for π^+ and π^- (direct and exchange) scattering.

In Table III corresponding experimentally determined quantities are given. Comparison of Tables II and III shows a rather remarkable over-all agreement. This agreement is not so good for the preliminary results on $\pi^\pm$ differential cross sections at 38° (c.m.) (lines 1 and 4 of Tables II and III). However, these points should be noted. As far as the π^- cross section is concerned one is here in a region where the Coulomb interference plays an important role and the cross section is very sensitive to changes in phases or angle (compare Fig. 4). Thus set (B) of the phases at the same angle gives a differential cross section of 0.57 mb per steradian. Accurate experiments in this region may ultimately provide a very sensitive test of accurately calculated phases. As for the π^+ cross section, where the disagreement is most serious, one is here also in the region where Coulomb interference effects are important. However, it so happens that for this choice of angle and energy the cross section is very insensitive to changes of the phases and it seems difficult to obtain such a large cross section with *any* set of phases which would appear reasonable. Finally, it should be remarked that the over-all agreement of calculated and measured data may not persist when accurate calculations or further measurements become available.

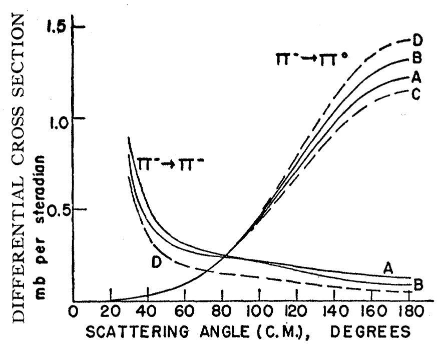

FIG. 4. Differential cross section (c.m.) for $\pi^- - p$ scattering (direct and exchange) at 40 Mev in mb per steradian. (A) ERA; (B) first-iteration solution; (D) Bethe-deHoffmann phases. Curve (C), the ERA using ω_p^*, is not shown as it is indistinguishable from (B).