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Meson Production by Mesons*

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A calculation of meson production in a meson-nucleon collision has been carried out for the processes $\pi^++p\rightarrow\pi^++\pi^++n$ and $\pi^++p\rightarrow\pi^++\pi^0+p$, by means of the method of Low. Use has been made of an approximate form of the Chew-Low transition matrix for scattering off the energy shell. This matrix exhibits explicitly the resonance behavior of the isotopic spin 3/2, angular momentum 3/2 scattering state. At 350-Mev incident-meson kinetic energy, the cross section for π^+ production is 0.86 mb and that for π^0 production is 5.5 mb. The angular distribution of each final state meson is of the form $A+\cos^2\theta$. For π^+ production, at 250-Mev incident-meson kinetic energy, $A=0.64$. The two final-state pions seem to share the available kinetic energy. The probability for a spin flip of the nucleon in the process is about 1.5 times the probability for no spin flip.

INTRODUCTION

THERE has been increasing interest in the production of mesons in meson-nucleon collisions at high energies. We have felt that it would be of interest to calculate the production cross section for energies near threshold. There is evidence indicating strong P-wave production of mesons. Such evidence consists of the small cross section for π^0 production in p-p collisions, the rapid increase with energy of the total cross section for π^+ production in these collisions, and the center-of-mass angular distribution for the π^+ of $0.29+\cos^2\theta$.[1,2] The (3/2, 3/2) resonant state in meson-nucleon scattering appears to be of importance in the production process. The hypothesis that meson production takes place through excitation of a nucleon into the (3/2, 3/2) resonant state is capable of explaining the small cross section for π^+ production in n-p collisions relative to that in p-p collisions.[3] The hypothesis has also been used to explain the π^+/π^- production ratio in nucleon-nucleus collisions.[4] We have assumed production from the (3/2, 3/2) state and have investigated the processes $\pi^++p\rightarrow\pi^++\pi^++n$ and $\pi^++p\rightarrow\pi^0+\pi^++p$.

The procedure has been to derive the amplitude describing the process by the method of Low.[5] The method deals with a perturbation-like expansion in terms of exact eigenstates of the total Hamiltonian. This feature makes possible an improvement of the ordinary perturbation-theory treatment, since it is possible to describe part of the process as meson-nucleon scattering and to use here the scattering transition matrix developed by Chew and Low.[6,7] We will be dealing with a transition matrix for scattering off the energy shell. Chew and Low have derived an approximate expression for this matrix in a form that exhibits the resonant behavior of its (3/2, 3/2) part. It was this possibility which led to the calculation presented here, in the hope that, in spite of the approximations, the form of this matrix off the energy shell would give reasonable results in the computation of a cross section for meson production. Chew and Low have shown that, on the energy shell, their matrix gives a good description of low-energy scattering.[6]

CALCULATIONS

Following Low,[5] one starts with Dyson's S-matrix between initial and final bare-particle states,

$$\langle f|S|i\rangle=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots dt_n\langle\Phi_{p'}|a_j(k_1)a_l(k_2)P[H_I(t_1)\cdots H_I(t_n)]a_i^*(k)|\Phi_p\rangle, \tag{1}$$

where $\hbar=c=1$, $x=(\mathbf{x},x_0)$, and $k=(\mathbf{k},\omega)$. H_I is the usual interaction Hamiltonian for pseudoscalar meson theory.[5]

Assume $k\neq k_1\neq k_2$; $p\neq p'$. The $a_i^*(k)$ is commuted through the P-bracket to the left and the $a_l(k_2)$ and $a_j(k_1)$ are commuted through to the right, making use of

$$\langle\Phi_{p'}|a_i^*(k)=a_l(k_2)|\Phi_p\rangle=a_j(k_1)|\Phi_p\rangle=0. \tag{2}$$

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[1] A. H. Rosenfeld, Phys. Rev. 96, 130 (1954).
[2] Frank S. Crawford, University of California Radiation Laboratory Report UCRL-2187, 1953 (unpublished), and M. Lynn Stevenson, University of California Radiation Laboratory Report UCRL-2188, 1953 (unpublished).
[3] G. B. Yodh, Phys. Rev. 98, 268(A) (1955).
[4] D. C. Peaslee, Phys. Rev. 94, 1085 (1954); 95, 1580 (1954).
[5] F. Low, Phys. Rev. 97, 1392 (1955).
[6] G. F. Chew and F. E. Low (to be published).
[7] G. C. Wick, Revs. Modern Phys. 27, 339 (1955).

The resulting expression is transformed by making use of the identity

$$\left\langle \Phi_{p'} \left| \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots dt_n P[H_I(t_1)\cdots H_I(t_n) A_i(x) A_j(y)\cdots] \right| \Phi_p \right\rangle = \langle \Psi_{p'} | P[\bar{A}_i(x)\bar{A}_j(y)\cdots] | \Psi_p \rangle, \quad (3)$$

where on the left side all operators are in the interaction representation and on the right side $\bar{A}_i(x)$ has the Heisenberg time dependence,

$$\bar{A}_i(\mathbf{x},x_0) = e^{iHx_0} A_i(\mathbf{x},0) e^{-iHx_0}, \quad (4)$$

and the $|\Psi_p\rangle$ are exact single-nucleon eigenstates of the total Hamiltonian (free-field plus interaction). The S-matrix is then

$$\begin{aligned}
\langle f|S|i\rangle &= \frac{(-i)^3}{(8\omega\omega_1\omega_2)^{\frac{1}{2}}} \int d^4x d^4y d^4z e^{ikx-ik_1y-ik_2z} \langle \Psi_{p'} | P[\bar{A}_i(x)\bar{A}_j(y)\bar{A}_l(z)] | \Psi_p \rangle \\
&+ \frac{(-i)^2\lambda}{(8\omega\omega_1\omega_2)^{\frac{1}{2}}} \int d^4x d^4y e^{ikx} \{ e^{-i(k_1+k_2)y} \langle \Psi_{p'} | P[(2\bar{\phi}_j(y)\bar{\phi}_l(y) + \delta_{jl}\bar{\phi}_m(y)\bar{\phi}_m(y))\bar{A}_i(x)] | \Psi_p \rangle \\
&+ e^{-ik_2x-ik_1y} \langle \Psi_{p'} | P[\bar{A}_j(y)(2\bar{\phi}_i(x)\bar{\phi}_l(x) + \delta_{il}\bar{\phi}_m(x)\bar{\phi}_m(x))] | \Psi_p \rangle \\
&+ e^{-ik_1x-ik_2y} \langle \Psi_{p'} | P[\bar{A}_l(y)(2\bar{\phi}_i(x)\bar{\phi}_j(x) + \delta_{ij}\bar{\phi}_m(x)\bar{\phi}_m(x))] | \Psi_p \rangle \} \\
&\qquad + \frac{(-i)\lambda}{(8\omega\omega_1\omega_2)^{\frac{1}{2}}} \int d^4x e^{i(k-k_1-k_2)x} \langle \Psi_{p'} | 2\bar{\phi}_l(x)\delta_{ij} | \Psi_p \rangle \\
&= \frac{(-i)}{(2\omega)^{\frac{1}{2}}} \int d^4x e^{ikx} \langle p'k_1k_2jl | A_i(x) | p \rangle, \quad (5)
\end{aligned}$$

where $A_i(x)$ is given by Low's quantity $O_i(x)$.[5] The quantity $\langle p'k_1k_2jl|A_i(x)|p\rangle$ is defined by the above equation for all values of p, p', k_1, and k_2. By the commutation process, it may be shown to be identical with

$$\left\langle \Phi_{p'} \left| a_j(k_1) a_l(k_2) \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \cdots dt_n P[H_I(t_1)\cdots H_I(t_n) A_i(x)] \right| \Phi_p \right\rangle = \langle \Psi^{(-)}{}_{p',k_1,k_2} | \bar{A}_i(x) | \Psi_p \rangle. \quad (6)$$

On the energy shell, $p'+k_1+k_2=p+k$, this quantity is the S matrix with the delta function of energy and momentum left out. However, off the energy shell, the S-matrix is not defined, whereas $\langle p'k_1k_2jl|A_i(x)|p\rangle$ is defined by the integrand on the left side of Eq. (6).

We will be interested in the static limit. We therefore drop terms in λ and consider

$$\begin{aligned}
&\langle p'k_1k_2jl | A_i(x) | p \rangle \\
&= \frac{(-i)^2}{(4\omega_1\omega_2)^{\frac{1}{2}}} \int d^3\mathbf{z} dz_0 d^3\mathbf{y} dy_0 \exp[-i(\omega_1 y_0 - \mathbf{k}_i\cdot\mathbf{y} + \omega_2 z_0 - \mathbf{k}_2\cdot\mathbf{z})] \langle \Psi_{p'} | P[\exp[i(Hz_0 - \mathbf{P}\cdot\mathbf{z})] A_l(0) \\
&\times \exp[-i(Hz_0-\mathbf{P}\cdot\mathbf{z})] \exp[i(Hy_0-\mathbf{P}\cdot\mathbf{y})] A_j(0) \exp[-i(Hy_0-\mathbf{P}\cdot\mathbf{y})] \exp[i(Hx_0-\mathbf{P}\cdot\mathbf{x})] A_i(0) \\
&\qquad \times \exp[-i(Hx_0-\mathbf{P}\cdot\mathbf{x})]] | \Psi_p \rangle, \quad (7)
\end{aligned}$$

where we have utilized invariance under the translation (momentum) operator $\mathbf{P}$,

$$A_i(\mathbf{x}) = e^{-i\mathbf{P}\cdot\mathbf{x}} A_i(0) e^{i\mathbf{P}\cdot\mathbf{x}}. \quad (8)$$

We now perform the indicated space and time integrations, setting $\mathbf{x}=x_0=0$ in the final result. This is best done as in the following example: Introducing complete sets of exact eigenstates $|\Psi_n\rangle$ and $|\Psi_m\rangle$, and considering a particular time ordering $z_0>y_0>x_0$, we have

$$\begin{aligned}
&-(4\omega_1\omega_2)^{\frac{1}{2}} \langle p'k_1k_2jl | A_i(x) | p \rangle \\
&= \sum_{n,m} \int d^3\mathbf{z} d^3\mathbf{y} \exp[-i\mathbf{k}_1\cdot\mathbf{y} - i\mathbf{k}_2\cdot\mathbf{z}] \int_{-\infty}^{\infty} dz_0 \eta_+(z_0-y_0) e_i^{\omega_2 z_0} \int_{-\infty}^{\infty} dy_0 \eta_+(y_0-x_0) e^{i\omega_1 y_0} \\
&\times \langle \Psi_{p'} | \exp[i(Hz_0-\mathbf{P}\cdot\mathbf{z})] A_l(0) \exp[-i(Hz_0-\mathbf{P}\cdot\mathbf{z})] | \Psi_n \rangle \\
&\times \langle \Psi_n | \exp[i(Hy_0-\mathbf{P}\cdot\mathbf{y})] A_j(0) \exp[-i(Hy_0-\mathbf{P}\cdot\mathbf{y})] | \Psi_m \rangle \\
&\qquad \times \langle \Psi_m | \exp[i(Hx_0-\mathbf{P}\cdot\mathbf{x})] A_i(0) \exp[-i(Hx_0-\mathbf{P}\cdot\mathbf{x})] | \Psi_p \rangle
\end{aligned}$$

$$= \sum_{n,m}\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} dz_0 e^{i(\omega_2+\omega_{p'}-E_n)z_0} \int_{-\infty}^{\infty}\frac{da\, e^{ia(z_0-y_0)}}{a-i\epsilon}\int_{-\infty}^{\infty} dy_0 e^{i(\omega_1+E_n-E_m)y_0}\int_{-\infty}^{\infty}\frac{db\, e^{ib(y_0-x_0)}}{b-i\epsilon}$$

$$\times\int d^3\mathbf{z}d^3\mathbf{y} \exp[i(-\mathbf{k}_1-\mathbf{p}_n-\mathbf{p}_m)\cdot\mathbf{y}]\exp[i(-\mathbf{k}_2-\mathbf{p}'+\mathbf{p}_n)\cdot\mathbf{z}]\langle\Psi_{p'}|A_l(0)|\Psi_n\rangle \times\langle\Psi_n|A_j(0)|\Psi_m\rangle\langle\Psi_m|A_i(0)|\Psi_p\rangle$$

$$=-\sum_{n,m}\frac{\langle\Psi_{p'}|A_l(0)|\Psi_n\rangle\langle\Psi_n|A_j(0)|\Psi_m\rangle\langle\Psi_m|A_i(0)\Psi_p\rangle\delta(\mathbf{p}_m-\mathbf{p}_n-\mathbf{k}_1)\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{k}_2)}{(E_n-\omega_2-\omega_{p'}-i\epsilon)(E_m-\omega_2-\omega_1-\omega_{p'}-i\epsilon)}. \tag{9}$$

Here we have used

$$\eta_+(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{da\, e^{iax_0}}{a-i\epsilon}$$
$$=1,\quad x_0>0$$
$$=0,\quad x_0<0.$$

This procedure is carried out for each time-ordering. The full amplitude is then given by

$$\langle p'k_1k_2jl|A_i(0)|p\rangle=\frac{1}{(4\omega_1\omega_2)^{\frac{1}{2}}}\sum_{n,m}\left\{\frac{\langle\Psi_{p'}|A_l(0)|\Psi_n\rangle\langle\Psi_n|A_j(0)|\Psi_m\rangle\langle\Psi_m|A_i(0)|\Psi_p\rangle\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{k}_2)\delta(\mathbf{p}_m-\mathbf{p}_n-\mathbf{k}_1)}{(E_n-\omega_2-\omega_{p'}-i\epsilon)(E_m-\omega_2-\omega_1-\omega_{p'}-i\epsilon)}\right.$$
$$+\frac{\langle\Psi_{p'}|A_j(0)|\Psi_n\rangle\langle\Psi_n|A_i(0)|\Psi_m\rangle\langle\Psi_m|A_l(0)|\Psi_p\rangle\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{k}_2)\delta(\mathbf{p}_m-\mathbf{p}+\mathbf{k}_1)}{(E_n-\omega_1-\omega_{p'}-i\epsilon)(E_m+\omega_2-\omega_p-i\epsilon)}$$
$$\left.+\frac{\langle\Psi_{p'}|A_i(0)|\Psi_n\rangle\langle\Psi_n|A_j(0)|\Psi_m\rangle\langle\Psi_m|A_l(0)|\Psi_p\rangle\delta(\mathbf{p}_m-\mathbf{p}_n-\mathbf{k}_1)\delta(\mathbf{p}_n-\mathbf{p}-\mathbf{k}_2)}{(E_n+\omega_1+\omega_2-\omega_p-i\epsilon)(E_m-\omega_p+\omega_2+i\epsilon)}\right\}, \tag{10}$$

plus three additional terms obtained by interchanging the subscripts j and l and 1 and 2.

If the $|\Psi_p\rangle$, $|\Psi_{p'}\rangle$, $|\Psi_n\rangle$, and $|\Psi_m\rangle$ were replaced by bare-nucleon spinors, the above equation would yield lowest-order perturbation theory for the six diagrams shown in Fig. 1. In this case, although $|\Psi_p\rangle$ and $|\Psi_{p'}\rangle$ represent physical nucleon states, the $|\Psi_n\rangle$ and $|\Psi_m\rangle$ may be any members of a complete orthonormal set of eigenstates of H, a physical nucleon, or a physical nucleon plus any number of real mesons. Another interesting feature is that although $|\Psi_p\rangle$ is to represent the initial physical proton in our process, and although $A_i(0)$—since it is derived from an interaction which represents the absorption of a π^+ by a neutron—contains the operator $\tau_+=\tau_1+i\tau_2$, yet $A_i(0)|\Psi_p\rangle\neq 0$. Operation on a bare proton with τ_+ yields zero, but the physical proton is a superposition of a bare-proton amplitude, a bare-neutron-plus-meson amplitude, etc. Equation (11) contains only energy denominators of the form ($E_{\text{intermediate}}$-E_{final}). Since the physical-nucleon self-energy appears in both states, this unobservable quantity does not appear in the equation.

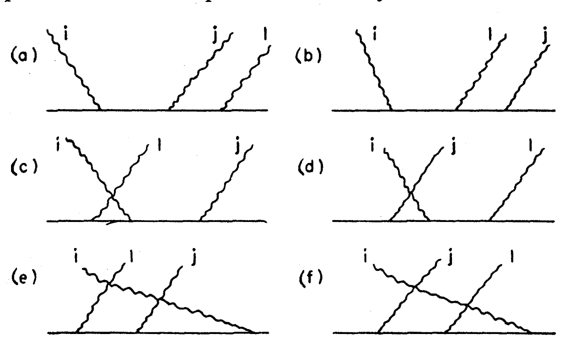

FIG. 1. Feynman diagrams for meson production in the Born approximation.

We wish to treat the equation for $\langle p'k_1k_2jl|A_i(0)|p\rangle$ in the static limit:

$$A_i(0)\rightarrow\frac{i(4\pi)^{\frac{1}{2}}f\boldsymbol{\sigma}\cdot\mathbf{k}\tau_i}{\mu(2\omega)^{\frac{1}{2}}}=A_k,$$
$$A_j(0)\rightarrow\frac{-i(4\pi)^{\frac{1}{2}}f\boldsymbol{\sigma}\cdot\mathbf{k}_1\tau_j}{\mu(2\omega_1)^{\frac{1}{2}}}=A_{k_1}{}^*, \tag{11}$$
$$A_l(0)\rightarrow\frac{-i(4\pi)^{\frac{1}{2}}f\boldsymbol{\sigma}\cdot\mathbf{k}_2\tau_l}{\mu(2\omega_2)^{\frac{1}{2}}}=A_{k_2}{}^*.$$

Here f is the unrenormalized, unrationalized pseudovector coupling constant. $\tau_i=(\sqrt{2}\tau_+,\ \sqrt{2}\tau_-,\ \tau_3)$. In the double sum over the two intermediate states, we keep those terms in which both intermediate states contain only a physical nucleon, and those terms in which one intermediate state contains only a physical nucleon and the other state contains a nucleon, plus one meson. These terms correspond to the nine diagrams shown in

Fig. 2, plus nine more obtained by interchanging k_1 and k_2.

Let us consider first the process $\pi^+ + p \to \pi^+ + \pi^+ + n$. Since there exist neither doubly charged nor negatively charged physical nucleons, our initial physical proton cannot make a transition to another state of the physical nucleon by absorbing a π^+ or by emitting two successive π^+. Therefore the matrix elements corresponding to diagrams (a), (c), (f), and (g) of Fig. 2 vanish.

We are left with two pairs of diagrams, (d) and (e), and (h) and (i). These correspond, respectively, to resonance scattering in the first and in the second intermediate state. In Fig. 2, the open diagrams, (d) and (h), contain the energy denominators corresponding to the resonating intermediate state. The energy controlling the resonance in diagram (d) (the singularity introduced by the energy denominator) is the total energy of the incident meson, whereas in diagram (h) it is the total energy of the second outgoing meson. We have investigated diagram (d) and its crossed counterpart, diagram (e), and we find that, since $\omega_1+\omega_2$ is well above the (3/2, 3/2) resonance energy, the contribution of these diagrams to the matrix element is less than 10% of that from diagrams (h) and (i). We shall consider only the latter two diagrams.

Our amplitude is then

$$\langle\beta k_1k_2|A_k|\alpha\rangle = \left\{\frac{\langle\beta|A_{k_1}{}^*|\gamma\rangle\langle\gamma|A_k|\lambda\rangle}{-\omega_1} + \frac{\langle\beta|A_{k_1}{}^*|\omega'\gamma\rangle\langle\omega'\gamma|A_k|\lambda\rangle}{\omega'-\omega_1-i\epsilon} + \frac{\langle\beta|A_k|\omega'\gamma\rangle\langle\omega'\gamma|A_{k_1}{}^*|\lambda\rangle}{\omega'+\omega_1+\omega_2}\right\}\frac{\langle\lambda|A_{k_2}{}^*|\alpha\rangle}{\omega_2}$$
$$+\text{the same expression with 1 and 2 interchanged.} \qquad (12)$$

In the new notation, $|\alpha\rangle$ ($|\beta\rangle$) represents the initial (final) physical proton (neutron) states; $|\lambda\rangle$ and $|\gamma\rangle$ represent the four states of the physical nucleon, and these are to be summed over; $|\omega\rangle$ represents the nucleon, one-meson states. One sums over the spin and isotopic spin of the nucleon and the isotopic spin of the meson, and one integrates over the meson momentum from zero up to the cutoff value. The delta functions of the momenta of intermediate and initial and final states are taken into account by integrating over intermediate nucleon momenta, the nucleon in the static limit being infinitely heavy, and hence the absorber of any amount of momentum required for conservation at a vertex.

In Eq. (12), we now neglect the ω_2 in the denominator of the last term in the curly brackets and the ω_1 in the denominator of the same term in the curly brackets of the same expression with 1 and 2 interchanged. Our

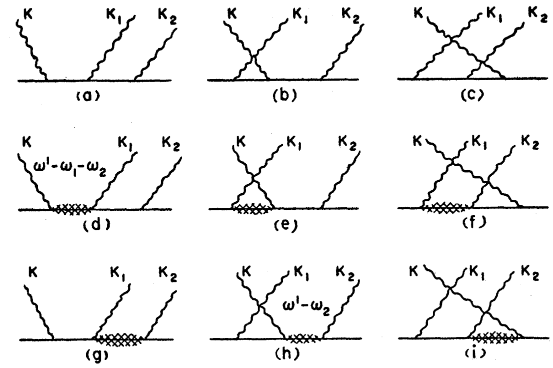

FIG. 2. Feynman diagrams for meson production in the one-meson approximation.

approximations are made in the terms coming from the "crossed" diagrams [(i) of Fig. 2]. These matrix elements contain no poles and will therefore be small compared with the matrix elements coming from the "open" diagrams [(h) of Fig. 2]. More explicitly, the approximation is made in a term of the form

$$\int_\mu^{\omega_{\max}} p\omega' d\omega' \frac{\langle\lambda|A_{k_1}{}^*|\omega'\gamma\rangle\langle\omega'\gamma|A_k|\alpha\rangle}{\omega'+\omega_1+\omega_2}, \qquad (13)$$

where

$$p=(\omega'^2-\mu^2)^{\frac{1}{2}}.$$

Dropping ω_2 in the denominator gives rise to an error in the integrand of the order of $\omega_2/(\omega'+\omega_1+\omega_2)$, which, at most, is about $2\mu/(\mu+3\mu)=\frac{1}{2}$. The error in the integral is less, since the integral diverges linearly, and hence the main contributions come from ω', several times μ. This can be seen from the following:

$$\langle\omega'\gamma|A_k|\alpha\rangle = -\sum_n \left\{\frac{\langle\gamma|A_p{}^*|n\rangle\langle n|A_k|\alpha\rangle}{\omega'-\omega_1-i\epsilon} + \frac{\langle\gamma|A_k|n\rangle\langle n|A_p{}^*|\alpha\rangle}{\omega'+\omega_2}\right\}$$
$$\sim (1/\omega')\{\langle\gamma|A_p{}^*|n\rangle\langle n|A_k|\alpha\rangle + \langle\gamma|A_k|n\rangle\langle n|A_p{}^*|\alpha\rangle\}, \qquad (14)$$

where

$$A_p{}^* = -i(4\pi)^{\frac{1}{2}} f\boldsymbol{\sigma}\cdot\mathbf{p}\tau_i/\mu(2\omega')^{\frac{1}{2}}.$$

Therefore $\langle\omega'\gamma|A_k|\alpha\rangle \propto 1/\sqrt{\omega'}$, and the above integral is proportional to $\omega_{\max}$.

Our amplitude now has the following form:

$$\langle\beta k_1k_2|A_k|\alpha\rangle = \left\{\frac{\langle\beta|A_{k_1}{}^*|\gamma\rangle\langle\gamma|A_k|\lambda\rangle}{-\omega_1} + \frac{\langle\beta|A_{k_1}{}^*|\omega'\gamma\rangle\langle\omega'\gamma|A_k|\lambda\rangle}{\omega'-\omega_1-i\epsilon} + \frac{\langle\beta|A_k|\omega'\gamma\rangle\langle\omega'\gamma|A_{k_1}{}^*|\lambda\rangle}{\omega'+\omega_1}\right\}\frac{\langle\lambda|A_{k_2}{}^*|\alpha\rangle}{\omega_2}. \qquad (15)$$

The term in parentheses is the one-meson approximation for the Chew-Low scattering matrix, $T_k(k_1)$ (in the notation of Chew[6]). It describes the scattering of a π^+ of momentum (energy) $k(\omega)$ into a π^+ of momentum (energy) $k_1(\omega_1)$, with the physical neutron going from state α to state β.

A similar equation may be derived for the process $\pi^+ + p \rightarrow \pi^+ + \pi^0 + p$. In this case, the matrix element corresponding to diagram (f) in Fig. 2 does not vanish. We neglect this diagram, since the energy denominators give rise to no poles. There is, in addition, a nonvanishing matrix element for the Born approximation represented by diagram (c). The contribution from this diagram is given by

$$\frac{\langle\beta|A_k|\gamma\rangle\langle\gamma|A_{k_2}{}^*|\lambda\rangle\langle\lambda|A_{k_1}{}^*|\alpha\rangle}{\omega_1(\omega_1+\omega_2)}. \tag{16}$$

We neglect ω_1 in the quantity $(\omega_1+\omega_2)$. The part

$$\frac{\langle\beta|A_k|\gamma\rangle\langle\gamma|A_{k_2}{}^*|\lambda\rangle}{\omega_2}$$

is then the "crossed" contribution to the Born approximation in the Chew-Low scattering matrix. The error in the Born term is of the order of $\omega_2/(\omega_1+\omega_2)$, which is quite small compared with the complete contribution to the matrix element from the Born terms plus the one-meson terms. The Born term can be treated exactly; however, we feel that to do so would alter the cross section by less than 10%.

In Eq. (15), the matrix element $\langle\lambda|A_{k_2}{}^*|\alpha\rangle$ is of the form

$$\langle\text{physical nucleon}|f\boldsymbol{\sigma}\cdot\mathbf{k}\tau_i|\text{physical nucleon}\rangle.$$

Evaluating this matrix element is equivalent to taking the vertex operator between bare nucleon states and renormalizing the coupling constant, $f\rightarrow f_r$.[7,8]

The scattering matrix may be expanded in terms of projection operators for the various states of total isotopic spin and angular momentum. We keep only the $I=3/2$, $J=3/2$ part. Then

$$T_k(k_1)=\frac{4\pi}{(4\omega\omega_1)^{\frac{1}{2}}}kk_1h_3(k_1)P_3Q_3. \tag{17}$$

Here P_3 and Q_3 are projection operators for the $J=3/2$ and $I=3/2$ states, respectively. P_3 is given explicitly by

$$P_3=\frac{1}{3kk_1}\{2\mathbf{k}\cdot\mathbf{k}_1+i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}_1)\}. \tag{18}$$

The expression for $h_3(k_1)$ in the "effective range approximation" of Chew and Low is given by

$$h_3(k_1)=\frac{\lambda_3}{\omega_1(1-\omega_1/\omega_3)-i\lambda_3k_1{}^3}, \tag{19}$$

[8] T. D. Lee, Phys. Rev. 95, 1329 (1954).

Here, $f_r{}^2$ is the renormalized, unrationalized coupling constant; ω_3 is the "resonance" energy. We take $f_r{}^2=0.08$; $\omega_3=2.1\mu$.[6]

The matrix element for $\pi^+ + p \rightarrow \pi^+ + \pi^+ + n$ is now

$$\begin{aligned}T_k(k_1,k_2)=\sum_\chi\Big\langle\chi_f\Big|&\frac{4\pi}{2(\omega_1\omega)^{\frac{1}{2}}}\left\{\frac{\lambda_3}{\omega_1(1-\omega_1/\omega_3)-i\lambda_3k_1{}^3}\right\}\\&\times\{2\mathbf{k}\cdot\mathbf{k}_1+i\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}_1)\}Q_3\Big|\chi\Big\rangle\\&\times\Big\langle\chi\Big|\frac{-i(4\pi)^{\frac{1}{2}}f_r\boldsymbol{\sigma}\cdot\mathbf{k}_2\sqrt{2}\tau_-}{\sqrt{2}\mu\omega_2{}^{\frac{3}{2}}}\Big|\chi_i\Big\rangle\\&+\text{the same expression with subscripts}\\&\quad 1\text{ and }2\text{ interchanged,}\end{aligned} \tag{20}$$

where $\chi_i(\chi_f)$ is the initial (final) bare-proton (nucleon) spinor, spin up or down, and the intermediate bare spinors are summed over spin and charge states.

The matrix element for π^0 production is given by the above formula with the $\sqrt{2}\tau_-$ in the perturbation term changed to τ_3 when the subscripts 1 and 2 are interchanged.

RESULTS AND DISCUSSION

The results for the square of the matrix element averaged and summed over initial and final spin states follow.

Spin flip:

$$\begin{aligned}|T_kk_1k_2|^2=&\frac{4(4\pi)^3f_r{}^6(kk_1k_2)^2}{81\mu^6(\omega_1\omega_2\omega)}\\&\times[A_0\{4\sin^2\theta_1\cos^2\theta_2+\sin^2\theta_2\cos^2\theta_1\\&+\tfrac{1}{2}\sin2\theta_1\sin2\theta_2\cos(\phi_2-\phi_1)\}\\&+B_0\{1\rightarrow2,\ 2\rightarrow1\}\\&+C_0\{2.5\sin2\theta_1\sin2\theta_2\cos(\phi_2-\phi_1)\\&\qquad+4\sin^2\theta_2\cos^2\theta_1+4\sin^2\theta_1\cos^2\theta_2\}].\end{aligned}$$

No spin flip:

$$\begin{aligned}|T_k(k_1k_2)|^2=&\frac{4(4\pi)^3f_r{}^6(kk_1k_2)^2}{81\mu^6(\omega_1\omega_2\omega)}\\&\times[(A_0+B_0)\{4\cos^2\theta_1\cos^2\theta_2+\sin^2\theta_1\sin^2\theta_2\\&-\sin2\theta_1\sin2\theta_2\cos(\phi_1-\phi_2)\}\\&+C_0\{8\cos^2\theta_1\cos^2\theta_2\\&+2\sin^2\theta_1\sin^2\theta_2\cos2(\theta_1-\theta_2)\\&\qquad-2\sin2\theta_1\sin2\theta_2\cos(\phi_1-\phi_2)\}],\end{aligned} \tag{21}$$

where, for π^+ production,

$$A_0=\frac{1}{\omega_2^2\{\omega_1^2(1-\omega_1/\omega_3)^2+\lambda_3^2k_1^6\}},$$

$$B_0=\frac{1}{\omega_1^2\{\omega_2^2(1-\omega_2/\omega_3)^2+\lambda_3^2k_2^6\}},$$

$$C_0=\frac{\omega_1\omega_2(1-\omega_1/\omega_3)(1-\omega_2/\omega_3)+\lambda_3^2k_1^3k_2^3}{(\omega_1\omega_2)\{\omega_1^2(1-\omega_1/\omega_3)^2+\lambda_3^2k_1^6\}\{\omega_2^2(1-\omega_2/\omega_3)^2+\lambda_3^2k_2^6\}}.$$

The coefficients for π^0 production are

$$A_0'=4.5A_0,\quad B_0'=2B_0,\quad C_0'=3C_0.$$

The differential cross section is given by

$$d\sigma=(k/\omega)^{-1}|T_k(k_1k_2)|^2 \, 2\pi\delta(\omega-\omega_1-\omega_2)k_1k_2\omega_1\omega_2d\omega_1d\omega_2d\Omega_1d\Omega_2/(2\pi)^6. \quad (22)$$

Spin flip:

$$\sigma=(5/4)(A+B)+2C.$$

No spin flip:

$$\sigma=A+B+C. \quad (23)$$

For π^+ production,

$$A(B)(C)=Z\int_\mu^T d\omega_1k_1^3k_2^3A_0(B_0)(C_0),$$

where $Z=0.7f_r^6k/\mu^6$, $\omega_2=\omega-\omega_1$, $k_2=(\omega_2^2-\mu^2)^{\frac{1}{2}}$, and T=incident meson kinetic energy$=\omega-\mu$.

For π^0 production replace A_0 by A_0', etc.

The angular distributions for each meson are given by the following:

No spin flip:

$$d\sigma/d(\cos\theta)=a\cos^2\theta+b, \quad (24)$$

where

$$a=\tfrac{3}{8}(A+B)+\tfrac{3}{2}C,\quad b=\tfrac{3}{8}(A+B).$$

Spin flip:

$$d\sigma/d(\cos\theta)=g\cos^2\theta+f,$$

where

$$g=\tfrac{3}{4}[(7/4)A-\tfrac{1}{2}B+C],\quad f=\tfrac{3}{4}(\tfrac{1}{4}A+B+C).$$

The integrations were performed numerically. At 250-Mev incident-meson kinetic energy, the differential cross section for π^+ production as a function of the polar angle between either outgoing meson and the incident direction is of the form $0.64+\cos^2\theta$. At 350 Mev, the angular distribution is of the form $0.7+\cos^2\theta$. The dip at 90° is more pronounced than that obtained by Miyachi[9] for the angular distribution of the π^+ from the process $\pi^-+p\rightarrow\pi^++\pi^-+n$ at 210 Mev. The latter calculation was done by using covariant perturbation theory with pseudovector coupling. The total cross sections for the two processes are listed in Table I.

At 350 Mev, the cross section for production of an additional meson is about 12% of the total cross section.[10] The sharp increase in cross section at 350 Mev is due to the resonance behavior of the scattering part of the matrix element when one or both of the mesons may come out with kinetic energy approaching the resonance kinetic energy. At 500-Mev incident-meson kinetic energy, the production cross section becomes of the order of the total cross section of 20 mb, indicating the breakdown of this approximation.

Comparison with definite experimental results is not yet possible. However, Blau and Caulton[11] have observed inelastic collisions of 500-Mev π^- mesons in emulsions. They have estimated the cross section for the production of an additional charged meson, averaged over π^--p and π^--n collisions, to be between 3.5 and 10 mb. We have taken the value of 3.5 mb and have deduced, on the basis of the isobar model, a lower limit on the production cross section in π^+-p collisions of about 6 mb. Yuan and Lindenbaum[10] have recently indicated, on the basis of their total-cross-section measurements, that meson production in π^+-p collisions is likely to become important above 300 Mev.

TABLE I.

Incident-meson kinetic energy (Mev)	Energy above threshold (c.m.) (Mev)	$\sigma(\pi^+\pi^+)$ (mb)	$\sigma(\pi^+\pi^0)$ (mb)	$\sigma(\pi^+\pi^+)+\sigma(\pi^+\pi^0)$ (mb)
250	60	0.071	0.44	0.51
350	130	0.86	5.5	6.4

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