Relation between Nucleon Density and Nuclear Potential*

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The relationship between nuclear density and potential is determined, taking account of the saturating character and the finite range of the two-body forces. The former characteristic alone leads to the nonlinear relation between nuclear density and potential pointed out by Wilets, and accounts for an increase of potential radius over density radius of about 5×10^{-14} cm. The finite-range effect is estimated by using pseudopotentials to represent the two-body interaction operators generated by the two-body potentials. The combined effects of nonlinearity and finite range give an increase in the rms potential radius which is very close to that observed experimentally. The fall-off distance of the potential surface predicted is, however, somewhat greater than that empirically determined. It is pointed out that this discrepancy may be removed if proper account is taken of the nonlocal character of the nuclear potential.

I. INTRODUCTION

SINCE the studies of the μ-mesonic atom and of electron scattering first indicated that the nuclear charge density corresponded to a mean radius close to $R=1.20\times10^{-13}A^{\frac{1}{3}}$, it has not been easy to reconcile this value with the considerably larger radii of (1.33 to 1.45) $\times10^{-13}A^{\frac{1}{3}}$ cm deduced from nuclear interactions. The discrepancy in radius is about (1.5 to 2)$\times10^{-13}A^{\frac{1}{3}}$ cm or in lead, for example, (0.9 to 1.2)$\times10^{-13}$ cm.

A possible explanation of the larger potential radius was given by Teller and Johnson,[1] who pointed out possible effects in nuclei with considerable neutron excess which would lead to greater radii for neutrons than for protons. These arguments do not apply in nuclei with $N=Z$, where the large radii are found just as is the heavier nuclei; furthermore, recent studies by Wilets[2] of the neutron and proton densities have shown that the effects discussed by Teller and Johnson are nearly compensated in heavy nuclei by the effects of the nuclear symmetry energy. This compensation is a result of the tendency for neutron and proton densities to be equal; it is energetically unfavorable for regions of large neutron or proton excess to occur.

More recently Drell[3] has examined the possible effects of the finite nuclear force range on the nuclear potential. Although this leads to some extension of the potential relative to the density distribution, Drell's studies seem to indicate that for nuclear potentials of the form indicated by theory and by experiment, the increase in the nuclear radius is too small to account for the aforementioned discrepancies. Drell's estimates neglected exchange effects and treated the interactions by perturbation methods; thus his results are somewhat inconclusive.

It is the purpose of this note to elaborate upon an effect pointed out by Wilets[4] and to show that, when combined with the effects of the finite range of nuclear forces, it leads to an increase in potential radius comparable to that deduced empirically. The essential point made by Wilets in his statistical analysis of nuclear structure is that because of the saturating character of nuclear forces, the potential seen by a neutron moving in nuclear matter cannot be a linear function of the density. Not only must the potential saturate at high densities, but in general it can be expected to fall off with decreasing density less rapidly than the density. This nonlinear relation automatically leads to an increase in the nuclear potential radius over that of the density. Wilets' estimate was that the increase in radius would be of the order of 0.8×10^{-13} cm.

We shall determine the magnitude of the Wilets effect using an analysis of the origin of the neutron-nucleus potential based on the detailed methods developed in the saturation studies made by the author and others.[5-7] We shall then attempt to estimate the joint contribution of the Wilets effect and the finite force range to the difference in radii of potential and density.

II. POTENTIAL DENSITY RELATIONSHIP IN A UNIFORM MEDIUM

We first determine the potential-density relationship in a uniform medium. We shall follow the several methods of the saturation work but as a simple example also give results based on some simple models. In the saturation studies, the determination of the interaction energy of a nucleus with its neighbors in a region of uniform density is

$$V(1)=\sum_i(K_{1i,1i}-K_{1i,i1}). \qquad (1)$$

The reaction matrix K is determined by the solution of the integral equation

$$K_{kl,1i}=v_{kl,1i}+\sum_{mn} v_{kl,mn}\frac{1}{E_1+E_i-E_m-E_n}K_{mn,1i}, \qquad (2)$$

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[1] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).
[2] L. Wilets, Phys. Rev. **101**, 1805 (1956).
[3] S. D. Drell, Phys. Rev. **100**, 97 (1955).
[4] R. A. Berg and L. Wilets, Phys. Rev. **101**, 201 (1956).
[5] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).
[6] K. A. Brueckner, Phys. Rev. **96**, 508 (1954).
[7] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).

where

$$E_k = (k^2/2M) + V(k), \tag{3}$$

and $V(k)$ is the interaction of a particle in the state k with its neighbors. The matrix elements are taken with respect to a set of states appropriate to uniform density, i.e., plane waves. We obtain the solution to this set of coupled equations, as in the saturation studies, by making the best fit to $V(k)$ of the form,

$$V(k) = V(0) + bk^2 + ck^4, \tag{4}$$

and determining the self-consistent values of b and c. We shall not give the details of this computation, which follows closely the procedures of reference 7. The result for the potential energy of interaction is given in Fig. 1 as a function of density. The very nonlinear relation between the potential and density is evident. Analysis of the origin of this result indicates that it depends most strongly on the great sensitivity of the solution of the self-consistency equations to the presence of a repulsive core. The relevance of the short range of the repulsion compared to the lower range of the attraction in the two-body interaction has previously been emphasized by Drell.[3]

As a second illustration of the Wilets effect, we consider simple two-body saturating interactions. We do not regard these two-body interactions as the actual two-body potentials, which should not be treated in first order, but rather as simple representation of the actual two-body reaction matrices which determine the binding energies.[8] Thus they should be related to the correct two-body potentials by coupled integral equations similar to Eqs. (2) and (3). Accordingly, while we choose ranges of the order of those of the usual two-body forces, we will regard the strengths as adjustable and adjust them to give on the average a potential energy of interaction of nucleon with nucleus equal to that determined empirically.

The examples we choose to consider are

$$v^{(1)}(r) = -V_1(1+4P_r)e^{-2\mu r}, \quad \text{exponential,}$$

$$v^{(2)}(r) = -V_2(1+4P_r)\frac{e^{-\mu r}}{r}, \quad \text{Yukawa,} \tag{5}$$

$$v^{(3)}(r) = -V_3(1+4P_r)\frac{e^{-2\mu r}}{r} + V_4\frac{e^{-4\mu r}}{r}(4+P_r), \quad \text{“hard core” Yukawa,}$$

where P_r is the space exchange operation and $1/\mu$ is the meson Compton wavelength. The saturation in the last case in a consequence of both the exchange character and the short-range "repulsive core."

[8] A similar assumption has been made independently by T. H. R. Skyrme in an analysis of nuclear surface structure (to be published).

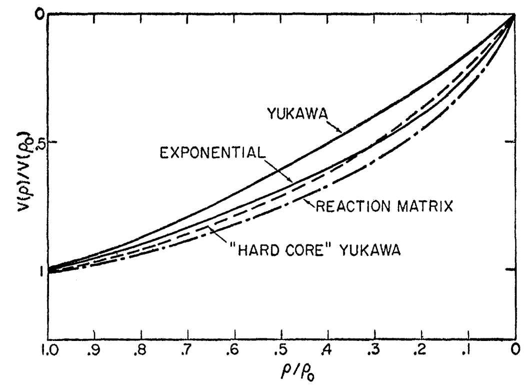

FIG. 1. Interaction energy of a particle moving near the Fermi momentum, as a function of density (assumed uniform). The curve labeled "reaction matrix" is the prediction of Eq. (1). The density ρ_0 corresponds to a radius parameter $R=1.20\times10^{-13}A^{1/3}$ cm.

The interaction energy of a particle in a state "1" with its neighbors is

$$V(1) = \sum_i \int \varphi_1^*(1)\varphi_i^*(i)v(|\mathbf{r}-\mathbf{r}_i|) \times[\varphi_1(1)\varphi_i(i) - \varphi_1(i)\varphi_i(1)]d\mathbf{r}_1 d\mathbf{r}_i, \tag{6}$$

where the sum runs over the filled states. In a uniform medium, the states are plane waves

$$\varphi_i(r_i) = \frac{1}{V^{\frac{1}{2}}} e^{i\mathbf{k}_i\cdot\mathbf{r}_i} \times \text{spin functions,} \tag{7}$$

where V is the normalization volume. Writing the interactions as

$$v(r) = f(r)(a+bP_r), \tag{8}$$

and going from summation over the momentum state to integration, we find

$$V(1) = \frac{A}{V}\int d\mathbf{r}_1 f(\mathbf{r}_1-\mathbf{r}_i)\left[(a-\tfrac{1}{4}b)+(b-\tfrac{1}{4}a) \times D(k_F|\mathbf{r}_1-\mathbf{r}_i|)\frac{\sin(k_1|\mathbf{r}_1-\mathbf{r}_i|)}{k_1|\mathbf{r}_1-\mathbf{r}_i|}\right], \tag{9}$$

where

$$D(x) = \frac{3}{x^2}\left(\frac{\sin x}{x} - \cos x\right). \tag{10}$$

The Fermi momentum k_F is determined by the equation

$$k_F = 1.52\mu/\eta, \tag{11}$$

where η is a parameter related to the density by

$$V/A = (4/3)(\pi/\mu^3)\eta^3. \tag{12}$$

Thus at a radius $R=1.40\times10^{-13}A^{\frac{1}{3}} = A^{\frac{1}{3}}/\mu$, η is equal to one.

The momentum k_1 of the interacting particle will be close to the Fermi momentum if the particle is in a

state close to the last filled level, so that we will replace k_1 by k_F. Thus we determine the interacting energy of particle "1" with the rest of the system by integrating an "effective potential" of interactions between particle "1" and particle "i":

$$v_e(r_{1i})=f(r_{1i})\left[(a-\tfrac{1}{4}b)+(b-\tfrac{1}{4}a)D(k_F r_{1i})\frac{\sin k_F r_{1i}}{k_F r_{1i}}\right] \quad (13)$$

over the density distribution (uniform) of particle "i." To carry out the integration in an elementary way, it is convenient to replace the density-dependent factor $D(x)\sin x/x$ by the very good approximation $(\sin\beta r/\beta r)^2$, where $\beta=0.90k_F$. We then find for the three choices of potential given in Eq. (5) the potential energy of interaction

$$V^{(1)}=-\frac{4\pi A}{V}\frac{15}{4}\frac{V_1}{4\mu}\frac{1}{\mu^2+\beta^2},$$

$$V^{(2)}=-\frac{4\pi A}{V}\frac{15}{4}\frac{V_2}{\beta^2}\ln\left(1+\frac{\beta^2}{\mu^2}\right), \quad (14)$$

$$V^{(3)}=-\frac{4\pi A}{V}\frac{15}{4}\left[\frac{V_3}{\beta^2}\ln\left(1+\frac{\beta^2}{4\mu^2}\right)-\frac{V_4}{16\mu^2}\right].$$

In the last case it is necessary to specify a ratio of V_4/V_3. This is taken to be 2.40 to give a minimum in the total energy at close to normal density. These results are given graphically in Fig. 1 together with the results of the saturation calculation; the qualitative similarity is apparent. The potential as a function of variable density can be obtained if these results are assumed to hold in a region of varying densities. Taking the density distribution determined by the electron scattering and computing the potential, gives the result of Fig. 2. The increase in potential over density radius due to this nonlinear potential-density relation is about 0.5×10^{-13} cm. This corresponds to an increase in the radius parameter (at $A^{\frac{1}{3}}=6$) from 1.12 to 1.20×10^{-13} cm. It is to be emphasized that the finite-range effect is not yet included in this result; we consider this in the next section.

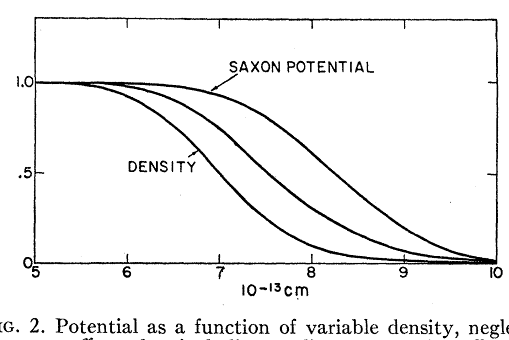

FIG. 2. Potential as a function of variable density, neglecting finite-range effects, but including nonlinear saturation effect. The intermediate curve gives the result for the reaction matrix. The curve labeled "Saxon potential" is taken from the work of Saxon *et al.*[9]

III. FINITE RANGE EFFECT

We next attempt to extend the results of the previous section to media of nonuniform density. If the change in density occurs slowly, the interaction of particle 1 with the remaining bound particles of the medium, is correctly given by the effective interaction $v_e(|\mathbf{r}_1-\mathbf{r}_i|)$ of Eq. (13) with k_F determined by the local value of the density. We shall assume that this relationship still holds even when the density change is rapid and evaluate k_F at an average value of the density. This procedure certainly is approximate, but it does allow a very simple determination of the varying interaction strength which is a consequence of the saturating character of the forces at the same time that the finite range effect is included. Our procedure then is the following; we determine the interaction energy of particle 1 by evaluating the integral

$$V(r_1)=\int\rho(r_2)v_e(|\mathbf{r}_1-\mathbf{r}_2|)d\mathbf{r}_2, \quad (15)$$

where k_F, which enters in v_e, is evaluated at the density $\rho(\frac{1}{2}|\mathbf{r}_1+\mathbf{r}_2|)$. This integral is not easily evaluated in general; we shall approximate to it by taking simple forms for $v_e(r)$. The correction due to the exclusion effect primarily alters the effective range of the potential, and can be approximately represented by simply altering the range of the interaction. This is evident from the form of $v_e(r)$ given in Eq. (13), the correction due to the exchange term vanishing at $r=0$ and increasing monotonically with r (out to $0.9k_F r=\pi$). The correction we make is to replace the range of the interaction by a density-dependent range, the value being taken to give the correct result for the interaction energy in the case of a uniform medium. This approximation will correctly reproduce the general effects of the exchange terms. The appropriate alteration in the ranges is apparent from Eq. (14); it is

$$\begin{aligned}
&2\mu\rightarrow2\mu[1+\beta^2/\mu^2]^{\frac{1}{2}}, \text{exponential},\\
&\mu\rightarrow\beta\left[\ln\left(1+\frac{\beta^2}{\mu^2}\right)\right]^{-\frac{1}{2}}, \quad \text{Yukawa},\\
&\left.\begin{aligned}&2\mu\rightarrow\beta\left[\ln\left(1+\frac{\beta^2}{4\mu^2}\right)\right]^{-\frac{1}{2}}\\ &4\mu\rightarrow4\mu\end{aligned}\right\}, \quad \text{"hard core" Yukawa.}
\end{aligned} \quad (16)$$

In this approximation the integrals are easily carried out numerically; the results are given in Fig. 3 for the exponential and "repulsive core" Yukawa. The result for the Yukawa well is intermediate between these cases. The rather peculiar shape of the curve in the "repulsive core" case is due to the short range of the repulsion compared with the attractive. In the surface region of the density, the short-range repulsion exerts its full effect, while the larger-range attraction is not

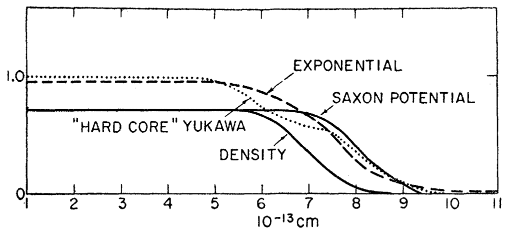

FIG. 3. Potential as a function of variable density, including finite-range effects. The curves are normalized to the same potential volume.

yet fully effective. On the other hand, well outside the matter, the long-range attraction alone acts. Thus the potential first increases rapidly as the matter distribution is approached, then less rapidly as the short-range repulsion exerts its full effect, and finally reaches its full value in the central region. This complex behavior does not occur in the case of the exponential since both attraction and repulsion have the same range.

To compare these results in detail with the semi-empirical results of Saxon,[9] it is necessary to choose a normalization, i.e., to choose a value for the interaction strengths V_1, V_2, V_3 of Eq. (5). A general measure of the range which is independent of the normalization, however, is the mean radius. This we define to be

$$\tfrac{3}{5}R^2=\int_0^\infty r^4v(r)dr\Big/\int_0^\infty r^2v(r)dr. \qquad (17)$$

For the case we have taken, the results are

$R=\ 8.98\times10^{-13}$ cm, density distribution,
$=10.47\times10^{-13}$ cm, Saxon potential,
$=10.73\times10^{-13}$ cm, predicted from exponential interaction,
$=10.76\times10^{-13}$ cm, predicted from "repulsive core" Yukawa interaction.

Thus the mean radius of the computed potential distribution is considerably larger than the density distribution and is close to that given by the Saxon results. To make a more detailed comparison, we normalize the distribution so that

$$\int_0^\infty V(r)r^2dr \qquad (18)$$

is the same as for the Saxon distribution. The normalized distributions are those given in Fig. 3. These differ considerably in shape, but there is general agreement in the surface regions so that the effective mean radius agrees with that of Saxon.

It is to be emphasized that the large mean radius of the computed potential is nearly the same for the two cases considered in detail. This is true although the "hard core" Yukawa has a considerably shorter range than the exponential. As we have shown, the difference in range of the two-body potential is not meaningful by itself but must be considered together with the origin of the saturation of the forces.

IV. NONLOCAL CHARACTER OF GENERAL POTENTIAL

An effect neglected in the preceding section arises from the nonlocal character of the correct effective interaction operation between the incoming and the nuclear particle. This is most easily seen if we consider the interacting energy between two particles, which to second order in the potential is

$$(\mathbf{r}_1{}^1\mathbf{r}_2{}^1|K|\mathbf{r}_1\mathbf{r}_2)=v(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_1{}^1+\mathbf{r}_2{}^1)$$
$$+v(\mathbf{r}_1{}^1-\mathbf{r}_2{}^1)(\mathbf{r}_1{}^1\mathbf{r}_2{}^1|G|\mathbf{r}_1\mathbf{r}_2)v(\mathbf{r}_1-\mathbf{r}_2), \qquad (19)$$

where G is the appropriate Green's function. Thus we cannot expect in general that the potential seen by a nucleon will be local; it will instead have the form

$$(\mathbf{r}_1{}^1|W|\mathbf{r}_1)=\sum_i\int\int\psi_i{}^*(r_i{}^1)(\mathbf{r}_1{}^1\mathbf{r}_i{}^1|K|\mathbf{r}_1\mathbf{r}_i)$$
$$\times\psi_i(r_i)d\mathbf{r}_id\mathbf{r}_i{}^1, \qquad (20)$$

neglecting exchange terms. The corresponding Schrödinger equation which determines the motion of particle 1 in the field of the nucleus thus will be

$$\frac{p^2}{2M}\psi(\mathbf{r})+\int d\mathbf{r}^1(\mathbf{r}|W|\mathbf{r}^1)\psi(\mathbf{r}^1)=E\psi(\mathbf{r}). \qquad (21)$$

The range of nonlocality of W will certainly be of the order of the range of the two-body interactions, i.e., not small compared with the nuclear edge depth. An even more complicated situation arises in the case of the imaginary part of the potential, which not only can be nonlocal, but also will not have the same shape as the real part of the potential. Thus it cannot have a precise meaning to make a comparison of "potential" shape (assumed local) with density shape, since the former can be at best a qualitative representation of the true nonlocal potential. It would seem desirable, consequently, to extend the range of potential considered in the analysis of nucleon-nucleus scattering to include such nonlocal interactions.

[9] Melkanoff, Moszkowski, Nodvik, and Saxon, Phys. Rev. 101, 507 (1956).