Theory of Plasma Resonance

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Starting from the Boltzmann transport equation, a formula is derived for the rate of change of electron density in a gas plasma. With its aid a study is made of the oscillations of electron density (about the steady state value) in a discharge confined between parallel plates. The oscillations are described in terms of a set of normal modes characteristic of the plasma under study. An expression is obtained for the impedance of the discharge as a function of frequency: for the one case calculated in detail, this formula gives a resonance in power absorption at a frequency of 0.7 of the plasma resonance frequency corresponding to the central electron density.

I.

RECENTLY several papers[1] have been published dealing with the electrical properties of gas plasmas. In all cases, however, important assumptions (such as neglect of diffusion or assumption of ambipolar conditions) were made in order to treat the behavior of the discharge. The necessity for such simplifications comes about because of the complexity of the basic plasma equations, which are highly nonlinear and hard to solve under even the best of circumstances. In the present work a different point of view will be adopted: It will be assumed from the start that the fields which are used to probe the plasma are small and that, in consequence, any fluctuations induced in the plasma density or other properties of the plasma will also be small. The equations of motion then become linear and, in this limit, can be solved exactly. Thus, this approach permits one to take account of effects that have been neglected in previous treatments. In addition, this point of view has an important conceptual advantage. It turns out that the solutions of the linearized equations can conveniently be expressed in terms of a set of normal modes that are characteristic of the discharge in question. These normal modes are analogous to those of a mechanical system so that, in the small-signal limit at least, the behavior of the discharge can be thought of in terms which are familiar from other branches of physics.

In performing actual calculations of the properties of a plasma, attention will be restricted to the case of a discharge between parallel plates. This geometry, which is about the simplest realizable experimentally, has the great advantage over others that the plasma density, space charge, etc., within it vary in only one dimension. Furthermore, in certain cases, at least, its steady state properties are well understood. In particular, Allis and Rose[2] have made a thorough investigation of the behavior of a discharge between parallel plates in an atmosphere of hydrogen gas. Their work includes detailed calculations of the ion and electron densities and space-charge fields within such a plasma. Extensive use of these results will be made in the succeeding development; the computations for the one case to be worked out in detail will be directly based upon the Allis-Rose work. The numerical results of this paper will thus only be applicable to a hydrogen plasma such as they studied. However, the problem will be formulated and solved in a general enough way to provide considerable qualitative insight into the behavior of gas plasmas in general.

The first step in the treatment of the plasma resonance problem is the derivation of equations of motion describing the dynamics of an ionized gas. This development, which proceeds from the fundamental transport equation, will be presented and discussed in the following section. Succeeding sections of the paper will then be concerned with the solution and application of these basic formulas. The latter, as was mentioned earlier, will be treated by assuming that any disturbances of the plasma are small fluctuations about the steady-state densities and fields (these are the quantities calculated by Allis and Rose for the case of H_2). The equations of motion then become linear and can be solved by standard techniques. In this way one arrives at a small-signal theory of plasma resonance, valid when the perturbing fields giving rise to the excitation are small compared to those normally present in the plasma.

In the final section of the paper a particular discharge, which is among those investigated by Allis and Rose, is studied in detail. This application is made with an experiment in mind and includes a calculation of the impedance of the discharge. The frequency variation of this quantity reveals possible oscillatory modes of the plasma: in the example treated, there is a resonance in power absorption whose width and frequency are determined in terms of the steady state properties of the discharge.

II.

The transport equation,[3] from which equations describing the plasma oscillations of an ionized gas will be derived, is

$$\partial F/\partial t = C - \nabla_r \cdot \mathbf{v}F + \nabla_v \cdot (e\mathbf{E}/m)F. \tag{1}$$

[1] E. Everhart and S. C. Brown, Phys. Rev. **76**, 839 (1949). W. O. Schumann, Z. Naturforsch. **4a**, 486 (1949); **5a**, 181 (1950).

[2] W. P. Allis and D. J. Rose, Phys. Rev. **93**, 84 (1954).

[3] D. J. Rose and S. C. Brown, Phys. Rev. **98**, 310 (1955).

Here F is the distribution function of electrons in velocity and configuration space, C represents the effects of collisions on F, ∇_r and ∇_v are the gradient operators in configuration and velocity space, and v, m, and e are the velocity, mass, and charge of the electron. Besides this equation there should also, in principle, be a similar one describing the motion of positive ions. However, for perturbing fields of high enough frequency the large inertia (compared to electrons) of the ions prevents them from moving appreciably, and the major part of any disturbance in charge distribution is due to electron motion. It will be assumed in subsequent work that all perturbing fields are of this high-frequency type so that the positive-ion distribution, although not constant in space, is stationary in time. For frequencies of interest as far as plasma resonance is concerned, this approximation is quite accurate.

The first step in the treatment of Eq. (1) is the expansion of the distribution function, F, into spherical harmonics in velocity space

$$F=F_0+\mathbf{v}\cdot\mathbf{F}_1/v+\cdots. \tag{2}$$

Higher harmonics in this series can safely be neglected under a fairly wide range of conditions.[4] Within these limitations, the equations determining F_0 and $\mathbf{F}_1$ are

$$\frac{\partial F_0}{\partial t}+(\nu_x+\nu_i-q)F_0=-\frac{v}{3}\nabla_r\cdot\mathbf{F}_1+\frac{2}{3}\frac{e\mathbf{E}}{m}\cdot\frac{\mathbf{F}_1}{v}+\frac{1}{3}\frac{e\mathbf{E}}{m}\cdot\frac{\partial\mathbf{F}_1}{\partial v}+C_0, \tag{3}$$

and

$$\frac{\partial\mathbf{F}_1}{\partial t}+\nu_c\mathbf{F}_1=-v\nabla_rF_0+\frac{e\mathbf{E}}{m}\frac{\partial F_0}{\partial v}. \tag{4}$$

Here ν_c, ν_x, and ν_i are the collision frequencies of an electron for momentum transfer, excitation, and ionization, respectively. C_0 represents the effect of elastic collisions on F_0,[5] and the quantity qF_0 is the rate of appearance of new electrons at low energies as a result of excitation and ionization. As Rose and Brown[3] point out, these equations may be integrated over velocity space to yield formulas which determine the spatial variation of electron density. The result of performing this operation on Eq. (3) is

$$\frac{\partial N}{\partial t}-\langle\nu_i\rangle N=-\int_0^\infty\frac{4\pi v^3}{3}\nabla\cdot\mathbf{F}_1dv. \tag{5}$$

where N is the electron density and $\langle\nu_i\rangle N$, the total ionization rate, is defined in reference 3. The latter, however, is usually small and will be dropped from subsequent equations. Differentiating Eq. (5) gives

$$\frac{\partial^2N}{\partial t^2}=-\frac{4\pi}{3}\int_0^\infty v^3\nabla_r\cdot\frac{\partial\mathbf{F}_1}{\partial t}dv. \tag{6}$$

Substituting for $\partial\mathbf{F}_1/\partial t$ from Eq. (4), one obtains the result

$$\frac{\partial^2N}{\partial t^2}=\frac{4\pi}{3}\int_0^\infty v^3\nu_c\nabla_r\cdot\mathbf{F}_1dv+\frac{4\pi}{3}\int_0^\infty v^4\nabla_r^2F_0dv-\frac{4\pi}{3}\int_0^\infty v^3\nabla_r\cdot\left(\frac{e\mathbf{E}}{m}\frac{\partial F_0}{\partial v}\right)dv. \tag{7}$$

For hydrogen ν_c is approximately constant so

$$\frac{4\pi}{3}\int_0^\infty v^3\nu_c\nabla_r\cdot\mathbf{F}_1dv=\frac{4\pi\nu_c}{3}\int_0^\infty v^3\nabla_r\cdot\mathbf{F}_1dv=-\nu_c\partial N/\partial t. \tag{8}$$

Using this expression in Eq. (12) and integrating the last term by parts yields

$$\frac{\partial^2N}{\partial t^2}+\nu_c\frac{\partial N}{\partial t}=\int_0^\infty\frac{4\pi}{3}v^4\nabla_r^2F_0dv+\nabla_r\cdot\left(\frac{e\mathbf{E}N}{m}\right). \tag{9}$$

Finally, if one assumes that the average energy of the electrons is constant throughout the plasma the term $\int_0^\infty(4/3)\pi v^4\nabla_r^2F_0dv$ may be written in the form $\frac{1}{3}\langle v^2\rangle\nabla^2N$. This condition is substantially satisfied in the types of plasma of interest for the present work. Equation (9) then takes the form

$$\frac{\partial^2N}{\partial t^2}+\nu_c\frac{\partial N}{\partial t}=\frac{\langle v^2\rangle}{3}\nabla^2N+\nabla_r\cdot\left(\frac{e\mathbf{E}N}{m}\right). \tag{10}$$

This formula is the one which will be used to discuss the dynamics of electron motion in the plasma.[6]

III.

The exact solution of Eq. (10) is made exceedingly difficult by the fact that its last term is nonlinear. For this reason, a perturbation theory approach will be adopted and N and E will be expanded in the form

$$\begin{aligned}N&=N^{(0)}+N^{(1)}+\cdots,\\E&=E^{(0)}+E^{(1)}+\cdots.\end{aligned} \tag{11}$$

with the basic assumption that $N^{(1)}$ and $E^{(1)}$ are

[4] S. C. Brown and A. D. MacDonald, Phys. Rev. **76**, 1629 (1949).

[5] An explicit form for C_0 is given in the paper of Rose and Brown (reference 3).

[6] The author is grateful to D. J. Rose for suggesting the method used to derive Eq. (10). A formula similar to (10), but without the diffusion term, $\frac{1}{3}\langle v^2\rangle\nabla^2N$, may be derived in a very simple way by eliminating the current, $\mathbf{J}$, between Newton's law, $m(d\mathbf{v}/dt+\nu_c\mathbf{v})=(m/eN)(d\mathbf{J}/dt+\nu_c\mathbf{J})=e\mathbf{E}$, and the continuity equation, $\nabla\cdot\mathbf{J}+\partial(Ne)/\partial t=0$. It has been used to discuss the reflection of radio waves from meteor trails [T. R. Kaiser and R. L. Closs, Phil. Mag. **43**, 1 (1952); R. E. B. Makinson and D. M. Slade, Australian J. Phys. **7**, 268 (1954)]. For the types of plasmas investigated by Allis and Rose, however, the diffusion term is comparable in size to others appearing in the equation and cannot be neglected.

small. The functions $N^{(0)}$ and $E^{(0)}$ are the steady-state electron density and field in the plasma. They are known, for the case of the hydrogen discharge, through the work of Allis and Rose. $N^{(1)}$ and $E^{(1)}$, on the other hand, represent small amplitude fluctuations about the steady values caused by small external fields applied across the plates which confine the plasma. The first-order equation relating $N^{(1)}$ and $E^{(1)}$ may be obtained in a straightforward way from Eq. (10). For the particular geometry under consideration here the functions $N^{(0)}$, $E^{(0)}$, $N^{(1)}$, and $E^{(1)}$ depend only upon a single variable, x, which is the distance coordinate measured normal to the parallel plates. Because of this fact Eq. (10) reduces, for a harmonic applied field of frequency, ω, to the ordinary differential equation

$$(-\omega^2+i\omega v_c)N^{(1)}=\frac{\langle v^2\rangle}{3}\frac{d^2N^{(1)}}{dx^2}+\frac{d}{dx}\left[\frac{eE^{(1)}N^{(0)}}{m}+\frac{eE^{(0)}N^{(1)}}{m}\right]. \tag{12}$$

Since it has been assumed that the perturbing fields vary too rapidly to alter the positive ion distribution, the only fluctuation in charge density is that due to changes in electron density. Thus, $E^{(1)}$ and $N^{(1)}$ are also related through Poisson's equation,

$$dE^{(1)}/dx=-4\pi eN^{(1)}. \tag{13}$$

The boundary conditions which apply to this density fluctuation are the same as those which hold for $N^{(0)}$; namely, that $N^{(1)}$ should vanish on the walls of the tube containing the plasma.

Equation (12), as it stands, is an integro-differential equation for $N^{(1)}$ since $E^{(1)}$, which appears in the last term of the formula, is related to $N^{(1)}$ through Eq. (13). However, by integrating the whole equation (the integral being from x to d minus that from $-d$ to x) and making use of the boundary conditions on $N^{(0)}$ and $N^{(1)}$, one arrives at a simple differential equation for $E^{(1)}$;

$$\Omega^2[E^{(1)}-G]=\frac{\langle v^2\rangle}{3}\frac{d^2E^{(1)}}{dx^2}+\frac{eE^{(0)}}{m}\frac{dE^{(1)}}{dx}-\frac{4\pi e^2N^{(0)}E^{(1)}}{m}-\frac{\langle v^2\rangle}{6}\left[\frac{d^2E^{(1)}(d)}{dx^2}+\frac{d^2E^{(1)}(-d)}{dx^2}\right]. \tag{14}$$

Here Ω^2 is used as an abbreviation for the quantity $\omega^2-i\omega v_c$ and $G=4\pi\sigma$ is the field due to the surface charge, σ, on the plates that confine the discharge.

To solve Eq. (14), consider first the eigenvalue problem

$$\frac{\langle v^2\rangle}{3}\frac{d^2E_i}{dx^2}+\frac{eE^{(0)}}{m}\frac{dE_i}{dx}-\frac{4\pi e^2N^{(0)}}{m}E_i+\omega_i^2E_i=0,\qquad \frac{dE_i}{dx}(\pm d)=0. \tag{15}$$

The functions, E_i, that satisfy these equations form a complete, orthonormal set on the interval $(-d, d)$; consequently, the solution of Eq. (15) can be written as a sum of the form

$$E^{(1)}=\sum_j\beta_jE_j. \tag{16}$$

Letting

$$-\frac{\langle v^2\rangle}{6}\sum_i\beta_i[E_i''(d)+E_i''(-d)]=\sum_i\frac{\beta_j\omega_j^2}{2}[E_j(d)+E_j(-d)]=K, \tag{17}$$

one obtains

$$\sum_j(\omega_j^2-\Omega^2)\beta_jE_j=K-\Omega^2G. \tag{18}$$

From this equation, a formula for β_j may be obtained in the usual way by multiplying by WE_j (W is the weighting function for the eigenfunctions, E_j) and integrating from $x=-d$ to $x=+d$. The result is

$$\beta_j=\frac{\alpha_j}{(\omega_j^2-\Omega^2)}(K-\Omega^2G), \tag{19}$$

where

$$\alpha_j=\int_{-d}^{d}WE_jdx, \tag{20}$$

and

$$\sum_j\alpha_jE_j=1 \tag{21}$$

for $-d\leq x\leq d$. Combining these equations, one obtains finally the result

$$E^{(1)}=G\sum_i\left[\frac{\alpha_jE_j}{\omega_j^2-\Omega^2}\right]\bigg/\sum_i\left[\frac{\alpha_j[E_j(d)+E_j(-d)]}{2(\omega_j^2-\Omega^2)}\right]. \tag{22}$$

This formula represents the solution of the plasma oscillation problem for a harmonic exciting field; its implications will be discussed in connection with an example to be treated in the next section.

IV.

In the preceding section, a formal solution was obtained to the plasma oscillation problem for the particular geometry of a discharge confined between parallel plates. Unfortunately, however, the result [Eq. (22)] is rather complicated in nature and does not lend itself readily to physical intuition. The present section, therefore, will be devoted to a detailed examination of this solution for a particular discharge. The results of this investigation will, it is hoped, illustrate the nature of the formulas and give some feeling for the behavior of the sort of plasma being studied here.

The first step in the evaluation of Eq. (22) is the calculation of the eigenfunctions, E_j, from Eq. (15). For this purpose one must know $E^{(0)}$ and $N^{(0)}$ which are the space-charge field and electron density in the unperturbed plasma. In the example to be considered here, these quantities were taken from the work of

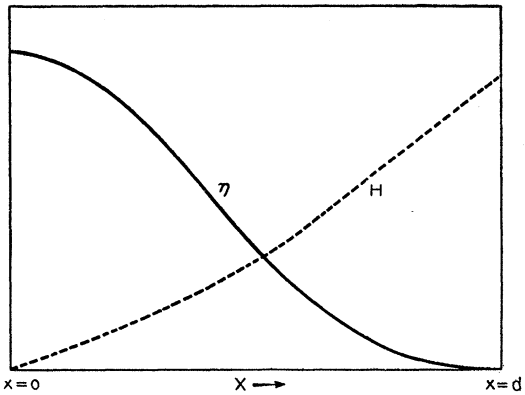

FIG. 1. Electron density (η) and field (H) as functions of position for the steady-state hydrogen plasma.

Allis and Rose. Curves of H and η (these are dimensionless variables, proportional to $E^{(0)}$ and $N^{(0)}$, defined by Allis and Rose) are illustrated in Fig. 1. For a tube with plate separation of 1 cm and $D_-/\mu_-=2$ volts, the value of η at $x=0$ corresponds to an electron density of $\sim 5\times 10^7$ cm^{-3}. Having chosen the various coefficients which appear in Eq. (15), one is then in a position to compute the E_j's. This was done with the aid of a differential analyzer. Since only even eigenfunctions (with respect to the transformation $x\rightarrow -x$) play a role in determining $E^{(1)}$, the differential equation (15) was integrated from $x=0$ to $x=d$ with the boundary conditions

$$\frac{dE_i}{dx}(0)=\frac{dE_i}{dx}(d)=0. \tag{23}$$

Figure 2 illustrates the first few eigenfunctions obtained in this way. The corresponding eigenvalues, ω_j^2, and expansion coefficients, α_j, are listed below, values of ω_j^2 being given in terms of the plasma frequency

$$\omega_p^2=4\pi e^2N^{(0)}(0)/m,$$

which is the natural unit of frequency appearing in the problem.

Eigenfunction	ω_j^2/ω_p^2	α_j
1	0.056	2.13
2	0.91	0.441
3	1.47	0.085
4	2.58	0.0066

As this table shows, the values of α_j fall off rapidly with increasing j—so rapidly, as a matter of fact, that only the first two of them are large enough to make an appreciable contribution to $E^{(1)}$. This situation, which arises from the oscillatory nature of the eigenfunctions for large j values, has here been shown to occur for a particular discharge. Other plasmas will, undoubtedly, behave somewhat differently but one would anticipate that for them, as for the present case, the expansion coefficients, α_j, would fall off with increasing j. These facts suggest that the field, $E^{(1)}$, can often be represented by the formula

$$E^{(1)}=G\left[\frac{\alpha_1E_1(x)}{\omega_1^2-\Omega^2}+\frac{\alpha_2E_2(x)}{\omega_2^2-\Omega^2}\right]\Big/\left[\frac{\alpha_1E_1(d)}{\omega_1^2-\Omega^2}+\frac{\alpha_2E_2(d)}{\omega_2^2-\Omega^2}\right]. \tag{24}$$

For frequencies low compared to ω_p (or ω_2) the terms in this formula involving E_2 are small and $E^{(1)}$ goes to the limit

$$E^{(1)}\simeq GE_1(x)/E_1(d). \tag{25}$$

Thus, as can be seen from the graph of E_1 in Fig. 2, the main effect of the plasma at these low frequencies is to screen out the external electric field and prevent it from penetrating into the center of the tube. Qualitatively speaking, one would expect the depth of penetration to be of the order of the Debye length which, in our example, is about $\frac{1}{5}$ the tube size if one uses for the electron density the value of $N^{(0)}$ at $x=0$. This figure is in approximate agreement with the value one would deduce from the shape of E_1.

In the high-frequency limit ($\omega\gg\omega_p$), the formula for $E^{(1)}$ takes the form

$$E^{(1)}\simeq\left(\frac{\alpha_1E_1(x)+\alpha_2E_2(x)}{\alpha_1E_1(d)+\alpha_2E_2(d)}\right), \tag{26}$$

which, because of Eq. (21), reduces to $E^{(1)}=G$. At high frequencies, therefore, an external field penetrates into the plasma without loss of intensity.

The intermediate region where $\omega\sim\omega_2$ is more complicated and will now be considered in detail. For this purpose it is useful to have a formula expressing the impedance of the plasma (with its associated capacity due to the parallel plates) in terms of the α_j's and E_j's. This quantity is also of particular interest from the experimental point of view, since it could well be measured for the sort of plasma under consideration here and thus afford an experimental check of the theory. The impedance of the plasma is given by the ratio

$$Z=V/j, \tag{27}$$

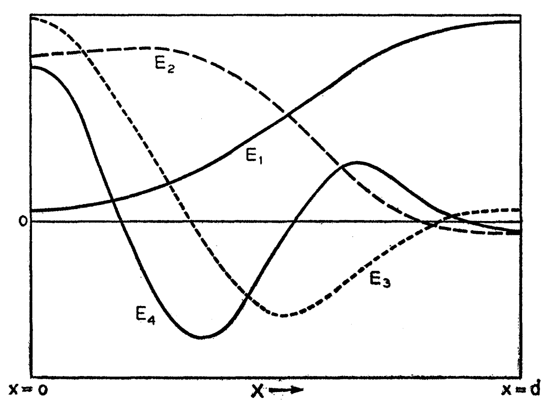

FIG. 2. Eigenfunctions of hydrogen plasma.

where V is the voltage across the plates and j the current in the external circuit. The current j is made up of two parts; a current due to motion of electrons in the gas, and a second part representing charge piling up on the surfaces of the plates. Thus, one may write

$$\begin{aligned} j &= j_{\text{plasma}}(d)+\dot{\sigma} \\ &= j_{\text{plasma}}(d)+(i\omega G/4\pi e). \end{aligned} \tag{28}$$

The quantity j_{plasma}, which is the current carried by electrons moving in the gas, may be evaluated by multiplying Eq. (4) by $4\pi v^3/3$ and integrating over velocity space. The result (for harmonic time dependence) is

$$\begin{aligned} (i\omega+\nu_c)\int_0^\infty \frac{4\pi v^3}{3}\mathbf{F}_1 dv &= -(i\omega+\nu_c)\frac{\mathbf{j}_{\text{plasma}}}{e} \\ &= -\frac{\langle v^2\rangle}{3}\nabla_r N-\frac{e\mathbf{E}N}{m}. \end{aligned} \tag{29}$$

The first-order terms from this equation are

$$\frac{j_{\text{plasma}}^{(1)}}{e}=\frac{-1}{4\pi e}\frac{1}{(i\omega+\nu_c)}\left(\frac{\langle v^2\rangle}{3}\frac{d^2E^{(1)}}{dx^2}+\frac{eE^{(0)}}{m}\frac{dE^{(1)}}{dx}-\frac{4\pi e^2N^{(0)}E^{(1)}}{m}\right). \tag{30}$$

Therefore, from Eq. (15),

$$\frac{j_{\text{plasma}}^{(1)}}{e}=\frac{-1}{4\pi e}\frac{1}{(i\omega+\nu_c)}\left(-\Omega^2(E^{(1)}-G)+\frac{\langle v^2\rangle}{3}\frac{d^2E^{(1)}(d)}{dx^2}\right). \tag{31}$$

At the wall,

$$\frac{j_{\text{plasma}}^{(1)}}{e}=\frac{-1}{4\pi e}\frac{1}{(i\omega+\nu_c)}\frac{\langle v^2\rangle}{3}\frac{d^2E^{(1)}(d)}{dx^2} \tag{32}$$

$$=\frac{1}{4\pi e}\frac{G}{(i\omega+\nu_c)}\left\{\Omega^2+\left[1\Big/\sum_i\left(\frac{\alpha_jE_j(d)}{\omega_j^2-\Omega^2}\right)\right]\right\}, \tag{33}$$

and the total current therefore is

$$j=\frac{1}{4\pi}\frac{G}{(i\omega+\nu_c)}\Big/\sum_i\frac{\alpha_jE_j(d)}{(\omega_j^2-\Omega^2)}. \tag{34}$$

The voltage, on the other hand, is given by

$$\begin{aligned} V &= 2\int_0^d E^{(1)}(x)dx \\ &= G\sum\left(\frac{\gamma_j}{\omega_j^2-\Omega^2}\right)\Big/\sum_i\left(\frac{\alpha_jE_j(d)}{\omega_j-\Omega^2}\right), \end{aligned} \tag{35}$$

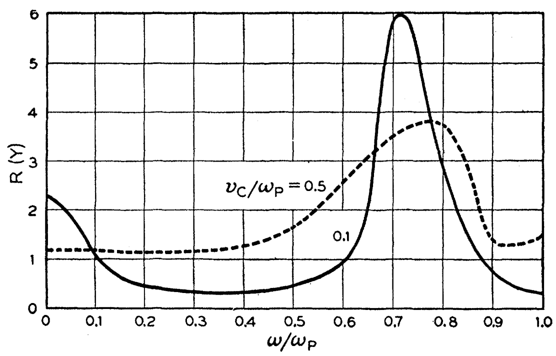

FIG. 3. Real part of admittance *vs* frequency.

where

$$\gamma_j=2\alpha_j\int_0^d E_j(x)dx. \tag{36}$$

The final result for the impedance per unit plate area is the rather simple relation

$$Z=\frac{V}{j}=4\pi(i\omega+\nu_c)\sum_i\left(\frac{\gamma_j}{\omega_j^2-\Omega^2}\right). \tag{37}$$

In the high-frequency limit, this formula approaches the value

$$Z_\infty=\frac{4\pi}{i\omega}\sum_i\gamma_j=\frac{8\pi d}{i\omega}, \tag{38}$$

which is the impedance, per unit area, of a parallel plate condenser with plate separation of $2d$. This, of course, is just the result one would expect from the earlier discussion of the behavior of the plasma for $\omega\gg\omega_p$.

For the particular example discussed earlier in this section, only the first two terms in the sum in Eq. (37) are of importance. The γ_j values upon which this conclusion is based are

$$\gamma_1=0.48,\quad \gamma_2=0.38,\quad \gamma_3=0.01.$$

The impedance of this plasma, as well as that of others whose γ values fall off sufficiently rapidly with increasing j, can therefore be expressed in the form

$$Z=4\pi(i\omega+\nu_c)\left(\frac{\gamma_1}{\omega_1^2-\Omega^2}+\frac{\gamma_2}{\omega_2^2-\Omega^2}\right). \tag{39}$$

Values of the real and complex parts of the admittance ($Y=1/Z$) calculated from this formula, using the γ_j and ω_j^2 values tabulated above, are plotted in Figs. 3 and 4 for two values of ν_c/ω_p. The most striking feature of these graphs is the resonance in the real part of Y (which measures power adsorption) at a frequency of about $0.7\omega_p$. This is an effect which should be measurable, and whose observation would provide strong confirmation of the theory developed here. The

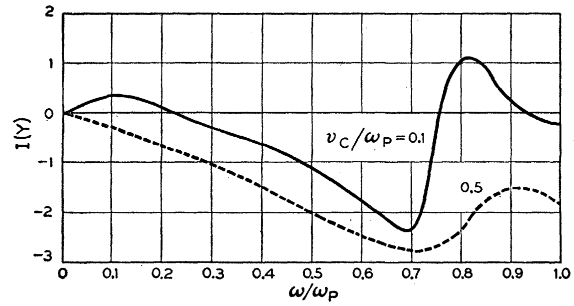

FIG. 4. Imaginary part of admittance *vs* frequency.

presence of this resonance can be fairly well understood from the physical point of view. In the present example (where γ_3, γ_4, etc., are negligible) the behavior of the plasma is characterized by the degree of excitation of the two modes, E_1 and E_2. In particular, for values of ω^2 between ω_1^2 and ω_2^2 the two modes are excited 180° out of phase with respect to one another. Under these circumstances, the voltage across the tube is low since the field E_1 is opposed to E_2; for the appropriate choice of ω^2 the potentials due to the two modes can be made to almost cancel. Thus there is a certain frequency at which, to achieve a given degree of excitation of the plasma, only a small voltage is required across the tube. At this point, the impedance of the discharge is low and the power adsorption goes through a maximum as shown in Figs. 3 and 4.

In addition to the total power absorption, whose variation with frequency is illustrated in Fig. 3, it is also of interest to investigate where in the plasma the absorption takes place. The power absorbed per unit volume is given by the formula

$$P(x)=\tfrac{1}{4}\,\mathrm{Re}[j^{(1)}(x)E^{(1)}(x)], \tag{40}$$

which may be evaluated by substituting for $j^{(1)}$ and $E^{(1)}$ the expressions given in Eqs. (22) and (30). In the low- and high-frequency limits, $P(x)$ assumes simple forms which are of particular interest. For $\omega\to 0$ the current density is constant [this statement, which is evident from the continuity equation, may also be verified by substituting Eq. (22) into (30)] and $P(x)$ is proportional to $E^{(1)}$, which equals E_1 at low frequencies. Thus, for small ω, power is absorbed near the walls of the tube in a layer of plasma whose thickness is about equal to the Debye length. For high frequencies ($\omega\gg\omega_p$), on the other hand, the field $E^{(1)}$ is constant throughout the tube and the current density, $j^{(1)}(x)$, as may be seen from Eq. (30), varies proportional to $N^{(0)}(x)$. In this case, therefore, the major part of the power absorption occurs near the center of the tube where the steady-state electron density has its largest value. These results are quite general in nature and should apply to a wide variety of plasmas. For intermediate frequencies, however, the power absorption distribution depends upon the details of the plasma density and could vary considerably from case to case. In the particular discharge under consideration here, $P(x)$ changes smoothly, as one increases the frequency, from a function peaked at the walls, through a distribution which is approximately constant across the tube to the high-frequency limit in which it is proportional to $N^{(0)}(x)$.

V.

The previous discussion of the properties of gas plasmas has laid emphasis on a particular example whose characteristics were investigated in detail. It is clear, however, that the sort of analysis presented there could be applied, almost without change, to other plasmas of that type. In particular, the notion that a confined discharge has a set of normal modes, such as the E_j's discussed in Secs. III and IV, seems a fairly general one. It might be expected, therefore, that the electrical behavior of a plasma could usually be represented by a formula of the general type of Eq. (37). Furthermore, whenever the excitation is caused by a constant electric field, as in the example discussed above, the γ_j values should fall off quite quickly with increasing j. The impedance would then be given by a formula like Eq. (39) and the frequency dependence of the admittance would be like that shown in Figs. 3 and 4. On the other hand, if excitation of the plasma were carried out in some other way (such, for example, as by electron bombardment), one might excite higher modes of the plasma and thus observe characteristic frequencies quite different from those that manifest themselves in the sort of experiments discussed above. The plasma problem for the case of particle bombardment is, however, much more difficult than for field excitation, and its solution will not be attempted here.

In conclusion, the author would like to express his thanks to D. J. Rose, who, by discussion and suggestion, has contributed greatly to the solution of the problems treated in this paper. Thanks are also due R. W. Hamming who assisted with the machine computations to determine the E_j's.