Californium-254 and Supernovae*

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It is suggested that the spontaneous fission of Cf^{254} with a half-life of 55 days is responsible for the form of the decay light-curves of supernovae of Type I which have an exponential form with a half-life of 55 nights. The way in which Cf^{254} may be synthesized in a supernova outburst, and reasons why the energy released by its decay may dominate all others are discussed. The presence of Tc in red giant stars and of Cf in Type I supernovae appears to be observational evidence that neutron capture processes on both a slow and a fast time-scale have been necessary to synthesize the heavy elements in their observed cosmic abundances.

OBSERVATIONAL DATA

A CHARACTERISTIC feature of supernovae of Type I is that after an initial period of 50–100 days the light curve develops an exponential tail corresponding to about 0.0137 magnitudes daily, or a half-life of 55 ± 1 days. Baade has analyzed the records of the supernovae *B* Cassiopeiae and *SN* Ophiuchi,[1] and has shown that their exponential decline is very closely similar to his own observations of the supernova in *IC* 4182, which was studied for about 600 days from maximum.[2] Although this study covered a period of more than ten times the half-life, no sensible deviations from a strictly exponential tail were found.

The total energy emitted in a supernova outburst is of the order of 10^{49}–10^{50} ergs, but the major part of this is emitted in the first few days, and by integrating under the exponential parts of the light curves we estimate that the total energy emitted in the decay curve is about 10^{47} ergs.

The most striking feature of the Type I supernova phenomena is the form of the decay curve. As Borst[3] originally pointed out, it is difficult to suggest any energy source other than a radioactive nucleus that could give this exponential decline, especially since the half-life involved is accurately the same for different supernovae. Be^7, Sr^{89}, and Cf^{254} all have half-lives near or equal to 55 days, and it is necessary to decide which of these is the most probable source of the energy.

DIFFICULTIES IN THE PRODUCTION OF Be^7 AND Sr^{89}

Be^7 has a half-life of 53–54 days,[4] and Borst[3] originally suggested that its decay by *K* capture would provide sufficient energy to explain the curve. He suggested that it was built by the endothermic reaction $He^4(\alpha,n)Be^7$, occurring at high temperatures. However, recent work[5–7] suggests that this is most unlikely, since He^4 would be destroyed by the exothermic Salpeter reaction in which C^{12} is produced. An alternative method of production is through spallation reactions of protons (with $E_p\geqslant 100$ Mev) on C, N, and O, which are known to give large yields of Li, Be, and B.

The mean energy made available in the decay of Be^7 is about 57 kev, so that to provide 10^{47} ergs, the total mass of Be^7 would be about 1.3×10^{31} grams. Since the incidence of Type I supernova outbursts is about 1 in 500 years, this suggests that in the life-history of our galaxy ($\sim5\times10^9$ years) the total production of Li^7 would have to amount to about 6.5×10^4 solar masses. However, this is about 100 times the observed abundance of this element in the cosmic abundance curve, and this argument alone probably rules out Be^7 as the energy source.

Sr^{89} has a half-life against β decay of 55 days and the mean electron energy released is 600 kev.[4] The most probable mechanism by which it can be built is through successive neutron captures on the abundant intermediate elements, or the Fe group. However there are two strong objections to using this as an energy source. In the first place the resultant production of Y^{89} in the Galaxy would amount to 7.5×10^4 solar masses or about 100 times the observed relative abundance of this element. Secondly, if a large flux of neutrons, sufficient to build Sr^{89}, was available there is no reason at all why the buildup should stop at Sr^{89}. It is clear that the buildup would take place over a wide range of radioactive heavy elements so that many other decays would release energy, thus destroying any agreement with the characteristic 55-day half-life curve.

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[1] W. Baade, Astrophys. J. **102**, 309 (1945).

[2] Baade, Burbidge, Hoyle, Burbidge, Christy, and Fowler, Publ. Astron. Soc. Pacific (to be published).

[3] L. B. Borst, Phys. Rev. **78**, 807 (1950).

[4] Hollander, Perlman, and Seaborg, Revs. Modern Phys. **25**, 469 (1953).

[5] E. E. Salpeter, Astrophys. J. **115**, 326 (1952).

[6] F. Hoyle, Astrophys. J. Suppl. **1**, 121 (1954).

[7] Fowler, Burbidge, and Burbidge, Astrophys. J. **122**, 271 (1955).

PRODUCTION AND SPONTANEOUS FISSION OF Cf^{254}

Cf^{254} has been produced in the November, 1952 thermonuclear test by the instantaneous irradiation of U by an intense neutron flux, and it has been found to decay by spontaneous fission with a half-life of 55 days.[8] About 200 Mev is released in each such decay, and this energy is carried mainly in the kinetic energy of the fragments. To produce 10^{47} ergs, about 1.2×10^{29} g of Cf^{254} must be produced in each supernova outburst. This suggests that a total mass of 600 solar masses of Cf^{254} has been built in the time scale of the Galaxy. This mass distributed among its fission products will give relative abundances of these elements not in disagreement with the observed abundances.

A further argument is suggested by the present status of stellar element synthesis theories. To build the intermediate elements between Na^{23} and Ti^{50} both a source of neutrons and charged particle reactions are required, while to build the elements heavier than those in the Fe peak a further supply of neutrons is demanded. It has been suggested that the reactions $Ne^{21}(\alpha,n)Mg^{24}$ and/or $C^{13}(\alpha,n)O^{16}$ may provide such a source.[7,9,10] These sources producing neutrons over time-scales of 10^5–10^7 years in red giant stars at temperatures of about 10^8 degrees cannot build certain isotopes of some elements, nor can they surmount the barrier of the short-lived alpha emitters to build the uranium isotopes. In order to do this a neutron capture chain taking only a few seconds is required so that nuclei with large neutron excesses can be built without appreciable β decay taking place. In a stellar synthesis theory, the obvious place in which suitable conditions may prevail is in a star just prior to or coincident with the early stages of a supernova outburst. If this viewpoint is correct, we might expect in supernova outbursts evidence for the presence of extremely neutron-rich nuclei such as Cf^{254}.

Three questions immediately suggest themselves:

(a) Why should the energy release in the fission of Cf^{254} dominate over all other processes?

(b) Wherein lies the source of the neutrons?

(c) How does the degradation of the energy and its release as light take place in order that the exponential form of the light curve is maintained as the supernova shell expands?

(a) Possible reasons for the dominance of Cf^{254} in the supernova outburst may lie in the systematics of spontaneous fission half-lives. The recent analysis of Swiatecki[11] based on experimental evidence such as that reviewed by Glass, Thompson, and Seaborg[12] is valuable in this respect. From Fig. 1 of Swiatecki's paper it appears that Cf^{252} and particularly Cf^{254} have anomalously short half-lives for spontaneous fission as compared with the other isotopes of Cf and other elements of similar atomic weight. This is reasonably to be associated with the closing of the shell of 152 neutrons at Cf^{250}. In the case of Cf^{252} the lifetime against spontaneous fission is 66 years, a relatively low value, but not as low as its alpha-decay lifetime of 2.1 years. However, for Cf^{254} the lifetime for fission is only 55 days, and no alpha or beta decays have been observed.[8] Moreover for Fm^{256} the fission lifetime is very short being $\sim$3 hours.[11] It is interesting to note the extrapolation of the data on the rate of decay of the fission products of the uranium isotopes suggests that the energy release in the beta- and gamma-ray activities of the Cf^{254} fission products do not become comparable with the initial fission energy release until a time of approximately 600 days has elapsed.[13] It would seem that Cf^{254} is thus unique[14] in decaying in 55 days with the release of some 200 Mev of fission energy as compared with the 5–10 Mev release in the first alpha decay of other isotopes. Shorter-period activities are of no importance as far as the exponential portion of the observed supernova light-curve is concerned, since the curve becomes exponential only after about 50 days. However, they may contribute appreciable energy in the early stages. On the basis of the arguments given above, longer-period activities do not seem to have a comparable energy release to that of Cf^{254} in the first 600 days.

In the fission decay of the californium fraction from thermonuclear debris,[8] in which the 2.1-year fission activity of Cf^{252} dominates after about 200 days, a simple calculation shows that the initial fission activity of Cf^{252} corresponded to a production of about 50 times as much Cf^{252} as Cf^{254}. This is entirely reasonable in a limited process of isotope buildup based on uranium with insufficient neutrons to carry the atomic weight from 252 to 254 and beyond. However, in supernova explosions we need only argue that Cf^{254} is produced in roughly equal amounts to its neighboring isotopes and elements and that sufficient neutrons are very rapidly supplied per Fe^{56} nucleus. The first neutrons are captured by these nuclei, and since the capture processes take place faster than the beta decay, the Fe builds up to an atomic weight at which it becomes neutron-unstable; i.e., neutrons are no longer bound and cannot

[8] Fields, Studier, Diamond, Mech, Inghram, Pyle, Stevens, Fried, Manning, Ghiorso, Thompson, Higgins, and Seaborg, Phys. Rev. **102**, 180 (1956). From the decay curve given in this reference we estimate an error in the half-life of ±5 days.

[9] J. L. Greenstein in *Modern Physics for the Engineer*, edited by L. N. Ridenour (McGraw-Hill Book Company, Inc., New York, 1954), p. 267.

[10] A. G. W. Cameron, Phys. Rev. **93**, 932 (1954); Astrophys. J. **121**, 144 (1955).

[11] W. J. Swiatecki, Phys. Rev. **100**, 937 (1955).

[12] Glass, Thompson, and Seaborg, J. Inorg. Nuclear Chem. **1**, 3 (1955).

[13] K. Way and E. G. Wigner, Phys. Rev. **73**, 1318 (1948).

[14] We have made an extensive but fruitless search for this unique property among isotopes of elements called by any other name than californium.

be captured. According to Smart,[15] this occurs for $N \simeq 0.7A$, $Z \simeq 0.3A$ and is roughly independent of A. At this point the process stops until a beta decay occurs, and for such neutron-rich isotopes Smart estimates that the lifetime for beta decay is about 0.1 second and is independent of A. Thus the build-up takes place along the $N = 0.7A$ line of isotopes and will be terminated only when beta-decay times becomes long, or some other process such as alpha decay becomes important. The time demanded for the beta decays to take place in building from Fe to Cf is then about 7 seconds.

The second alternative termination by alpha decay mentioned just above, does not seem very likely for such neutron-rich isotopes. However, Smart's calculations show a trend towards decreasing beta-decay energies, and thus to increasing lifetimes as A increases. At the closing of the $Z = 82$ proton shell there will be a sudden drop by about 1 Mev in the beta-decay energy, perhaps resulting in a lifetime of 10–100 seconds, and this might put an effective stop to the build-up process. Coulomb barrier effects will prevent the (p,n) process for Z as large as 82. For $Z \simeq 0.3A = 82$, $A \simeq 270$. This shows that the buildup may not go far beyond the progenitor of Cf^{254}. Furthermore, since the neutron-poor nuclei have the largest neutron-capture cross sections, we should expect a bunching of the products near the limiting point of the process. Thus in supernovae it is reasonable that Cf^{254} should constitute about 10% of the products of the synthesis based on Fe^{56}. If this is the case, Cf^{254} will dominate in the production of energy after the initial explosion of the star.

(b) It is our view that the building of Cf^{254} probably takes place in the regions just outside the core of the star. We have been able to arrive at a plausible mechanism for neutron production provided that the composition of the material is taken as follows: protons, alpha particles, light nuclei C^{12}, O^{16}, and Ne^{20} in approximately equal numbers, the Fe abundance being less than the abundance of the light nuclei by about 10^3.

This composition is not unreasonable from an astrophysical point of view, since we are probably dealing with stars in an advanced evolutionary stage in which there is a serious hydrogen deficiency as compared with the normal stellar material. In accordance with our ideas on stellar evolution, we regard the alpha particles and light nuclei as having been synthesized in the interior of the star itself, becoming mixed by some circulation process into the envelope. In contrast with this we assume that there has been no enrichment of the Fe content of the envelope, where the Fe abundance present at the time of formation of the star is still unchanged.

The process leading to the buildup of Cf^{254} can tentatively be separated into three parts:

(i) An initial raising of the temperature in the envelope to about 10^8 degrees, the energy being derived from the gravitational field of the star. The release of gravitational energy is taken to arise from the implosion of the central regions of the star, this being our view of the cause of the supernova itself. Such an implosion would lead to the material of the envelope also falling inwards with the consequent release of gravitational energy.

(ii) The onset of reactions of the type $C^{12}(p,\gamma)N^{13}$, $O^{16}(p,\gamma)F^{17}$, $Ne^{20}(p,\gamma)Na^{21}$. The mean energy production from these reactions may be taken as approximately 2 Mev per proton. With the composition as given above, the number of free protons per gram of material is of the order of 3×10^{22}, so that the energy production from the (p,γ) reactions is about 10^{17} ergs per gram. This energy is available for heating the material and thereby causing it to burst outward in explosion. The energy so released will also escape from the star partly in the form of radiation. A reasonable estimate might be that more than 10^{16} ergs per gram may be expected to be preserved in the ultimate outward velocity of expansion of the supernova, thereby suggesting a speed upward of 1000 km/sec. This is of the order of the velocity of expansion observed in the Crab nebula which is thought to be the debris from a supernova of Type I. It thus appears that the energy released by (p,γ) reactions is capable of explaining the outburst of the supernova and gives the correct order of magnitude for the velocity of the outburst.

(iii) The dimension of the envelope before explosion is probably smaller than the radius of the sun, perhaps of the order of 10^9–10^{10} cm. The time-scale of the explosion is accordingly in the range 10–100 seconds. This means that an appreciable fraction of the Na^{21} produced from Ne^{20} by the (p,γ) reaction is able to undergo beta decay (half-life $\simeq 23$ seconds) before the explosion gets far underway, i.e., before the temperature of the material declines appreciably. For an energy production of 10^{17} ergs per gram the temperature attained by the material is about 10^9 degrees, and at this temperature the exoergic reaction $Ne^{21}(\alpha,n)Mg^{24}$ takes place in a time of 1 second. We therefore expect a neutron production of the same order as the number of neon nuclei present in the material. With the Fe content given above, this would lead to a supply of several hundred neutrons per Fe nucleus which is of the order required to explain the buildup of Cf^{254}.

Two further remarks should be made. A temperature of 10^9 degrees resulting from an energy production of 10^{17} ergs per gram is obtained assuming that the energy content of the resulting radiation field is less than the energy resident in the matter. Neutron addition to Fe depends on the (n,γ) reaction for Fe possessing a cross section at least a thousand times greater than those of the (n,γ) reactions on the light nuclei. This is the case for all light nuclei except N^{14}. Neutrons will be lost in the reaction $N^{14}(n,p)C^{14}$, but provided that the

[15] J. S. Smart, Phys. Rev. **75**, 1379 (1949). The operation of the (γ,n) process will decrease N and increase Z slightly.

abundance of N^{14} is appreciably less than that of Ne^{21} there can be no serious interference with the buildup of Cf^{254}, since the decay period of C^{14} is very long.

(c) The problem of the physical conditions in the expanding supernova envelope is extremely complex, particularly as it seems clear that the visible radiation is not thermal in origin. This is shown by the color-index curve of the supernova in *IC* 4182, which from 30 to 80 days after maximum became bluer and not redder as would be expected for the thermal emission of an expanding envelope of declining luminosity.

The spectra of two Type I supernovae, covering periods of about a year, have been published by Minkowski.[16] They consist of a number of very broad emission features, some of which showed the same general appearance during most of this time (apart from a steady progression in wavelength), while others showed considerable changes. No satisfactory identification of any of these features has yet been made, with the exception of narrow emission lines of [*OI*] that appeared about 200 days after maximum.

Three possible sets of physical conditions in the envelope may be briefly mentioned. The material into which the fission fragments are ejected may be very hot. Forbidden atomic transitions between low-excitation states of highly ionized atoms might then provide energy in the visible spectrum region, by analogy with the solar corona. But in this case the large reservoir of energy contained in high ionization potentials would exceed the energy emitted in the visible light and would be so great as to outweigh the energy received from the decay of Cf^{254}. The 55-day half-life could then scarcely be preserved in the light curve.

Secondly, the material might be cool enough for the atoms to be overwhelmingly neutral, but not cool enough for molecules to form in an appreciable degree. Slowing of the fission fragments of Cf^{254} would then produce ionization, and recombination spectra would be emitted. This raises the serious difficulty that such spectra should have readily identifiable features. Also the optical depth in permitted transitions would be large and once again it is doubtful whether the rate of decay of the Cf^{254} would be able to control the rate of emission from the envelope.

Thirdly, the material might be very cool, with the atoms of C, N, and O mainly in molecular form. Fission fragments could then cause dissociation or ionization of the molecules and the spectra would consist of overlapping bands. The heat content of the material might well remain small and the problem of the envelope could be less serious.

Finally, we might mention that there is some evidence suggesting that a flux of fairly high-energy particles may arise in the first few days of a Type I supernova outburst. It is known that electromagnetic activity on a very large scale is currently taking place in the Crab nebula. This is manifested in the form of synchrotron emission which demands both a flux of very high-energy particles and a magnetic field. It has been estimated[17] that the total energy currently contained in the electrons some 900 years after the outburst is about 10^{47} ergs.[18] To produce a component of the visible radiation in the first few days of a supernova outburst by the synchrotron mechanism would demand very high-energy particles and strong magnetic fields (electron energies near 500 Mev in magnetic fields of 150 gauss might be plausible).

[16] R. Minkowski, Astrophys. J. **89**, 143 (1939).

SUPERNOVAE OF TYPE II

It is of interest to refer back to the process of neutron production discussed above. An essential feature of the process was that the number of protons, alpha particles, and light nuclei must be comparable with one another, a requirement that demands a large hydrogen deficiency compared with normal stellar material. The question that now arises is: What happens if there is no such large deficiency of hydrogen? What happens if the number of protons appreciably exceeds the number of light nuclei?

We have not been able to find a process of neutron production in this case. The (p,γ) reactions occur on a large scale. Thus in addition to the reaction $Ne^{20}(p,\gamma)Na^{21}$, we have the reactions $Na^{21}(p,\gamma)Mg^{22}(\beta^+,\nu)Na^{22}(p,\gamma)Mg^{23}(p,\gamma)Al^{24}(\beta^+,\nu)\cdots$ etc., where the beta processes require a time of about a second. The repeated (p,γ) reactions prevent Na^{21} being formed in large concentrations, and hence the reactions $Na^{21}(\beta^+,\nu)Ne^{21}(\alpha,n)Mg^{24}$ do not occur to an appreciable degree. We have not been able to find any other (α,n) or (p,n) reactions which are important. It would seem to follow that when hydrogen is present in excess concentration: (i) there is no neutron production and hence no buildup of Cf^{254}; (ii) on account of the profusion of (p,γ) reactions the energy production is not limited to the 10^{17} ergs per gram calculated above.

The results which can be deduced from these conclusions may be compared with the observed properties of supernovae of Type II.[19] These are considered to be members of stellar population I, and Type I supernovae are members of population II. Type II supernovae show evidence of high explosive speeds of about 5000 km/sec which are significantly greater than the probable explosion speeds of Type I supernovae, suggesting a greater energy production. This is also suggested by the very high surface temperatures, the emission very likely being thermal. The $H\alpha$ line has been tentatively identified in the spectra, and this would be expected if hydrogen were present in appreciable concentration. Most significantly, the light curves

[17] J. H. Oort and T. Walraven, Bull. Astron. Soc. Neth. **12**, 285 (1956).
[18] G. R. Burbidge (to be published).
[19] R. Minkowski, Publ. Astron. Soc. Pacific **53**, 224 (1941).

of supernovae of Type II do not show the 55-day half-life; the curves fall more steeply, suggesting that radioactive nuclei with decay periods longer than a few days are absent. There is considerable variety in the light curves of different Type II supernovae.

We therefore identify two cases, one in which the hydrogen concentration is deficient and the other in which it is present in excess concentration. These we tentatively associate with supernovae of types I and II, respectively.

CONCLUSION

In conclusion we wish to emphasize that the production of Cf^{254} in the November, 1952 thermonuclear test stands as clear evidence for the terrestrial production on a fast time-scale of heavy elements by neutron-capture processes. Our argument in this paper would indicate that this process is occurring on a large scale and has contributed to the synthesis of the heavy elements. In a similar manner, the existence of Tc in certain stars demonstrates that neutron-capture processes on a slow time-scale are occurring in stars. It is our point of view that neutron-capture processes on both a fast and a slow time-scale have been necessary to synthesize the heavy nuclei in their observed abundances.

We should like to express our thanks to Dr. W. Baade for many stimulating discussions, and for providing us with very valuable unpublished data on supernovae.

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Modification of the Brillouin-Wigner Perturbation Method

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By using the method of Goldhammer and Feenberg, a simple, generally valid prescription for improving the Brillouin-Wigner perturbation procedure is derived.

GOLDHAMMER and Feenberg have recently proposed an interesting modification of the Brillouin-Wigner perturbation method.[1] They illustrate their method with several examples, in each of which their refinement produces a correction factor of the same simple form. Its repeated appearance suggests that this simple correction factor may be rather generally applicable, and in this note we wish to show that this is indeed the case.

Assuming a familiarity with the material and notation of reference 1, we recall that the wave function has the form

$$\psi^{(n)}=\psi_0+G_1\sum{}'\psi_a\frac{V_{a0}}{E-E_a}+G_2\sum{}'\psi_b\frac{V_{ba}V_{a0}}{(E-E_b)(E-E_a)}+\cdots+G_n\sum{}'\frac{\psi_l V_{lk}\cdots V_{a0}}{(E-E_l)(E-E_k)\cdots(E-E_a)}. \quad (1)$$

The associated expression for the energy is

$$E=E_0+V_{00}+2\sum_{i=1}^{n}G_i\epsilon_{i+1}+\sum_{i,j=1}^{n}G_iG_j(\epsilon_{i+j+1}-\epsilon_{i+j}), \quad (2)$$

where

$$\epsilon_2=\sum_a{}'\frac{V_{0a}V_{a0}}{E-E_a},$$

$$\epsilon_3=\sum_{a,b}{}'\frac{V_{0a}V_{ab}V_{b0}}{(E-E_a)(E-E_b)},$$

and so on. In the Brillouin-Wigner method, all G_i's equal 1, whereas, according to Goldhammer and Feenberg, the G_i's in (2) are varied to make E a minimum. A formal discussion of the general case, for arbitrary n, is given in their paper.

It is instructive, however, to consider two special cases. In the first, we put all G_i's equal to 1 except G_n. The best choice of G_n is then given by

$$G_n=(1-\epsilon_{2n+1}/\epsilon_{2n})^{-1}, \quad (3)$$

while the energy becomes[2]

$$E=E_0+V_{00}+\sum_{i=2}^{2n}\epsilon_i+\frac{\epsilon_{2n+1}}{(1-\epsilon_{2n+1}/\epsilon_{2n})}. \quad (4)$$

Our second case is slightly more general: we put all G_i's equal to 1 except G_{n-1} and G_n. We then find, for the best choice of G_{n-1} and G_n:

$$G_{n-1}=\frac{\epsilon_{2n-2}(\epsilon_{2n}-\epsilon_{2n+1})-\epsilon_{2n-1}(\epsilon_{2n-1}-\epsilon_{2n})}{(\epsilon_{2n-2}-\epsilon_{2n-1})(\epsilon_{2n}-\epsilon_{2n+1})-(\epsilon_{2n-1}-\epsilon_{2n})^2},$$
$$G_n=\frac{\epsilon_{2n-2}\epsilon_{2n}-\epsilon_{2n-1}^2}{(\epsilon_{2n-2}-\epsilon_{2n-1})(\epsilon_{2n}-\epsilon_{2n+1})-(\epsilon_{2n-1}-\epsilon_{2n})^2}. \quad (5)$$

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[1] P. Goldhammer and E. Feenberg, Phys. Rev. **101**, 1233 (1956).

[2] More generally, if all G_i's equal 1 except G_k, optimizing G_k leads to

$$G_k=1-\epsilon_{k+n+1}/(\epsilon_{2k+1}-\epsilon_{2k}),$$
$$E=E_0+V_{00}+\sum_{i=2}^{2n+1}\epsilon_i+\epsilon_{k+n+1}^2/(\epsilon_{2k}-\epsilon_{2k+1}).$$

Thus, regardless of k, the correction to the wave function is of order $n+1$ and the correction to the energy of order $2n+2$, as one would expect. Also, both corrections approach zero as n increases, again regardless of k.