

Effect of an Electric Field on Positronium Formation in Gases: Theoretical*†

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A theory is developed to explain the influence of an electric field on positronium formation in gases. The Boltzmann equation is used to describe the behavior of the positrons under the combined influence of the electric field and molecular collisions. A diffusion equation in velocity space is deduced wherein the annihilation of free positrons and the disappearance of positrons by positronium formation are absorption terms. From this equation the functional dependence of the increase in positronium formation on the experimental parameters is calculated. The result describes correctly the experimental observations and yields information on the pertinent atomic processes. In particular, the cross sections for the elastic scattering of positrons by helium, neon, and argon are found to be $0.023\pi a_0^2$, $0.12\pi a_0^2$, and $1.5\pi a_0^2$, respectively, to about $\pm 25\%$. These values are considerably smaller than the corresponding quantities for electrons, and are in agreement with simple theoretical estimates.

I. INTRODUCTION

POSITRONS traversing matter will eventually annihilate with the electrons they encounter, the energy of each annihilating positron-electron pair being given off as gamma radiation. Many of the interesting things which have been learned about positrons by studying various aspects of the annihilation radiation under different experimental conditions are included in the recent review articles of Deutsch¹ and DeBenedetti and Corben,² which give historical background and experimental and theoretical details.

The probability for annihilation is sufficiently small, so that in many ways a positron can take part in ordinary atomic processes, unaffected by the fact that it will ultimately annihilate. For positrons in a gas, which is the situation of interest here, these processes include ionization and excitation of the gas molecules, elastic collisions, and, in particular, collisions in which a positron captures an electron from a gas molecule to form positronium, which is a bound hydrogen-like system consisting of a positron and an electron held together by their Coulomb interaction. The possible existence of such a system was suggested as early as 1934, but experimental evidence for positronium was not obtained until much later.³

The amount of positronium formation in various gases has been studied both experimentally^{1,4,5} and

theoretically.⁶⁻⁹ Deutsch and Brown¹⁰ observed that the yield of positronium is increased in a strong radio-frequency electric field, and subsequently an increased yield was also observed in a static electric field.¹ The effect of a static electric field has been studied in detail by Marder, Hughes, Wu, and Bennett.¹¹ It is the intent of this paper to provide a theoretical interpretation of the observed influence of an electric field on positronium formation.

It became apparent quite soon that the theoretical considerations required were in many aspects analogous to those encountered in the classic problem of the energy distributions of electrons in a gas subjected to an electric field, and thus that the cross section for elastic scattering of a positron by a gas molecule plays a dominant role. In effect, the experiments measure this cross section, which in the cases considered is much smaller than the corresponding quantity for electrons. This apparently surprising result is due primarily to the difference in sign of the interaction potential. As will be seen, the dependence of the increase in positronium formation in a particular gas on the experimental variables (applied electric field and gas density) can be described quantitatively with only the elastic scattering cross section and the lifetime for the annihilation of free positrons as parameters.

II. GENERAL DISCUSSION

A. Positronium Formation in Gases

A qualitative description of positronium formation has been given by Ore.⁶ Four regions of *positron* energy

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¹ M. Deutsch, *Progr. Nuclear Phys.* **3**, 131 (1953).

² S. DeBenedetti and H. C. Corben, *Annual Review of Nuclear Science* (Annual Reviews, Inc. Stanford, 1954), Vol. 4, p. 191.

³ M. Deutsch, *Phys. Rev.* **82**, 455 (1951).

⁴ T. A. Pond, *Phys. Rev.* **85**, 489 (1952).

⁵ S. DeBenedetti and R. Siegel, *Phys. Rev.* **94**, 955 (1954).

⁶ A. Ore, Universitetet i Bergen Arbok 1949, Naturvitenskapelig rekke No. 9.

⁷ A. Ore, Universitetet i Bergen Arbok 1949, Naturvitenskapelig rekke No. 12.

⁸ H. S. W. Massey and C. B. O. Mohr, *Proc. Phys. Soc. (London)* **A67**, 695 (1954).

⁹ C. B. O. Mohr, *Proc. Phys. Soc. (London)* **A68**, 342 (1955).

¹⁰ M. Deutsch and S. C. Brown, *Phys. Rev.* **85**, 1047 (1952).

¹¹ Marder, Hughes, Wu, and Bennett, *Phys. Rev.* **103**, 1258 (1956), preceding paper.

are distinguished (Fig. 1). These are determined by the minimum energy required to ionize a gas molecule (E_{ion}), the minimum energy required to excite a gas molecule (E_{exc}), and the minimum energy required to form positronium (E_{thr}). The threshold for positronium formation is smaller than E_{ion} by exactly the binding energy of positronium. For the rare gas atoms it turns out that $E_{\text{thr}} < E_{\text{exc}}$, but in the case of more complicated molecules it is possible and in fact likely that $E_{\text{thr}} > E_{\text{exc}}$, so that region III does not exist.

An energetic positron from a radioactive source makes inelastic collisions in the gas and rapidly loses most of its energy. The important events that can occur in collisions of a positron with a gas molecule depend on the energy of the positron. In region I only energy loss by ionization and excitation are significant. Positronium can be formed, but only with sufficient kinetic energy so that it can break up by impact ionization. In region II positronium formation competes with energy loss by excitation, while in region III only positronium formation is important. Elastic collisions can occur at all energies, but are only significant in region IV where no inelastic processes compete. In an elastic collision a positron loses on the average $2m/M$ of its energy, where m is the positron mass and M is the mass of the gas molecule; hence a positron must make many collisions before an appreciable change of its energy takes place. Further, a positron can annihilate with an electron of a gas molecule without forming positronium. Such annihilation of free positrons can also occur at all energies, but its likelihood is sufficiently small so that most of the positrons which do not form positronium will reach region IV and will thermalize or almost thermalize before annihilating.

From the above elementary considerations it is possible to estimate that

$$\frac{E_{\text{exc}} - E_{\text{thr}}}{E_{\text{exc}}} < \left[\frac{\text{fraction of positrons}}{\text{forming positronium}} \right] < \frac{E_{\text{ion}} - E_{\text{thr}}}{E_{\text{ion}}}.$$

The underestimate is obtained by neglecting positronium formation in region II, while the overestimate is obtained by neglecting inelastic collisions in this region. In both cases it is assumed that the last inelastic collision leaves the positrons uniformly distributed in energy. Experimental observations^{1,4,5} indicate that the fraction of positrons which form positronium falls within the estimated limits for the gases studied.

Mohr⁹ has considered the hypothetical case of positrons in a gas of atomic hydrogen. Using a Monte-Carlo method, he finds that at high pressures a fraction 0.35 of the positrons form positronium. This result falls within the limits of $\frac{1}{3}$ to $\frac{1}{2}$ obtained for atomic hydrogen from the previous elementary considerations. Mohr predicts that as the pressure is decreased below about one atmosphere an increasing amount of parapositronium should be observed. This occurs because the

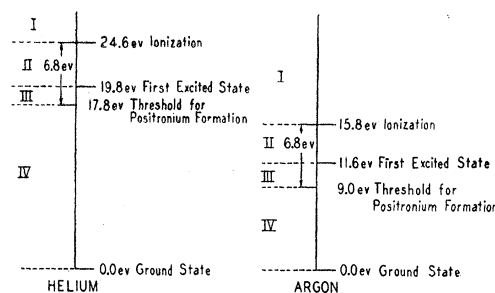


FIG 1. Energy regions (shown for positrons in He and A).

annihilation of parapositronium atoms formed from positrons in region I can compete successfully with impact ionization of these atoms. The observed amount of orthopositronium should be similarly affected at pressures about a thousand times smaller. This annihilation of positronium formed from positrons in region I represents a departure from the assumptions of the previous elementary considerations.

B. Effect of Electric Field

The qualitative considerations of the previous section can be extended to include the effect of electric fields. Accelerated by such fields, positrons in region IV can gain sufficient energy to reach region III, where they have the possibility of forming positronium.¹⁰ The alternate modes of decay—(1) annihilation of free positrons and (2) formation of positronium followed by two quantum or three quantum annihilation—compete, the second process being favored by high fields. The functional dependence of the increase in positronium formation on the applied electric field can be discussed qualitatively and treated quantitatively by utilizing certain results obtained for the analogous problem of electrons in a gas subjected to an electric field.

Under the influence of an energy field \mathcal{E} , a positron can gain or lose energy between collisions of the order of $e\mathcal{E}l$, where e is the charge of the positron and l is the mean free path of the positron in the gas. For the fields in question, $e\mathcal{E}l$ is small (~ 0.1 eV), so that only events in region IV need be reconsidered. Instead of tending to thermal equilibrium, a positron will now tend towards an equilibrium distribution¹² for which the probability of having a velocity between v and $v+dv$ is proportional to

$$v^2 dv \exp \left[- \int \frac{mvdv}{kT + \frac{1}{3}M(e\mathcal{E}l/mv)^2} \right],$$

or the probability for having an energy between E

¹² S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1952), second edition, Chap. 18.

($=\frac{1}{2}mv^2$) and $E+dE$ is proportional to

$$E^{\frac{1}{2}}dE \exp\left[-\frac{dE}{kT + \frac{2}{3}(M/m)(e\mathcal{E}l)^2/E}\right].$$

For $\mathcal{E}=0$, this just reduces to the Maxwell-Boltzmann distribution. For values of \mathcal{E} appropriate to the experiment, kT can be neglected. In the special case l proportional to v (mean free time $\tau=l/v$ independent of v), this is still a Maxwell-Boltzmann distribution, but with an "effective temperature" determined by the electric field. For the low-energy positrons under consideration l is practically independent of velocity so that the distribution becomes a Druyvesteyn distribution,

$$E^{\frac{1}{2}}dE \exp[-(E/E^*)^2]$$

with $E^* = (\frac{1}{3}M/m)^{\frac{1}{2}}e\mathcal{E}l$.

The experimentally observed influence of an electric field on positronium formation can be understood in terms of the equilibrium distribution to which the positrons thermalize. Any particular positron will continually change its energy under the combined influence of the electric field and of elastic collisions. In this way it may be able to reach region III and form positronium, instead of remaining in region IV where it would eventually annihilate with an electron of a gas molecule. For \mathcal{E} small, positronium formation by this process does not compete successfully with annihilation in region IV. Increasing \mathcal{E} shifts the mean energy of the distribution closer to E_{thr} . Because of the increased likelihood of a positron being above the threshold for positronium formation, more of the positrons form positronium. For \mathcal{E} sufficiently large, all the positrons from region IV form positronium and further increase of the electric field has no effect.

The fraction of the positrons which reach region IV under the influence of the electric field and then form positronium depends on the relative size of the probability per unit time for forming positronium and the probability per unit time for the annihilation of free positrons. In a preliminary report of this work,¹³ the probability per unit time for forming positronium was computed from the appropriate average of the formation cross section over the equilibrium distribution. This average is an integral over the "tail" of the equilibrium distribution above E_{thr} and is essentially proportional to the size of the tail and hence a very sensitive function of \mathcal{E} . These calculations based on the static equilibrium distribution yield a correct description of the basic features of the experimental results, but do not give detailed agreement.

In the following sections, the random "motion" in energy of a positron is treated dynamically by a diffusion equation. Such a description can be expected

in general for a process involving many random steps.^{14,15} In this formulation, annihilation of free positrons and positronium formation are considered as absorption processes. It is to be noted that in the latter case the pertinent event to be counted as the disappearance of a positron is the *formation* of positronium, and not the ultimate *annihilation* of the positronium atom, which serves only to identify the event. Since the lifetime for annihilation of free positrons is much longer than the relaxation time for reaching equilibrium, the main effect taken into account by the diffusion equation is the deviation of the quasi-equilibrium distribution from a Druyvesteyn (equilibrium) distribution, which occurs because of positronium formation in region III.

Two cases are considered. If the cross section for positronium formation is taken to be small compared to the cross section for elastic collisions, then the results of the previous calculations¹³ are reproduced. If, more realistically, the cross section for positronium formation (in region III) is taken to be of the order of or larger than the cross section for elastic collisions, then region III can be treated as completely absorbing, so that the threshold for positronium formation is considered as an absorbing barrier. Hence the calculations involve only the cross section for elastic collisions and the cross section for annihilation of free positrons. Very good agreement with experiment is obtained for reasonable values of these cross sections.

III. DIFFUSION EQUATION

A. Derivation

The problem is formulated by assuming an initial distribution (at $t=0$) of positrons in region IV and asking for the fraction of those positrons which form positronium. This fraction I depends on the electric field and is proportional to the experimentally observed increase in positronium formation.

Let $f'(\mathbf{v}, t)$ be the function which describes the distribution of positrons in velocity space at any time t , i.e., the integral over space of the complete distribution function. Normalization will be introduced explicitly where needed but for the sake of discussion it can be assumed that $f'(\mathbf{v}, t)d_3v$ gives the fraction of positrons which, at the time t , are in the volume element (in velocity space) d_3v about \mathbf{v} . This quantity can then be interpreted alternately as the probability that a particular positron is in the indicated volume at the time t . The function f' satisfies the Boltzmann equation.

$$\left. \frac{\partial f'}{\partial t} \right]_{\text{drift}} = \left. \frac{\partial f'}{\partial t} \right]_{\text{collision}}$$

The drift term is the substantial derivative in velocity

¹³ W. B. Teutsch and V. W. Hughes, Phys. Rev. **98**, 1174(A) (1955).

¹⁴ S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

¹⁵ M. C. Wang and G. E. Uhlenbeck, Revs. Modern Phys. **17** 233 (1945).

space,

$$\frac{\partial f'}{\partial t} \Big|_{\text{drift}} = \frac{\partial f'}{\partial t} + \mathbf{a} \cdot \nabla_v f'.$$

The uniform acceleration \mathbf{a} is just that due to the electric field, $\mathbf{a} = e\mathbf{E}/m$, and ∇_v is the gradient with respect to the velocity coordinates. The collision term consists of three parts,

$$\frac{\partial f'}{\partial t} \Big|_{\text{collision}} = \frac{\partial f'}{\partial t} \Big|_e + \frac{\partial f'}{\partial t} \Big|_a + \frac{\partial f'}{\partial t} \Big|_f,$$

which take into account the scattering of positrons from one region of velocity space to another by elastic collisions, the removal of positrons from the distribution by annihilation, and the removal of positrons from the distribution by positronium formation, respectively.

If there are N gas molecules per unit volume and the cross section for annihilation per molecule is σ_a , the annihilation term is

$$\frac{\partial f'}{\partial t} \Big|_a = -N\sigma_a v f' = -\lambda_a f'.$$

If σ_a is taken inversely proportional to v , $\lambda_a (=N\sigma_a v)$ is independent of velocity, and the effect of annihilation of free positrons can be completely taken into account by the transformation

$$f'(\mathbf{v}, t) = e^{-\lambda_a t} f(\mathbf{v}, t).$$

Similarly, the formation term is

$$\frac{\partial f'}{\partial t} \Big|_f = -N\sigma_f v f',$$

or

$$\frac{\partial f}{\partial t} \Big|_f = -N\sigma_f v f,$$

where σ_f is the formation cross section. In region II, σ_f vanishes and there remains only

$$\frac{\partial f}{\partial t} \Big|_{\text{drift}} = \frac{\partial f}{\partial t} \Big|_e,$$

i.e., the transformed function obeys a Boltzmann equation involving only elastic collisions. For electrons in a gas in the presence of an electric field, this problem has been treated by Morse, Allis, and Lamar¹⁶ and reviewed by Chapman and Cowling.¹²

Their procedure can be applied equally well to positrons, and leads, with the recognition of the near isotropy of f and the smallness of m/M , from the Boltzmann equation to a diffusion equation for $f^{(0)}$, the isotropic part of f . Only the energy distribution will be

of interest, so $f^{(0)}$ provides all the desired information, since the probability of a positron having a velocity between v and $v+dv$ is

$$\int_{\Omega} d\Omega v f'(\mathbf{v}, t) = 4\pi v^2 dv e^{-\lambda_a t} f^{(0)}(v, t).$$

With the inclusion of the term describing the "absorption" of positrons through positronium formation, the desired diffusion equation becomes

$$\frac{\partial f^{(0)}}{\partial t} = -\frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{1}{3} \frac{\partial f^{(0)}}{\partial v} + \frac{m}{M} \frac{v^4}{l} f^{(0)} \right] - \frac{\sigma_f v}{\sigma_e l} f^{(0)},$$

where $l = 1/N\sigma_e$.

The cross section σ_e which appears here is loosely termed the cross section for elastic collisions in this paper, but is more rigorously the cross section for momentum transfer. For a differential scattering cross section $\sigma(\theta)$, the total elastic cross section is $2\pi \int_0^\pi \sigma(\theta) \sin\theta d\theta$, while the momentum-transfer cross section is $2\pi \int_0^\pi (1 - \cos\theta) \sigma(\theta) \sin\theta d\theta$, which is the quantity which actually appears in the above equation. For low-energy positrons, S -wave scattering predominates, and thus $\sigma(\theta)$ is practically independent of θ , particularly for scattering from the spherically symmetric rare gas atoms, and the two expressions do not differ appreciably.

The above diffusion equation without the absorption term should be compared to the radial part of the usual diffusion equation with drift:

$$\partial W / \partial t = \nabla \cdot (D \nabla W + \mathfrak{g} W),$$

where D is the diffusion constant and \mathfrak{g} is minus the drift velocity. The comparison is even more striking if the equation is rewritten

$$\frac{\partial f^{(0)}}{\partial t} = -\frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{1}{2} \frac{v}{l} \frac{\partial^2 f^{(0)}}{\partial v^2} + \frac{v}{l} \frac{m}{M} v f^{(0)} \right],$$

and it is noted that random walk theory indicates that a simple random walk^{14,15} of many steps can be described in terms of a diffusion equation, with a diffusion constant equal to [one-half the rate at which the steps are being taken] times [the average value of the square of the step size], and with a drift velocity equal to [the rate at which the steps are being taken] times [the average value of the step size]. In analogy to the usual diffusion current,

$$i(v, t) = -4\pi v^2 \left(\frac{1}{3} \frac{a^2 l}{v} \frac{\partial f^{(0)}}{\partial v} + \frac{m}{M} \frac{v^2}{l} f^{(0)} \right)$$

represents the "radial" flow of particles. It can also be noted that the equilibrium distribution mentioned in

¹⁶ Morse, Allis, and Lamar, Phys. Rev. 48, 412 (1935). See also W. P. Allis and H. W. Allen, Phys. Rev. 52, 703 (1937).

the introduction,

$$f_0 = \exp \left[- \int \frac{mvdv}{\frac{1}{3}M(e\mathcal{E}l/mv)^2} \right]$$

satisfies the equation

$$df^{(0)}/dt = 0.$$

B. Formal Solution

For a given initial distribution $f_i(v) = f^{(0)}(v, 0)$, the function $f^{(0)}(v, t)$ is completely determined by the diffusion equation with the obvious boundary conditions $i(0, t) = 0$ and $i(\infty, t) = 0$. It should be remembered that the function $f^{(0)}(v, t)$ does not represent the complete solution since it does not contain the effect of the annihilation of free positrons. However, this can be taken into account by reintroducing the factor $e^{-\lambda_a t}$ where pertinent. Besides the variables and universal constants, the diffusion equation contains \mathcal{E} and N , which are the experimental variables, and σ_e and σ_f , which can be calculated theoretically, at least in principle. These two cross sections as well as σ_a will be treated as parameters, using where necessary the functional dependence of these quantities on v discussed in Sec. IV.

With the substitution

$$f^{(0)} = g^{(0)} f_0 = g^{(0)} \exp \left[- \int \frac{mvdv}{\frac{1}{3}M(al/v)^2} \right],$$

the diffusion equation becomes

$$\frac{\partial g^{(0)}}{\partial t} f_0 = \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{1}{3} a^2 v l f_0 \frac{\partial g^{(0)}}{\partial v} \right] - \frac{\sigma_f v}{\sigma_e l} f_0 g^{(0)}.$$

The initial condition is

$$g^{(0)}(v, 0) = g_i(v) = f_i(v)/f_0(v),$$

and the boundary conditions are

$$i(v, t) = - (4/3) \pi v a^2 l f_0 \partial g^{(0)} / \partial v = 0 \quad \text{at } v = 0, \infty.$$

The variables can be separated by the substitution

$$g^{(0)}(v, t) = g(v) e^{-\lambda t},$$

which yields the Sturm-Liouville equation¹⁷

$$\frac{d}{dv} \left[p(v) \frac{dg}{dv} \right] + [q(v) + \lambda r(v)] g = 0$$

with

$$p(v) = (4/3) \pi a^2 v l f_0, \quad q(v) = -4 \pi v^2 (\sigma_f / \sigma_e) (v/l) f_0,$$

and

$$r(v) = 4 \pi v^2 f_0.$$

In this form, the boundary conditions are

$$p(v) dg/dv = 0 \quad \text{at } v = 0, \infty.$$

In terms of the eigenfunctions g_n of this equation and the corresponding eigenvalues λ_n , the formal solution can be written down:

$$g^{(0)}(v, t) = \sum_n c_n g_n(v) e^{-\lambda_n t}$$

where

$$c_n = \int_0^\infty r(v) g_i(v) g_n(v) dv / \int_0^\infty r(v) [g_n(v)]^2 dv.$$

In the expression for the coefficient c_n , the normalization of the g_n is explicitly exhibited, but the normalization of g_i ,

$$\int_0^\infty g_i(v) r(v) dv = 4 \pi \int_0^\infty f_i(v) v^2 dv = 1,$$

is tacitly assumed.

The total probability that a positron will form positronium is the desired quantity I , and is obtained by integrating the product of [the probability that a positron has a velocity between v and $v+dv$] and [the probability that a positron with a velocity v will form positronium in the time interval between t and $t+dt$] over all velocities and times, i.e.,

$$\begin{aligned} I &= \int_0^\infty dt \int_0^\infty dv q(v) g^{(0)}(v, t) e^{-\lambda_a t} \\ &= \sum_n \frac{c_n}{\lambda_n + \lambda_a} \int_0^\infty q(v) g_n(v) dv = \sum_n C_n \frac{\lambda_n}{\lambda_n + \lambda_a}. \end{aligned}$$

The C_n can be expressed as

$$C_n = c_n \int_0^\infty r(v) g_n(v) dv,$$

by making use of the differential equation and boundary conditions which the g_n obey. From the definition of c_n and the normalization of g_i , it follows that

$$\sum_n C_n = 1.$$

The same result can be obtained by observing that in the absence of annihilation of free positrons all of the positrons will eventually form positronium, i.e., $\lambda_a = 0$ implies $I = \sum_n C_n = 1$.

If the lifetime for the annihilation of free positrons is much longer than the relaxation time for reaching equilibrium, so that $\lambda_a \ll \lambda_1$, then

$$I = C_0 \frac{\lambda_0}{\lambda_0 + \lambda_a} + \sum_{n \neq 0} C_n = C_0 \frac{\lambda_0}{\lambda_0 + \lambda_a} + 1 - C_0.$$

This expression involves only the initial distribution and the quasi-equilibrium distribution.

¹⁷ P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. 6. This text has been used extensively throughout this paper and its notation has been followed wherever convenient.

Case of $\sigma_f=0$

It is useful to consider the limiting case of no absorption by positronium formation described by $\sigma_f=0$. The solution of the diffusion equation in this case agrees with that obtained in the limit $\mathcal{E} \rightarrow 0$ for the general case, independent of the magnitude of σ_f .

In particular, if the mean free path l is proportional to the velocity (mean free time $\tau=l/v$ independent of velocity), then the eigenvalue equation can be solved in terms of Hermite polynomials of odd order (H_{2n+1}), yielding

$$g_n = v^{-1} H_{2n+1} \left[\frac{mv}{e\mathcal{E}\tau} \left(\frac{3m}{2M} \right)^{\frac{1}{2}} \right]$$

and

$$\lambda_n = 2n(m/M)\tau^{-1}.$$

More generally, however, it can be observed that the equilibrium situation is described by $g_0 = \text{const}$ and $\lambda_0=0$, independent of the dependence of l on v . The next higher eigenvalue λ_1 determines the relaxation time for reaching the equilibrium distribution for an arbitrary initial distribution. For the particular situation of interest, *viz.* mean free path independent of velocity, the value of λ_1 is obtained in Appendix I. The results for the two particular situations considered suggest that the relaxation time for reaching equilibrium, *i.e.*, $1/\lambda_1$, is of the order of M/m times an appropriate average of τ over the equilibrium distribution.

Case of " σ_f small"

Secondly, consider the diffusion equation for $\sigma_f \ll \sigma_e$. In this case g_0 may be approximated by the corresponding function for the case of $\sigma_f=0$,

$$g_0 \approx \text{const.}$$

This approximation yields

$$\begin{aligned} C_n &= 1, & n=0 \\ &= 0, & n \neq 0 \end{aligned}$$

and there results simply

$$I = \lambda_0 / (\lambda_0 + \lambda_a).$$

If in addition the same approximation is used to evaluate λ_0 ,

$$\begin{aligned} \lambda_0 &= - \frac{\int_0^\infty g_0(v) \left[\frac{d}{dv} \left(p(v) \frac{dg_0}{dv} \right) + q(v) g_0(v) \right] dv}{\int_0^\infty r(v) [g_0(v)]^2 dv} \\ &\approx - \frac{\int_0^\infty q(v) dv}{\int_0^\infty r(v) dv} \\ &= 4\pi \int_0^\infty v^2 dv N \sigma_f v f_0 / 4\pi \int_0^\infty v^2 dv f_0, \end{aligned}$$

then the expression obtained previously¹³ is reproduced (see Appendix II).

Case of " σ_f large"

Finally, consider the solution of the diffusion equation for σ_f large, *i.e.*, $\sigma_f \gtrsim \sigma_e$ for energies above E_{thr} . Approximately then, this region can be taken to be completely absorbing, which is taken into account by replacing the boundary condition at $v = \infty$, *viz.* $i(\infty, t) = 0$, by one at $v = v_{\text{thr}}$ appropriate to an absorbing barrier, *viz.* $f^{(0)}(v_{\text{thr}}, t) = 0$. The other boundary condition is the same as in the absence of absorption, as is the diffusion equation, but the domain of the variable v is now restricted to region IV, *i.e.* $0 \leq v \leq v_{\text{thr}}$.

Strictly speaking, σ_f is zero at $v = v_{\text{thr}}$ and must rise smoothly, so that the desired inequality is not satisfied just above the threshold. Nevertheless, it is assumed that σ_f increases rapidly enough, so that the preceding argument need not be altered, though some deviation from this assumption could be taken into account by choosing the placement of the absorbing barrier an appropriate "distance" above the threshold.

The total probability that a positron forms positronium is obtained by integrating over time the probability that a positron is absorbed on the barrier in the time interval between t and $t+dt$, and is

$$I = \int_0^\infty dt i(v_{\text{thr}}, t) e^{-\lambda_a t} = \sum C_n \frac{\lambda_n}{\lambda_n + \lambda_a}.$$

By making use of the differential equation and the boundary conditions applicable to this case, it can be seen that the C_n which appear here are of exactly the same form as in the general case, except that the upper limit of the pertinent integrals is v_{thr} instead of ∞ .

The differential equation applicable to the case of σ_f large is of a form which is readily converted into an integral equation. Two integrations and use of the boundary conditions yields

$$g(x) = \lambda \int_x^{x_{\text{thr}}} [p(x')]^{-1} \int_0^{x'} r(x'') g(x'') dx'' dx'.$$

This form of the integral equation was found useful for numerical work and is equivalent to the more usual form

$$g(x) = \lambda \int_0^{x_{\text{thr}}} G(x, x') r(x') g(x') dx',$$

as an integration by parts will show.

C. Numerical Solution

In Sec. III B it was indicated that the increase in positronium formation could be described primarily in terms of the lowest eigenvalue of the appropriate differential equation. This eigenvalue represents the

rate at which positrons from the quasi-equilibrium distribution form positronium. In the case of σ_f small, the problem of finding this lowest eigenvalue has been reduced to the evaluation of integrals over the equilibrium distribution which is carried out in Appendix II. The corresponding problem for the case of σ_f large is treated in this section.

If the mean free path l is taken to be independent of v , it becomes convenient to use E^* as the unit of energy and to introduce the corresponding unit of velocity v^* , defined by

$$\frac{1}{2}m(v^*)^2 = E^* = (\frac{1}{3}M/m)^{1/2}e\mathcal{E}l.$$

The differential equation can then be written in terms of the dimensionless velocity variable $u = v/v^*$ as

$$\frac{d}{du} \left[u \exp(-u^4) \frac{dg}{du} \right] + 16\Lambda u^2 \exp(-u^4)g = 0,$$

where $\Lambda = \frac{1}{4}(l/v^*)(M/m)\lambda$ is a dimensionless eigenvalue. Alternate forms of the equation can be obtained by using the dimensionless energy variable $\eta = E/E^* = u^2$ or the variable $x = (E/E^*)^2 = \eta^2 = u^4$, yielding

$$\frac{d}{d\eta} \left(\eta \exp(-\eta^2) \frac{dg}{d\eta} \right) + 4\Lambda \eta^{3/2} \exp(-\eta^2)g = 0$$

or

$$\frac{d}{dx} \left(x e^{-x} \frac{dg}{dx} \right) + \Lambda x^{-1/2} e^{-x} g = 0.$$

In terms of these dimensionless variables, the Druyvesteyn distribution is

$$f_0 = \exp(-u^4) = \exp(-\eta^2) = e^{-x}.$$

For the purposes of this section, the form involving x will be the most convenient. The boundary conditions applicable to the case of σ_f large are

$$[x e^{-x} dg/dx]_{x=0} = 0 \quad \text{and} \quad g(x_{\text{thr}}) = 0,$$

where $x_{\text{thr}} = (E_{\text{thr}}/E^*)^2 = \eta_{\text{thr}}^2 = u_{\text{thr}}^4$. It is to be noted that while E_{thr} is fixed for any particular gas, the magnitude of x_{thr} varies in an inverse manner with the

applied electric field. Thus the problem of finding λ_0 as a function of \mathcal{E} becomes in the dimensionless form the problem of finding the lowest eigenvalue Λ_0 as a function of x_{thr} .

Approximate evaluation of Λ_0 can be attained by employing certain results of variational methods.¹⁸ Specifically, Λ_0 was approximated by

$$\underline{\Lambda} \leq \Lambda_0 \leq [\Lambda],$$

where

$$\underline{\Lambda} = \min \frac{-\frac{d}{dx} \left(x e^{-x} \frac{dg}{dx} \right)}{x^{-1/2} e^{-x} g},$$

$$[\Lambda] = \int_0^{x_{\text{thr}}} g \frac{d}{dx} \left(x e^{-x} \frac{dg}{dx} \right) dx / \int_0^{x_{\text{thr}}} x^{-1/2} e^{-x} g^2 dx,$$

and suitable trial functions to approximate the exact lowest eigenfunction g_0 were used.

The trial functions considered in this connection were of four types:

$$g_A(k) = x_{\text{thr}}^k - x^k,$$

$$g_B(k) = x_{\text{thr}}^k e^{x_{\text{thr}}} - x^k e^x,$$

$$g_C(k) = x_{\text{thr}}^k e^{\frac{1}{2}x_{\text{thr}}} - x^k e^{\frac{1}{2}x},$$

$$g_{D1} = \int_x^{x_{\text{thr}}} (x')^{-1} e^{x'} \int_0^{x'} (x'')^{-1/2} e^{-x''} dx'' dx'$$

$$= h(x_{\text{thr}}) - h(x),$$

$$g_{D2} = \int_x^{x_{\text{thr}}} (x')^{-1} e^{x'} \int_0^{x'} (x'')^{-1/2} e^{-x''} (1 - x''/x_{\text{thr}}) dx'' dx'$$

$$= (1 - 3/4x_{\text{thr}})[h(x_{\text{thr}}) - h(x)] + 4(x_{\text{thr}}^{3/2} - x^{3/2})/3x_{\text{thr}}.$$

The function $h(x)$ is defined in terms of an integral

$$h(x) = \int_0^x (x')^{-1} e^{x'} \int_0^{x'} (x'')^{-1/2} e^{-x''} dx'' dx'.$$

Various overestimates and underestimates were obtained,¹⁹ and intermediate values were selected as approximations to Λ_0 (see Table I in Sec. IV A). Some of these overestimates and underestimates divided by the approximate value of Λ_0 are plotted in Fig. 2 vs a dimensionless parameter ϵ :

$$\epsilon = x_{\text{thr}}^{-1/2} = E^*/E_{\text{thr}},$$

which is proportional to \mathcal{E} . It can be seen that the approximate value of Λ_0 must agree with the true value of Λ_0 within at least 5% except perhaps in the neighbor-

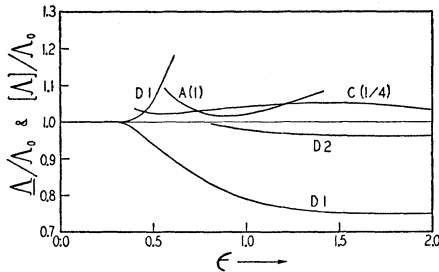


FIG. 2. Overestimates and underestimates for the eigenvalue Λ_0 divided by the approximate value of Λ_0 versus ϵ , a dimensionless parameter proportional to the applied electric field \mathcal{E} . The label on each curve corresponds to the subscript of the trial function which was used to obtain it.

¹⁸ Reference 17, Chap. 9.

¹⁹ Various mathematical tables were used, especially K. Pearson, *Tables of the Incomplete Γ -Function* (H. M. Stationery Office, London, 1922), and *Tables of Sine, Cosine, and Exponential Integrals* (National Bureau of Standards, Washington, D. C., 1940), Vol. 2.

hood of $\epsilon=0.7$ ($x_{thr}\sim 2$). In view of the rapid variation of Λ_0 with ϵ , this is quite adequate (see Fig. 3). For x_{thr} larger than about 7, $\Lambda_0=1/h(x_{thr})$ holds to better than one percent. Hence the behavior of Λ_0 for large x_{thr} (small \mathcal{E}) can be readily inferred from the asymptotic behavior of $h(x)$,

$$h(x)\sim\Gamma(\frac{3}{4})x^{-1}e^{-x}.$$

The increase in positronium formation as calculated for the case of σ_f large involves besides the eigenvalue λ_0 only the quantity C_0 , provided $\lambda_1\gg\lambda_a$. The equation

$$I=C_0\frac{\lambda_0}{\lambda_0+\lambda_a}+1-C_0$$

and the rapid variation of λ_0 with \mathcal{E} imply that C_0 plays a pertinent role only as long as $\lambda_0/(\lambda_0+\lambda_a)$ is small. This occurs only for small \mathcal{E} , specifically for ϵ less than about 0.5 for the gases considered. If the initial distribution and the function g_0 were known exactly, C_0 could, of course, be determined exactly. However, in view of the approximate constancy of g_0 except near the threshold which is suggested by the good trial functions, the expression

$$C_0=\frac{\int_0^{v_{thr}} r(v)g_i(v)g_0(v)dv \times \int_0^{v_{thr}} r(v)g_0(v)dv}{\int_0^{v_{thr}} r(v)g_i(v)dv \times \int_0^{v_{thr}} r(v)[g_0(v)]^2dv}$$

indicates that C_0 does not depend on the initial distribution or the electric field in a sensitive way and does not differ appreciably from unity for any reasonable initial distribution and for the electric fields in question.

An estimate of C_0 , using an initial distribution uniform in energy and the function g_{D1} as an approximation to g_0 , does indicate that C_0 varies only slowly with \mathcal{E} [compared to the marked variation of $\lambda_0/(\lambda_0+\lambda_a)$] and is very nearly one for pertinent values of \mathcal{E} . Further detailed considerations of C_0 are not warranted at present in view of the limits of accuracy of the available experimental data.

IV. CROSS SECTIONS

Consideration of the effect of an electric field on positronium formation involves the cross sections for the three pertinent atomic processes which can occur in a collision of a positron with a gas atom and which are elastic scattering, annihilation, and positronium formation. The assumptions which have been made concerning these cross sections follow from simple and rather general considerations, as will be discussed in this section. It is to be remembered that only the functional dependence of these quantities on v has been assumed and that their magnitudes have been left as parameters. The calculation of these cross sections from

fundamental principles forms the basis of another investigation.²⁰ Some preliminary results of that investigation are included in this section.

A. Elastic Scattering

The scattering of a positron from an atom is a many-body problem, involving the positron coordinates and all the electron coordinates even if the nucleus is assumed to remain fixed. However, as a first approximation, it is possible to regard the atom as a center of force unperturbed by the colliding positron. In this approximation, there is obtained the one-body problem of a positron scattered by a fixed central potential which represents the average potential of the nucleus and all the electrons. This potential is repulsive at all distances (for positrons).

It follows on rather general grounds²¹ that for such a repulsive potential, strong variations of the scattering cross section with energy cannot occur; moreover, at low energies the *S*-wave scattering always predominates. This situation is in distinction to the one for attractive potentials, where resonances and the Ramsauer effect can occur. These effects are closely associated with the existence of bound states, which are of course totally absent for a repulsive potential. Calculations for a square potential, which can be carried out exactly, illustrate these observations very clearly, and indicate also that the scattering cross section for a repulsive potential is in general smaller than that for the corresponding attractive one. If the effective radius of a repulsive scattering potential is less than the wavelength of the scattered particle, *S*-wave scattering predominates, and the cross section for scattering is practically independent of energy. Positrons with energies below the threshold for positronium formation satisfy this criterion, so that to a good approximation the scattering of positrons in this energy range by gas atoms is isotropic and energy-independent.

To obtain an estimate of the cross section for elastic scattering of positrons by atoms and especially to study the relative size of this cross section for various atoms, it is convenient to employ the simple model used by Allis and Morse²² for the corresponding problem for electrons. This model has a nucleus with a charge $+Ze_{eff}$ surrounded by Z_{eff} electrons on a spherical shell of radius r_a . The electrostatic potential of such a configuration is

$$V(r)=\begin{cases} Z_{eff}(r^{-1}-r_a^{-1}), & r\leq r_a, \\ 0, & r\geq r_a. \end{cases}$$

Calculation of the cross section for elastic scattering of positrons, using the Schwinger variational principle as

²⁰ W. B. Teutsch and V. W. Hughes (to be published).

²¹ N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, Chap. II.

²² W. P. Allis and P. M. Morse, *Z. Physik* **70**, 567 (1931).

TABLE I. Cross sections for elastic scattering of positrons.

Element	r_a/a_0	β	σ_e (units of πa_0^2)
He	0.55	0.80	0.07
Ne	0.75	1.71	0.4
A	1.4	2.7	2
He	hydrogenic potential		0.5

given by Morse and Feshbach²³ with the simple trial function $R(r)=r$, yields

$$\sigma_e = \pi r_a^2 \left[\frac{20}{18 + 15/\beta^2} \right]^2$$

in the limit of low positron energy. The quantity β is defined by

$$\beta = \frac{1}{2} Z_{\text{eff}} r_a / a_0,$$

where a_0 is the first Bohr orbit and r_a/a_0 corresponds to the r_0 of Allis and Morse. Table I lists the results so obtained with the values of β and r_a/a_0 which Allis and Morse used to fit the scattering data for electrons. That these results should be considered as order of magnitude calculations is emphasized by the different value of the cross section which is obtained for He using a potential arising from an electron distribution given in terms of a hydrogenic wave function.

The representation of an atom by a fixed central potential is, of course, only an approximation, since the presence of the positron distorts the atom, increasing the electron density in the vicinity of the positron. The net effect of this polarization is to make the potential seen by the positron less repulsive or possibly even attractive. A small amount of polarization makes the actual cross section somewhat smaller than the value which is calculated using the static potential of

the unperturbed atom. A large amount of polarization can actually make the potential attractive on the average and thus lead to completely different values for the cross section. For example, Ore⁶ uses a potential corresponding to the interaction of a helium atom with a (stationary) unit positive charge, thus overestimating the effect of polarization, and obtains roughly $\sigma_e = 4\pi a_0^2$.

B. Annihilation

The cross section for the annihilation of a positron with an electron has been calculated by Dirac²⁴ neglecting their electrostatic interaction. In the nonrelativistic limit, his result becomes

$$\sigma_a^{(D)} = \pi r_0^2 c / v,$$

where v is the relative velocity of the positron and electron, c is the velocity of light, and r_0 is the classical electron radius,

$$r_0 = e^2 / mc^2.$$

The cross section for annihilation per molecule can be written

$$\sigma_a = \mathfrak{z} \sigma_a^{(D)} = \mathfrak{z} \pi r_0^2 c / v,$$

where \mathfrak{z} is some effective number of electrons per molecule. The parameter \mathfrak{z} would be just the number of electrons Z per molecule if electrostatic interactions were absent. The repulsion by the nucleus tends to keep the positron away from the region of high electron density and thus tends to make \mathfrak{z} less than Z , whereas polarization tends to increase the electron density in the vicinity of the positron and so has the opposite effect. In general, \mathfrak{z} would not be expected to be independent of positron energy, but in the absence of any specific theoretical or experimental evidence, it is simplest to take \mathfrak{z} to be constant. It seems reasonable to make this approximation for the low-energy positrons in question.

Calculation of \mathfrak{z} with the simple function $R(r)=r$ leads to the result $\mathfrak{z}=Z$. A better estimate for He is obtained using the iterate of this trial function and the hydrogenic wave function for the electrons, and yields $\mathfrak{z} \approx 1$. The treatment of Ore yields $\mathfrak{z}=2.7$.

The only published experimental data on the lifetime of free positrons in gases seems to be that of Deutsch²⁵ on oxygen. From his value of the decay rate λ_a divided by the pressure P ,

$$\lambda_a / P = 5 \times 10^6 \text{ sec}^{-1} \text{ atmos}^{-1},$$

the value $\mathfrak{z} \cong 27 \cong 1.7Z$ can be inferred for molecular oxygen.

C. Positronium Formation

The cross section for positronium formation is of course zero for positron energies less than E_{thr} . For

TABLE II. Eigenvalues.

ϵ	x_{thr}	Λ_0	Λ^*	$\Lambda_0 \Gamma(\frac{1}{2}) e^{x_{\text{thr}}}$
0.250	16	1.47×10^{-6}	7.34×10^{-7}	14.8
0.258	15	3.44×10^{-6}	1.75×10^{-6}	13.8
0.267	14	8.76×10^{-6}	4.53×10^{-6}	12.8
0.277	13	2.18×10^{-5}	1.15×10^{-5}	11.8
0.289	12	5.45×10^{-5}	2.93×10^{-5}	10.9
0.302	11	1.34×10^{-4}	7.36×10^{-5}	9.8
0.316	10	3.27×10^{-4}	1.84×10^{-4}	8.8
0.333	9	7.86×10^{-4}	4.54×10^{-4}	7.8
0.354	8	1.87×10^{-3}	1.11×10^{-3}	6.8
0.378	7	4.33×10^{-3}	2.66×10^{-3}	5.8
0.408	6	9.78×10^{-3}	6.25×10^{-3}	4.84
0.447	5	2.17×10^{-2}	1.45×10^{-2}	3.90
0.500	4	4.67×10^{-2}	3.30×10^{-2}	3.12
0.577	3	9.87×10^{-2}	7.50×10^{-2}	2.42
0.707	2	2.08×10^{-1}	1.75×10^{-1}	1.89
0.816	1.50	3.32×10^{-1}	3.00×10^{-1}	1.81
1.000	1.00	5.40×10^{-1}	5.40×10^{-1}	1.80
1.155	0.75	7.45×10^{-1}	8.00×10^{-1}	1.93
1.414	0.50	1.03	1.23	2.08
2.000	0.25	1.98	2.80	3.10

²³ Reference 17, Chap. 12.

²⁴ P. A. M. Dirac, Proc. Cambridge Phil. Soc. 26, 361 (1930).

²⁵ M. Deutsch, Phys. Rev. 83, 866 (1951).

higher energies the energy dependence of the formation cross section can be separated into two factors

$$\sigma_f = \left(\frac{E - E_{\text{thr}}}{E} \right)^{\frac{1}{2}} \sigma_f'(E).$$

The factor $[(E - E_{\text{thr}})/E]^{\frac{1}{2}}$ is proportional to the ratio of the velocity of the outgoing positronium atom to the velocity of the incoming positron, and varies rapidly with positron energy in the vicinity of the threshold. The factor σ_f' is proportional to the square of the transition probability from the initial state to the final state.

Massey and Mohr⁸ have calculated σ_f for atomic hydrogen both by the Born approximation and by the method of distorted waves. The magnitude of the cross section in the Born approximation is indicated by the result $\sigma_f'(E_{\text{thr}}) = 3\pi a_0^2$ for atomic hydrogen and $\sigma_f'(E_{\text{thr}}) = 5\pi a_0^2$ for helium.²⁰ An adequate description for positron energies not too much larger than E_{thr} can

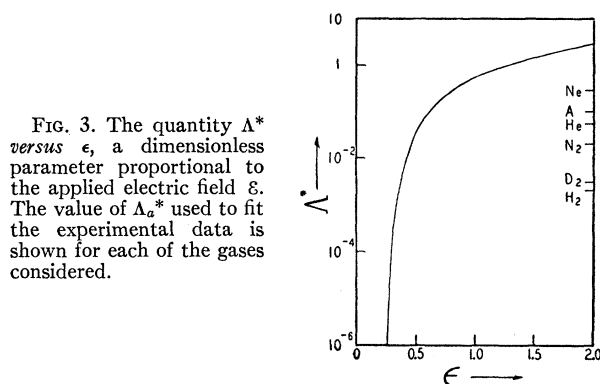


FIG. 3. The quantity Λ^* versus ϵ , a dimensionless parameter proportional to the applied electric field \mathcal{E} . The value of Λ_a^* used to fit the experimental data is shown for each of the gases considered.

be attained by writing

$$\sigma_f \approx \left(\frac{E - E_{\text{thr}}}{E} \right)^{\frac{1}{2}} \sigma_f'(E_{\text{thr}}).$$

V. RESULTS

A. Theoretical

In Sec. III B it was indicated that the dependence of the increase of positronium formation on the applied electric field \mathcal{E} could be described primarily in terms of the dependence of the eigenvalue λ_0 on \mathcal{E} . The values of the dimensionless eigenvalue Λ_0 (see Sec. III C) and the quantity Λ^* , defined by

$$\Lambda^* = \epsilon^{\frac{1}{2}} \Lambda_0 = \frac{1}{4} (l/v_{\text{thr}}) (M/m) \lambda_0,$$

are given in Table II, and Λ^* is plotted *vs* ϵ in Fig. 3. It is to be noted that on the scale used, the overestimates and underestimates for Λ^* would be indistinguishable. The increase in positronium formation can now be written as

$$I = \frac{\lambda_0}{\lambda_0 + \lambda_a} = \frac{\Lambda^*}{\Lambda^* + \Lambda_a^*}$$

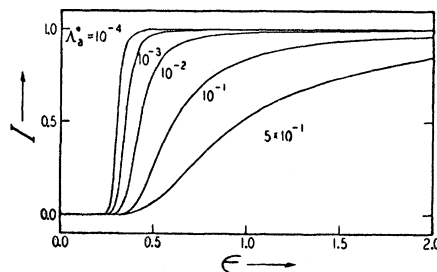


FIG. 4. Increase in positronium formation I versus ϵ , a dimensionless parameter proportional to the applied electric field \mathcal{E} , shown for a number of different values of Λ_a^* . The results depicted are obtained by taking σ_f large.

and is plotted *vs* ϵ in Fig. 4. (The corresponding plot for σ_f small is shown in Fig. 5.) In this form Λ^* carries all the dependence on \mathcal{E} and

$$\Lambda_a^* = \frac{1}{4} (l/v_{\text{thr}}) (M/m) \lambda_a$$

is a constant which depends only on the gas. The values of Λ_a^* used for the various gases is also indicated in Fig. 3. If $\lambda_a = N 3\pi r_0^2 c$ is inserted, there results

$$\Lambda_a^* = \frac{3\pi r_0^2 c}{4\sigma_e v_{\text{thr}}} \frac{M}{m} = \frac{3M}{\sigma_e} \left(\frac{\text{ry}}{E_{\text{thr}}} \right)^{\frac{1}{2}} \times 1.79 \times 10^{-4},$$

where ry is the energy corresponding to one rydberg, M is in amu and σ_e is in units of πa_0^2 .

The relationship between ϵ and the experimental variables is obtained in terms of its definition and is

$$\epsilon = \frac{E^*}{E_{\text{thr}}} = \left(\frac{1}{3} \frac{M}{m} \right)^{\frac{1}{2}} \frac{e \mathcal{E} l}{E_{\text{thr}}} = \frac{\mathcal{E}}{p \sigma_e} \left(\frac{\text{ry}}{E_{\text{thr}}} \right) \frac{M^{\frac{1}{2}}}{1303},$$

where \mathcal{E} is in volts/cm and p is the ratio of the density of the gas to the density of an ideal gas at NTP. The quantity p can then be written as

$$p = N/N_0,$$

where N_0 is Loschmidt's number, and if the gas in the experiment can be treated as an ideal gas, becomes

$$p = (P/P_0)(T_0/T),$$

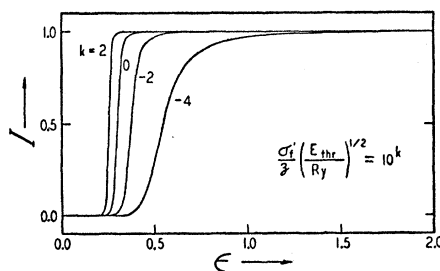


FIG. 5. Increase in positronium formation I versus ϵ , a dimensionless parameter proportional to the applied electric field \mathcal{E} , shown for a number of different values of $(\sigma_f'/3)(E_{\text{thr}}/\text{ry})^{\frac{1}{2}}$. The results depicted are obtained by taking σ_f small.

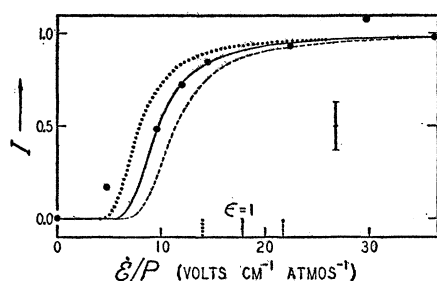


FIG. 6. Increase in positronium formation I versus \mathcal{E}/P , the applied electric field divided by the pressure, for helium. The theoretical curves are obtained by taking " σ_f large," and using $z=2$ and $\sigma_e=0.023\pi a_0^2$ —, $\sigma_e=0.018\pi a_0^2$ ···, $\sigma_e=0.028\pi a_0^2$ ---. The experimental points were taken at 2.7 atmos. The typical statistical error of the experimental data is indicated, as are the values of \mathcal{E}/P corresponding to $\epsilon=1$.

where P is the pressure, T the absolute temperature, and P_0 and T_0 are the pressure and temperature corresponding to NTP. If $T=300^\circ\text{K}$ is assumed, then

$$\epsilon = \frac{\mathcal{E}}{P\sigma_e} \left(\frac{\text{ry}}{E_{\text{thr}}} \right) \frac{M^{\frac{1}{2}}}{1184},$$

with P in atmospheres is obtained. It may be noted that the conversion from the theoretical variable ϵ to the experimental variable \mathcal{E}/P involves only known quantities and σ_e , which can be selected to obtain the best fit between the theoretical and experimental curves for the increase in positronium formation. In this sense the experiments can be considered to measure the elastic cross section σ_e .

B. Comparison with Experiment

Marder *et al.*¹¹ report their experimental results in terms of a quantity Φ which is to be interpreted as the ratio of the fraction of positrons which form positronium in the field \mathcal{E} to the fraction of positrons which form positronium in the absence of an electric field. For $\mathcal{E}=0$, the quantity Φ is unity and for large \mathcal{E} it attains

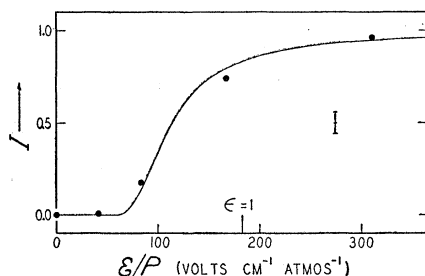


FIG. 7. Increase in positronium formation I versus \mathcal{E}/P , the applied electric field divided by the pressure, for neon. The theoretical curves are obtained by taking σ_f large, and using $z=10$ and $\sigma_e=0.12\pi a_0^2$. The experimental points were taken at 1.2 atmos. The typical statistical error of the experimental data is indicated, as is the value of \mathcal{E}/P corresponding to $\epsilon=1$. (Some experimental points at larger values of \mathcal{E}/P do not appear on this figure.)

a saturation value Φ_{max} . To compare theory and experiment the quantities Φ and I are related by

$$I = (\Phi - 1) / (\Phi_{\text{max}} - 1).$$

To fit a theoretical curve to the experimental data, there are two parameters available; one is a scale factor relating ϵ to \mathcal{E}/P and the other determines the steepness of the rise. In addition, there is a choice between the theoretical results for σ_f small and those for σ_f large. It is readily ascertained that the experimental data cannot be fitted with the theoretical curves for σ_f small for reasonable values of σ_f'/z , so that in the detailed comparison of theory and experiment only theoretical results for σ_f large are used.

In obtaining the experimental data for any particular gas, it was found that the reproducibility from run to run was not as good as the internal consistency of any particular run. For this reason, comparison with theory will be made only for the results of a single representative run for each gas. The run so chosen agrees with the average of all the runs to within experimental error. The results for the rare gases He, Ne, and Ar; the gases H_2 , D_2 , and N_2 ; and the other gases which were measured will be considered separately.

The Rare Gases

The various assumptions which were made in obtaining the theoretical description for the increase in positronium formation in a static electric field are expected to hold fairly well for the rare gases. The experiments can then be considered to measure σ_e and z , in the sense that a particular set of values for σ_e and z yields a theoretical curve which represents the experimental results.

It is to be noted that z enters only through Λ_e^* and that Λ_e^* is related to the steepness of the rise of the theoretical curves, but not in a sensitive way (see Fig. 4). Thus a range of values of z gives an acceptable fit of the experimental data, this range includes $z=Z$ and extends to values of z differing from Z by about a factor of two. It is encouraging that the experiment finds z to be in the range of values expected from rough theoretical arguments. In the absence of more particular indications than those given in Sec. IV B, it is felt best at present to take $z=Z$ and treat only σ_e as a parameter.

Experimental and theoretical results for the increase in positronium formation I as a function of \mathcal{E}/P are compared in Figs. 6–8 for helium, neon, and argon. The theoretical curve shown in each case is one with $z=Z$ and with σ_e adjusted to obtain a "best fit" of the experimental data. The experimental points shown are from a representative run for each gas; the typical statistical error of these points is indicated separately. The value of \mathcal{E}/P corresponding to $\epsilon=1$ is also shown so that the various graphs may be intercompared.

The data for helium have been fitted using $z=Z=2$

and $\sigma_e = 0.023\pi a_0^2$ as shown in Fig. 6. Curves corresponding to $z=2$ and $\sigma_e = 0.018\pi a_0^2$ (dotted) or $\sigma_e = 0.028\pi a_0^2$ (dashed) are also shown, and give some indication of the accuracy with which a value for σ_e can be selected. If the helium data are fitted with curves (not shown) corresponding to $z=1$ or $z=4$, agreement is noticeably poorer but still within experimental error; the value of σ_e which then gives the best fit is about 25% higher or lower, respectively.

The data for neon have been fitted using $z=Z=10$ and $\sigma_e = 0.12\pi a_0^2$ as shown in Fig. 7. Somewhat better agreement can be obtained using $z \sim 5$, which then yields a best fit with σ_e about 30% larger. The data for argon have been fitted using $z=Z=18$ and $\sigma_e = 1.5\pi a_0^2$ as is shown in Fig. 8.

Table III lists the values of σ_e obtained for He, Ne, and A using $z=Z$, together with other pertinent data²⁶ for these atoms. These values of the cross sections for the elastic scattering of positrons are radically smaller than the corresponding quantities for electrons, in agreement with the rough theoretical considerations of Sec. IV A. The detailed agreement of the theoretical curves and the experimental results for reasonable values of the parameters indicates that a correct

TABLE III. Summary of pertinent data for the rare gases.

Ele- ment	Z	M (amu)	E_{ion} (units of ry)	E_{exc} (units of ry)	E_{thr}	σ_e (units of πa_0^2) exp. theor.	Fig.
He	2	4	1.81	1.46	1.31	0.023 0.07	6
Ne	10	20	1.59	1.23	1.09	0.12 0.4	7
A	18	40	1.15	0.85	0.65	1.5 2	8

description for the increase of positronium formation in a static electric field has been attained.

Hydrogen, Deuterium, and Nitrogen

The qualitative considerations which served as background for the quantitative calculation of the increase of positronium formation in a static electric field no longer strictly apply to polyatomic molecules. In the case of the rare gas atoms, the threshold for positronium formation, E_{thr} , lies distinctly below the lowest excited state, E_{exc} . Positrons with energies below E_{exc} can lose energy in a collision with a gas atom only through the recoil of the gas atom. For polyatomic molecules the corresponding considerations are more complex because of the existence of low-lying energy levels.²⁷

In a polyatomic gas, positrons can lose energy by ionization or excitation of the gas molecules, as is of course true in a monatomic gas. As before, E_{ion} is

²⁶ *Atomic Energy Levels*, National Bureau of Standards Circular 467 (U. S. Government Printing Office, Washington, D. C., 1949).

²⁷ For considerations on energy loss by electrons in collisions with molecules see H. S. W. Massey and E. H. S. Burhop *Electronic and Ionic Impact Phenomena* (Clarendon Press, Oxford, 1952), Chap. IV.

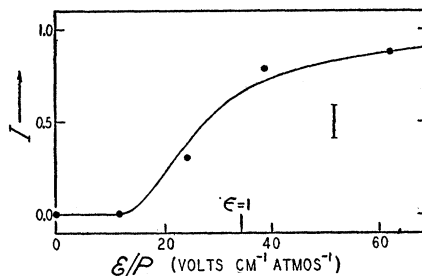


FIG. 8. Increase in positronium formation I versus ϵ/P , the applied electric field divided by the pressure, for argon. The theoretical curves are obtained by taking σ_f large, and using $z=18$ and $\sigma_e = 1.5\pi a_0^2$. The experimental points were taken at 2.7 atmos. The typical statistical error of the experimental data is indicated, as is the value of ϵ/P corresponding to $\epsilon=1$. (Some experimental points at larger values of ϵ/P do not appear in this figure.)

written for the minimum energy required to ionize the molecule, and $E_{thr} = E_{ion} - 0.5$ ry is the threshold for positronium formation, but now the various modes of excitation are considered separately. The minimum energy required to produce electronic excitation of the molecule is distinguished according to spin state of the appropriate level. In the pertinent case of H_2 , D_2 , and N_2 , $^1E_{exc}$ and $^3E_{exc}$ correspond to the lowest excited singlet and triplet level. The minimum energy required to produce vibrational and rotational excitations is sufficiently small (less than 1 ev in all cases) so that positrons of essentially all energies are able to produce such excitations. In view of this complication, it is clear that the theory as developed for the rare gases cannot be applied directly to yield z and σ_e .

However, it has been possible to select curves from the family obtained in Sec. V A (Fig. 4) which represent the experimental data for H_2 , D_2 , and N_2 . These curves are shown in Figs. 9–11 together with the experimental points. The experimental values for I are not obtained from Φ as before but from another quantity which is more directly related to the raw data and which has the same general dependence on ϵ/P as does Φ . The parameters which were adjusted (instead of z and σ_e)

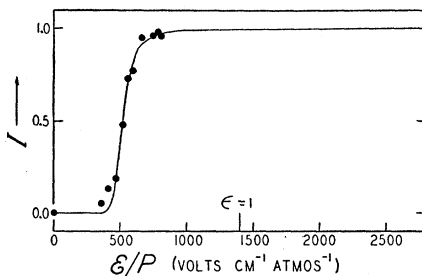


FIG. 9. Increase in positronium formation I versus ϵ/P , the applied electric field divided by the pressure, for H_2 . The theoretical curves are obtained by taking σ_f large, and using $\Lambda_a^* = 2 \times 10^{-3}$ and $\epsilon/P \div \epsilon = 1400$ volts cm^{-1} atmos $^{-1}$. The value of ϵ/P corresponding to $\epsilon=1$ is so indicated. The experimental points were taken at 2.7 atmos.

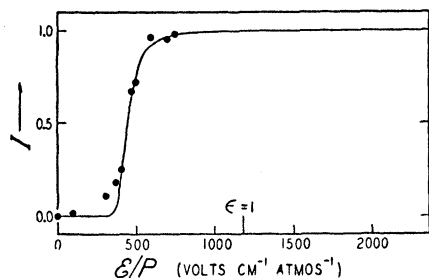


FIG. 10. Increase in positronium formation I versus E/P , the applied electric field divided by the pressure for D_2 . The theoretical curves are obtained by taking σ_f large, and using $\Lambda_a^* = 3 \times 10^{-3}$ and $E/P \div \epsilon = 1180$ volts cm^{-1} $atmos^{-1}$. The value of E/P corresponding to $\epsilon = 1$ is so indicated. The experimental points were taken at 2.7 atmos.

were simply Λ_a^* and the conversion factor from the dimensionless electric field variable to the corresponding experimental variable $E/P \div \epsilon$. As before for $\frac{1}{2}$ and σ_e , the quantity Λ_a^* is determined within about a factor of two and the choice of Λ_a^* has some influence on the value of $E/P \div \epsilon$, which is obtained. But because the curves here are steeper a change in Λ_a^* by a factor of two changes $E/P \div \epsilon$ by only about 10%. The values of Λ_a^* and $E/P \div \epsilon$ for the best fit of the theoretical curves and the experimental results for H_2 , D_2 , and N_2 are listed in Table IV together with other pertinent data²⁸ for these molecules. Except for a comparison of the corresponding values for H_2 and D_2 (below), no particular significance is attached to these parameters; nor is any inference drawn from the observation that the theoretical curves do fit the experimental data, in view of the manifold of such curves which is available.

Some understanding of the influence of low-lying energy levels on the effect of an electric field on positronium formation in polyatomic gases can be obtained from a comparison of the experimental data for the isotopes H_2 and D_2 . The only difference which should be observable will be due to the difference in their nuclear masses. It is to be noted that electronic, vibrational, and rotational energy levels depend differently on the mass (by factors of M^0 , $M^{-\frac{1}{2}}$, and M^{-1} , respectively). The electronic energy levels of these molecules is such that $^1E_{exc} > E_{thr} > ^3E_{exc}$. Since the ground state is a singlet state and positrons can cause a spin change only via their (weak) magnetic interaction, it follows that positrons cannot readily excite these molecules to triplet states. Electrons, of course, do excite such states very readily via exchange interaction. Moreover, because of the small mass of the positrons, excitation of vibrational and rotational states is also unlikely. However, the quantity to be considered is not the probability for a particular type of collision but the average energy loss which can be written

roughly as:

$$\langle \delta E \rangle = \sum \sigma_i \delta E_i / \sum \sigma_i,$$

where σ_i is the cross section for a particular type of collision, δE_i is the corresponding energy loss, and the summation is to be extended over all possible collisions including elastic collisions. Some understanding about $\langle \delta E \rangle$ may be obtained by comparing the value of $E/P \div \epsilon$ for hydrogen and deuterium. The theory which includes only energy loss by molecular recoil requires that $M^{\frac{1}{2}} E/P \div \epsilon$, as well as $M^{-1} \Lambda_a^*$, be the same for isotopes of a particular element. Comparison of $E/P \div \epsilon$ for H_2 and D_2 shows that this is not what is observed. However, if it is recalled that M appears because $\langle \delta E \rangle \propto M^{-1}$ when only elastic collisions occur, then the quantities which should be considered are $\langle \delta E \rangle^{-\frac{1}{2}} E/P \div \epsilon$ and $\langle \delta E \rangle \Lambda_a^*$. The values of $E/P \div \epsilon$ can then be related by taking $\langle \delta E \rangle \propto M^{-\frac{1}{2}}$; this is not to be deduced from the values of Λ_a^* which have been specifically adjusted to conform to this result. Hence it is clearly indicated that energy loss to molecular recoil (or rotation) does not predominate and strongly suggested that energy loss to molecular vibration does predominate; but other explanations are not precluded. The energy level structure of N_2 is similar to that of H_2 and D_2 , and hence the results of the above discussion should apply to N_2 also.

Other Gases

Experiments with CO_2 , CH_4 , C_2H_6 , CCl_2F_2 , and SF_6 show no increase in positronium formation up to the maximum electric fields used (approximately 2000 volts/cm at pressures of about 2 atmospheres). Thus in these gases energy losses in collisions still compete favorably with energy gain from the electric field even at the highest fields used and prevent the positrons from reaching the threshold for positronium formation before they annihilate as free positrons. This strong energy loss is presumably due to the existence of electronic energy levels which can be excited by positrons below the threshold for positronium formation. It is to be supposed that for sufficiently high values of E/P , even these gases would exhibit an increase in positronium formation, but it is not clear that the required values of E/P can be attained without complicating effects, e.g., breakdown, occurring. The negative observations for these gases clearly show why minute traces of such polyatomic gases strongly affect the

TABLE IV. Summary of pertinent data for hydrogen, deuterium, and nitrogen.

Molecule	Z	A (amu)	E_{ion}	$^1E_{exc}$ (units of ry)	$^3E_{exc}$	E_{thr}	Λ_a^*	$E/P \div \epsilon$ (volts cm^{-1} $atmos^{-1}$)	Fig.
H_2	2	2	1.13	0.82	0.33	0.63	2×10^{-3}	1400	9
D_2	2	4	1.13	0.82	0.33	0.63	3×10^{-3}	1180	10
N_2	14	28	1.15	0.63	0.45	0.65	2×10^{-2}	810	11

²⁸ G. Herzberg, *Molecular Spectra and Molecular Structure* (D. Van Nostrand Company, Inc., New York, 1950), second edition, Vol. I.

results in these gases which do exhibit an increase of positronium formation in a static electric field.

C. Discussion

From comparison of the theoretical dependence of the increase in positronium formation in a particular gas on the applied electric field and the gas density, as developed in Secs. III and V A, with the experimental results of Marder *et al.*¹¹ for the rare gases, it has been possible to deduce some information regarding the magnitudes of the cross sections for the pertinent atomic processes. It is found that the cross section for positronium formation is "large" and that the cross section per molecule for annihilation of a free positron with an electron of a gas molecule is about Z times the annihilation cross section per electron. The most accurately determined cross section is that for elastic scattering. The values obtained for this quantity are in general agreement with the results of simple theoretical considerations.

Agreement between theory and experiment was obtained by taking the energy region III as completely absorbing in the diffusion treatment of the positron energy distribution. This requires that in region III the formation cross section be sufficiently large, presumably of the order of the elastic cross section or larger. Because of the small values of the latter cross sections for any particular gas, the magnitude of the formation cross section so indicated was to be expected *a priori*.

The parameter β related to the cross section for the annihilation of free positrons has been determined to be $\beta \sim Z$ within about a factor of two. Since Z varies from 2 for He to 18 for Ar, this result indicates that the correct interpretation has been given to the parameter β . Because β enters into the present considerations in a rather insensitive way it cannot be determined very accurately. It may be remarked, however, that experiments of a quite different kind can measure directly the decay rate for the annihilation of free positrons in a gas and hence determine β . Moreover, the energy dependence of β can be tested by measuring this decay rate in an electric field. No such experiments on the rare gases seem to have been reported in the literature.

The quantity which is determined to any useful degree of accuracy is the cross section for the elastic scattering of positrons by the gas atoms. The principal uncertainty in this quantity arises from the uncertainty in β . If β were known, then σ_e would probably be determined to about 10% from the present data. However, in the absence of specific knowledge about β , the (experimental) values of σ_e listed in Table III should be considered uncertain by about $\pm 25\%$.

The detailed fit of theory and experiment suggests that the fundamental processes which are involved are correctly described. It may be remarked however that the general dependence of I on \mathcal{E} illustrated in Fig. 4 can be expected to hold for considerably more varied

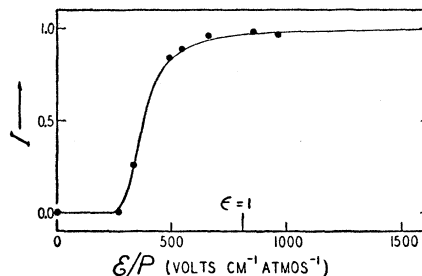


FIG. 11. Increase in positronium formation I versus \mathcal{E}/P , the applied electric field divided by the pressure, for N_2 . The theoretical curves are obtained by taking σ_f large, and using $\Lambda_e^* = 2 \times 10^{-2}$ and $\mathcal{E}/P \div \epsilon = 810$ volts $\text{cm}^{-1} \text{atmos}^{-1}$. The value of \mathcal{E}/P corresponding to $\epsilon = 1$ is so indicated. The experimental points were taken at 2 atmos.

conditions than indicated by the assumptions used in deriving that relationship. Thus if σ_e has a different dependence on v than was assumed, or if the energy loss is by excitation of rotational levels instead of by recoil, the indicated dependence of I on \mathcal{E} will be obtained but now a different significance must be attached to the parameters which appear.

The qualitative considerations of Sec. II suggest that the saturation of the increase in positronium formation at large values of the electric field corresponds to positronium formation by all the positrons in the gas. The experiments under discussion¹¹ indicate that only about $\frac{2}{3}$ of all the positrons form positronium at the high fields. This might be due to formation of positronium compounds²⁹ or to the annihilation of parapositronium atoms formed from positrons in region I.⁹ Clearly more direct experimental information on these questions would be desirable.

As further points of interest it may be noted that the effect of radio-frequency electric fields on positronium formation is readily taken into account in the Boltzmann equation (Sec. III A). The Boltzmann equation can also be altered to take into account mixtures of gases, and energy loss by vibrational and rotational excitation. Experiments show marked effects due to small admixtures of polyatomic gases in the rare gases, and from experiments of this type it may be possible to obtain useful information about such polyatomic molecules.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor H. Margenau, Professor L. Onsager, and Dr. G. Chester for helpful discussions; and Mr. D. Malone and Mr. S. Hawks for calculational assistance.

APPENDIX I. RELAXATION TIME FOR REACHING EQUILIBRIUM

In Sec. III B, considerable simplification resulted by assuming the relaxation time for reaching equilibrium

²⁹ J. A. Wheeler, Ann. N. Y. Acad. Sci. 48, 219 (1946); A. Ore, Phys. Rev. 73, 1313 (1948); A. Ore, Universitetet i Bergen Arbok 1952, Naturvitenskapelig rekke No. 5.

was much shorter than the lifetime of free positrons, so that the corresponding decay rates satisfied $\lambda_1 \gg \lambda_a$.

Consider first the case of $\sigma_f = 0$. Using the dimensionless notation of Sec. III C, the appropriate differential equation becomes

$$\frac{d}{du} \left[u \exp(-u^4) \frac{dg}{du} \right] + 16\Lambda u^2 \exp(-u^4) g = 0,$$

with the boundary conditions

$$u \exp(-u^4) dg/du = 0 \quad \text{at } x=0, \infty.$$

The transformation

$$g = \psi \exp(\frac{1}{2}u^4)$$

leads to

$$\frac{d^2\psi}{du^2} + \frac{1}{u} \frac{d\psi}{du} + [8u^2 - 4u^6 + 16\Lambda u] \psi = 0.$$

This equation should be compared with the equation

$$\frac{d^2\psi'}{du^2} + \frac{1}{u} \frac{d\psi'}{du} + [bu - \gamma^2 u^4 + 16\Lambda' u] \psi' = 0,$$

which has the eigenvalues

$$\Lambda_n' = \frac{3}{16}\gamma(2n+1) - \frac{1}{16}b.$$

The eigenvalues of the original equation may be underestimated by noting

$$\Lambda_n \geq \Lambda_n',$$

as long as b and γ are chosen so that

$$4u^6 - 8u^2 \geq \gamma^2 u^4 - bu$$

is satisfied for all positive values of u . The best underestimate using this method is obtained if b and γ are adjusted, subject to this restriction, so as to maximize Λ_n' .

This yields the desired underestimates:

$$\Lambda_n \geq \Lambda_n' = \frac{3^{\frac{1}{2}}(2K)^{\frac{1}{2}} (16+15K^2)^{\frac{1}{2}} - 16}{60 [(16+15K^2)^{\frac{1}{2}} - 4]^{\frac{1}{2}}},$$

with

$$K = 3(2n+1).$$

For $n=0$, where $\Lambda_0=0$ is known, the underestimate becomes

$$\Lambda_0 \geq -0.115.$$

For $n=1$, the quantity of interest Λ_1 is underestimated by

$$\Lambda_1 \geq 0.755.$$

To ascertain that this is a realistic estimate for Λ_1 , it is useful to obtain an overestimate for this quantity. In terms of the variable $x=u^4$, a simple trial function for g_1 is $1-(4/3)x$. The $4/3$ occurs because the trial

function must be orthogonal to $g_0 = \text{const}$,

$$\int_0^\infty x^{-\frac{1}{2}} e^{-x} (\text{const}) [1 - (4/3)x] dx = 0.$$

With this trial function, the Rayleigh-Ritz principle yields

$$\Lambda_1 \leq [\Lambda_1] = \frac{\int_0^\infty x e^{-x} \left[\frac{dg}{dx} \right]^2 dx}{\int_0^\infty x^{-\frac{1}{2}} e^{-x} g^2 dx} = \frac{1}{\Gamma(7/4)} = 1.088,$$

so that indeed $\Lambda_1 \cong 1$.

Turning to the interesting case of σ_f large, a rough underestimate for the rate of attaining quasi-equilibrium is simply

$$\Lambda_1[\sigma_f \text{ large}] \geq \Lambda_1[\sigma_f = 0].$$

That

$$\lambda_1 \gg \lambda_a$$

holds for pertinent values of \mathcal{E} may be noted by observing equivalently that

$$\epsilon^{\frac{1}{2}} \Lambda_1 \gg 0.755 \epsilon^{\frac{1}{2}} \gg \Lambda_a^*$$

holds, except for very small ϵ , which do not occur in the analysis because $E^* \gg kT$ is always assumed.

A somewhat more sophisticated underestimate for Λ_1 may be obtained by noting

$$\int_0^{x_{\text{thr}}} G(x, x) x^{-\frac{1}{2}} e^{-x} dx = h(x_{\text{thr}}) = \sum_n (1/\Lambda_n).$$

From this, there results first

$$h(x_{\text{thr}}) \geq 1/\Lambda_0 \quad \text{or} \quad 1/h(x_{\text{thr}}) \leq \Lambda_0,$$

as before, and secondly

$$h(x_{\text{thr}}) \geq 1/\Lambda_0 + 1/\Lambda_1 \quad \text{or} \quad \Lambda_1 \geq [h(x_{\text{thr}}) - 1/\Lambda_0]^{-1},$$

where in the last expression Λ_0 may be replaced by any overestimate without affecting the inequality.

APPENDIX II. NUMERICAL SOLUTION FOR σ_f SMALL

If σ_f is taken to be small, the increase of positronium formation in an electric field can be described in terms of λ_0 , which can be approximated by

$$\lambda_0 = 4\pi \int_0^\infty v^2 dv N \sigma_f v f_0 / 4\pi \int_0^\infty v^2 dv f_0,$$

as indicated in Sec. III B. Using the dimensionless energy variable $\eta = E/E^*$, so that $f_0 = \exp(-\eta^2)$ and $\eta_{\text{thr}} = E_{\text{thr}}/E^* = \epsilon^{-1}$, and writing

$$\sigma_f = \left(\frac{E - E_{\text{thr}}}{E} \right)^{\frac{1}{2}} \sigma_f' = \left(\frac{\eta - \eta_{\text{thr}}}{\eta} \right)^{\frac{1}{2}} \sigma_f',$$

with σ_f' taken to be constant, there results

$$\lambda_0 = N\sigma_f' v_{thr} \eta_{thr}^{-\frac{1}{2}} \frac{\int_{\eta_{thr}}^{\infty} \eta^{\frac{1}{2}} (\eta - \eta_{thr})^{\frac{1}{2}} \exp(-\eta^2) d\eta}{\int_{\eta_{thr}}^{\infty} \eta^{\frac{1}{2}} \exp(-\eta^2) d\eta}.$$

The denominator is readily integrated to yield $\frac{1}{2}\Gamma(\frac{3}{4})$. However, the integral in the numerator must be approximated. Putting $\xi = \eta - \eta_{thr}$, the numerator becomes

$$\begin{aligned} \int_{\eta_{thr}}^{\infty} \eta^{\frac{1}{2}} (\eta - \eta_{thr})^{\frac{1}{2}} \exp(-\eta^2) d\eta \\ = \exp(-\eta_{thr}^2) \int_0^{\infty} (\xi + \eta_{thr})^{\frac{1}{2}} \xi^{\frac{1}{2}} \exp(-2\eta_{thr}\xi - \xi^2) d\xi, \\ \approx \eta_{thr}^{-1} \exp(-\eta_{thr}^2) \Gamma(\frac{3}{4}) / 2\sqrt{2}, \end{aligned}$$

where it may be noted that the major contribution to

the integral occurs for values of η in the range

$$\eta_{thr} \leq \eta \lesssim \eta_{thr} + 1/2\eta_{thr},$$

and the approximations

$$\exp(-\xi^2) \approx 1, \quad (\xi + \eta_{thr})^{\frac{1}{2}} \approx \eta_{thr}^{\frac{1}{2}},$$

were used. This is completely adequate for the present purpose.

Using the values of the integrals indicated, and recalling that $\lambda_a = N\frac{4}{3}\pi r_0^2 c$, there results for the increase in positronium formation I in the present approximation

$$\begin{aligned} I &= \frac{\lambda_0}{\lambda_0 + \lambda_a} \\ &= \frac{1.3 \times 10^6 \eta_{thr}^{-\frac{1}{2}} \exp(-\eta_{thr}^2)}{1.3 \times 10^6 \eta_{thr}^{-\frac{1}{2}} \exp(-\eta_{thr}^2) + (\frac{4}{3}/\sigma_f')(E_{thr}/ry)^{-\frac{1}{2}}} \end{aligned}$$

with σ_f' in units of πa_0^2 . Figure 5 shows this result plotted vs the dimensionless field parameter $\epsilon = \eta_{thr}^{-1}$ for a number of different values of $(\sigma_f'/\frac{4}{3})(E_{thr}/ry)^{\frac{1}{2}}$.

Density Effect on the Ionization in Gases by Electrons*

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The density effect on the ionization produced by electrons has been observed by comparing their specific ionization in He and H₂ gases at one and ten atmospheres pressure. The measurements were made by sending primary electrons of energy variable from 2–35 Mev through an ionization chamber into a Faraday cup. No density effect is observed or expected at one atmosphere pressure at these energies, but at ten atmospheres pressure, the results indicate the onset of the density effect at about 10 Mev in H₂ and 20 Mev in He. At 35 Mev, the percentage reduction in ionization from that expected without density effect amounts to (8.0 ± 1) in H₂ and (3.5 ± 1.3) in He. These results are in near agreement with the calculations of Sternheimer.

I. INTRODUCTION

FOR some time, it has been known^{1,2} that the mass stopping power of any material for charged particles should be dependent on the density of the material. This dependence arises when the particle velocity approaches c and the relativistic extension of the particle's transverse field is affected by the polarization of the surrounding material. This effect has been demonstrated as influencing the energy loss of particles in solids,³ the ionization produced in gases,^{4,5} the grain

density in nuclear emulsions,⁶ and the light output of scintillators.⁷

In gases at atmospheric pressure, the density effect begins to occur when the kinetic energy of the charged particle is about 100 times its rest energy. In order to demonstrate the effect at lower particle energy, it is necessary to increase the gas pressure. The theory of the effect^{8,9} indicates that the correction to the rate of energy loss depends on the gas pressure P and particle momentum p only as a function of the product $(pP^{\frac{1}{2}})$.

The measurements described in this paper were made with an ionization chamber at 10 atmospheres pressure using electrons of up to 35-Mev energy. The observed

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