Alpha Decay of Spheroidal Nuclei*

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The consequences of spheroidal deformation of nuclei on the barrier transmission in alpha decay are considered. A set of coupled differential equations is derived relating the amplitudes of the various groups of alpha particles emitted from a nucleus described by the Bohr-Mottelson model. The cases of the decay of Th^{228} and Cm^{242} were studied numerically and from them information regarding the probability distribution of alpha particles on the nuclear spheroidal surface is obtained. It is found that the one-body model of an alpha particle in a well does not yield these distributions, and it is thus concluded that "alpha-particle clusters" have a short mean free path in nuclear matter. The shift in the surface distributions of Th^{228} and Cm^{242} may be explained qualitatively in terms of the order of nucleon orbital filling.

The over-all penetration factors for the spheroidal case are compared with those for the spherical case, and it is found that the resultant enhancement due to the deformation is not nearly as large as that predicted by Hill and Wheeler on the basis of a one-dimensional approximation.

INTRODUCTION

RECENTLY an impressive amount of data has been amassed demonstrating the existence of rotational spectra in regions far removed from closed-shell configurations.[1] The existence of such level schemes is predicted by the Bohr-Mottelson[2] strong-coupling model of the nucleus in which it is assumed that the nucleus has an appreciable spheroidal deformation.

In the region of heavy nuclei ($A \gtrsim 230$) where alpha decay is generally a prominent mode of decay, the rotational bands are particularly well developed, and some cases of alpha emission by even-even nuclei to members of the rotational band as high as the 8+ level have been observed.[3] Alpha decay of even-even nuclei to states other than the rotational band members has been observed only in the case of a few nuclides.

One of the most conspicuous features of the recent data involves the variation between nuclei of the relative intensities of the various alpha groups. Asaro[4] has calculated "hindrance" factors for all alpha groups, where the hindrance factor is defined as the ratio of the intensity of the alpha group leading to the ground state to the intensity of the alpha particles leading to the particular excited state, corrected for the energy difference between the states. For the energy dependence of the decay rate, he used Preston's[5] alpha-decay formula (for no spin change[6]). Figure 1, which is due to Asaro, summarizes the data. A more fundamental presentation of the experimental alpha intensity variations is to be found in the comprehensive study by Winslow.[7]

It is to be expected that the occurrence of large spheroidal deformations will have pronounced effects on the process of charged-particle emission. In contrast to the case of spherical nuclei, the electrostatic field of a spheroid is not central. The coupling resulting from the noncentral nature of the field will have a bearing on the relative amplitudes of particles emitted with different orbital angular momenta. It is one of the purposes of this note to see whether it is possible to explain the values and trends for the hindrance factors of the $l=2$ and $l=4$ waves in the decay of even-even nuclei in terms of the noncentral electrostatic field.

Another consequence of the distortion of the nucleus, earlier explored by Hill and Wheeler,[8] is a thinning out of the potential barrier in certain directions leading to directed alpha emission in those directions. They gave an approximate expression for the penetrability based

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[1] For a review of the experimental data and for an extensive list of references see: A. Bohr, *Rotational States of Atomic Nuclei* (Ejnar Munksgaard, Copenhagen, 1954), and A. Bohr and B. R. Mottelson in *Beta and Gamma Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955).

[2] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952); A Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[3] I. Perlman and F. Asaro, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 157.

[4] F. Asaro (unpublished, 1955).

[5] M. A. Preston, Phys. Rev. **71**, 865 (1947).

[6] It is of interest to compare the hindrance factors for alpha particles having angular momentum $l \leq 4$ to the reduction from the centrifugal barrier. R. G. Thomas [Progr. Theoret. Phys. (Japan) **12**, 253 (1954)] for example, has calculated the reduction factors, η_l, in the JWKB approximation and finds that for a 5-Mev uranium alpha emitter they are 0.59 for $l=2$ and 0.18 for $l=4$. The smaller results of J. J. Devaney [Phys. Rev. **91**, 587 (1953)] appear to be in error. From the approximate formulas of G. Gamow and C. L. Critchfield [*Theory of Atomic Nucleus and Nuclear-Energy Sources* (Oxford University Press, Oxford, 1949), p. 173] one can obtain the following simple expression for η_l, which agrees fairly well with Thomas' results:

$$\eta_l = \exp\left[-\frac{\hbar l(l+1)}{(MR)^{\frac{1}{2}}(Z-2)^{\frac{1}{2}}e}\right].$$

In this equation, Z is the atomic number, R the effective nuclear radius, and M the mass of the alpha particle. From the equation, it is evident that nowhere in the alpha emitter region will η_l be small enough to account for the observed $l \leq 4$ hindrance factors.

[7] G. H. Winslow, Argonne National Laboratory Report ANL-5381, 1955 (unpublished).

[8] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1134 (1953).

on a one-dimensional WKB integration through the "thinnest" part of the barrier. It is to be noted though that if the decay is highly directional with respect to the nuclear symmetry axis, it is necessary that components of the alpha waves with high l values[9] occur with large amplitudes. These would be the components leading to the higher rotational states. Since these components experience a much larger effective potential than the S wave, because of the centrifugal potential and the additional energy associated with rotation of the recoil nucleus, one might expect significant deviations from the penetration formula of Hill and Wheeler[8] based on a one-dimensional WKB integration through the thinnest part of the barrier. In the final section of this note, the total barrier penetrabilities for Cm^{242} and Th^{228} are calculated.

In the next section we derive the general equations governing alpha decay to a rotational band of the daughter nucleus. In the following section, equations for decay from an even-even nucleus are formulated in prolate spheroidal coordinates, and these equations serve as the basis for the subsequent exploratory numerical work.

FORMULATION OF THE ALPHA-DECAY PROCESS

To formulate the problem of alpha decay in the region external to the nuclear surface, it is necessary to take into account the electrostatic interaction between the alpha particle and the residual nucleus. The first question to be settled is which degrees of freedom of the nucleus are required for an appropriate description of the process. In the case of a spherical daughter nucleus it is easy to see that it is unnecessary to consider the Coulomb interaction between the alpha particle and the protons individually, as this force is very much smaller than nuclear forces. It thus suffices to consider only the interaction of the alpha particule with the nucleus as a whole, and the appropriate nuclear coordinates are those of the center of mass of the system. In the case of a deformed nucleus the interaction between the alpha particle and the quadrupole field of the nucleus is not small compared to the energy characterizing rotation. Here it is necessary to include in the description of the process the rotational coordinates of the nucleus. Alternatively, it is necessary to include in the total wave function the low-lying rotational states. The emitted alpha particle can then be thought to induce transitions between the rotational states through the quadrupole component of the field.

In general the Schrödinger equation for the system can be reduced to a system of coupled equations in the variable r by expanding the wave function in terms of some complete orthogonal set of functions in the remaining variables

$$\psi=\sum_\gamma R_\gamma(r)\Omega_\gamma(x_i,\theta,\phi), \tag{1}$$

[9] J. A. Rasmussen, University of California Radiation Laboratory Report UCRL-2431, 1953 (unpublished).

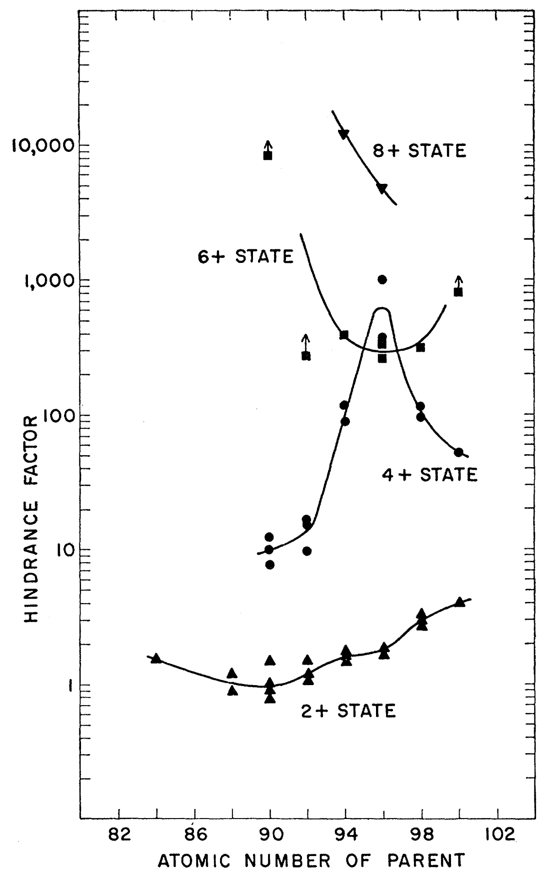

FIG. 1. Hindrance factors of alpha groups in even-even nuclei (defining the ground-state transition as unhindered) from Asaro.[4]

where x_i is the set of variables required to describe the recoil nucleus.

Multiplication by $\Omega_\gamma{}^*$ and integration over all variables except r reduces the partial differential equation to a set of ordinary differential equations in r.[10] Ω_γ can be expanded in terms of products of eigenfunctions of the residual nucleus and normalized spherical harmonics, $Y_{l,m}(\theta,\phi)$, in the angles of the alpha particle with respect to axes fixed in space. The set necessary to describe the decay process is limited by the constraints that the angular momentum of the parent nucleus I_i and its space projection M_i be conserved. The constraints are satisfied by the summation[11]

$$\Phi(I_i,M_i,I_f,l,\zeta)=\sum_m(I_fl\,M_i-m\,m|I_flI_iM_i)\times\chi(I_f,M_i-m,\zeta)Y_{l,m}(\theta,\varphi), \tag{2}$$

where $(I_fl\,M_i-m\,m|I_flI_iM_i)$ is a Clebsch-Gordan coefficient and $\chi(I_f,M_i-m,\zeta)$ is a normalized nuclear wave function for a state with angular momentum I_f and component $M_f=M_i-m$. The remaining quantum

[10] For example, see M. A. Preston, Phys. Rev. **75**, 90 (1949).
[11] See E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).

numbers used to describe the nuclear state are represented by ζ.

The set of equations obtained from (1) and (2) will contain coupling terms the strength of which will depend on the magnitude of the electric transition moments connecting the nuclear states involved. The outstanding examples of large electric moments between nuclear states are those resulting in the "fast" electric quadrupole transitions between members of a rotational band. Thus, we see that the low rotational bands, both because their members constitute the low-lying nuclear states and because band members are connected by large matrix elements, are the appropriate nuclear states for our problem. We shall in the following restrict ourselves to the decay to the states of one band.

For explicit nuclear wave functions we turn to the work of Bohr and Mottelson,[1] in which nuclear motion is approximately separated into "rotational" and "intrinsic" parts. In the Bohr-Mottelson model of strongly deformed nuclei the rapid individual particle motion is thought to take place in a deformed nuclear field (or well) which rotates nearly adiabatically. The deformed nuclear field is taken to have axial symmetry, although there may be cases where this is not true.[12] The wave function for the nucleus then approximately factors (apart from a symmetrization) into

$$\chi(I,M,K,\Omega)=\phi_\Omega(x_i')\left(\frac{2I+1}{8\pi^2}\right)^{\frac{1}{2}}D_M{}^I{}_K(\Theta_i), \qquad (3)$$

where $\phi_\Omega(x_i')$ is the state function for the particle structure, x_i' being the coordinates of the particles with respect to a frame of reference fixed in the nucleus. Ω is the particle-structure quantum number which for approximately independent particles is given by $\Omega=\sum_i\Omega_i$, where Ω_i is the angular momentum of the ith particle about the nuclear symmetry axis. The function

$$[(2I+1)/8\pi^2]^{\frac{1}{2}}D_M{}^I{}_K(\Theta_i)$$

is the normalized state of rotation (or wave function for a symmetrical top) having as arguments the Eulerian angles, $\Theta_i(=\theta_1,\theta_2,\theta_3)$; K denotes the projection of the angular momentum of the nucleus, $\mathbf{I}$, on the symmetry axis, and is in this model an approximate constant of the motion.

The Hamiltonian of which (3) is an eigenfunction is

$$H_{\mathrm{nucl}}=H_P(x_i')+H_{\mathrm{rot}}(\Theta_i), \qquad (4)$$

where $H_P(x_i')$ is the energy operator for the individual particles in the deformed well and $H_{\mathrm{rot}}(\Theta_i)$ is the rotational energy operator which gives rise to a spectrum of the form

$$E_I=(\hbar^2/2\mathfrak{J})I(I+1).$$

$\mathfrak{J}$, which is interpreted as a moment of inertia, depends on the details of the particle structure, in particular on the interparticle forces.[13] Its value is, in general, larger than that predicted from the irrotational fluid picture but smaller than the moment associated with a rigid rotation.

To obtain the complete Hamiltonian for the alpha decay problem in the center-of-mass system, we add to (4) the energy operator for the alpha particle in the region outside the short-range nuclear force field

$$H_\alpha=(\hbar^2/2\mu)\Delta+V_{\mathrm{el}}(\mathbf{r};\Theta_i). \qquad (5)$$

The inclusion of Θ_i in the electrostatic energy indicates that the field at a point in the space-fixed system varies as the daughter nucleus rotates. μ is the reduced mass of the system.

A consequence of the assumption that the nuclear well is axially symmetric is that the nucleons within the nucleus have an axially symmetric distribution. This will reflect itself in a corresponding symmetry for the alpha particles on the surface. To incorporate this symmetry into our formulation of the problem and also to effect a simplification of the electrostatic interaction we shall consider the description of the process in a system of coordinates fixed in the daughter nucleus, i.e., a coordinate system whose polar axis coincides with the nuclear symmetry axis. We then expand the wave function of the system as

$$\psi=\sum_{l,m}r^{-1}w_{l,m}(r)\Upsilon(I_i,M_i;l,m,K_f,\Omega), \qquad (6)$$

where $w_{l,m}\rightarrow e^{ikr}$ when $r\rightarrow\infty$,

$$\Upsilon(I,M;l,m,K,\Omega)=\phi_\Omega(x_i')(2I+1/8\pi^2)^{\frac{1}{2}}\times D_M{}^I{}_{K+m}(\Theta_i)Y_{l,m}(\theta',\varphi'), \qquad (7)$$

and (r,θ',ϕ') are the spherical polar coordinates of the alpha particle in the new system.

That (7) is the appropriate state for this problem—that is, contains the states of the daughter nucleus belonging to the single rotational band with quantum number K_f and has the proper transformation properties (i.e., represents a state with angular momentum I_i and component M_i)—can be demonstrated by transforming to the space-fixed frame of reference. For this purpose, we make use of the transformation properties of the spherical harmonics,

$$\begin{aligned}Y_{l,m}(\theta',\varphi')&=\textstyle\sum_m D_m{}^{l\dagger}{}_{m'}(\Theta_i)Y_{l,m'}(\theta,\varphi)\\&=\textstyle\sum_{m'}(-1)^{m'+m}D_{-m}{}^l{}_{-m}(\Theta_i)Y_{l,m'}(\theta,\varphi),\end{aligned} \qquad (8)$$

where $D^{l\dagger}$ is the Hermitian adjoint of D^l, and the Clebsch-Gordan expansion for the D's,

$$\begin{aligned}&D_{-m'}{}^l{}_{-m}(\Theta_i)D_M{}^I{}_{K+m}(\Theta_i)\\&=\sum_{I'=|I-l|}^{I+l}(IlM\;-m'|IlI'\;M-m')(Il\;K+m\;-m|IlI'K)\\&\qquad\times D_{M-m'}{}^{I'}{}_K(\Theta_i),\end{aligned} \qquad (9)$$

[12] B. Segall, Phys. Rev. **95**, 605(A) (1954).

[13] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955) and S. A. Moszkowski, Phys. Rev. **103**, 1328 (1956), this issue.

to obtain

$$\begin{aligned}\Upsilon(I,M;l,m,K,\Omega)&=\phi_\Omega(x')(2I+1/8\pi^2)^{\frac{1}{2}}\\ &\times\textstyle\sum_{I'}(-1)^m(I\,l\,K+m\;-m|I\,l\,I'\,K)\\ &\times\textstyle\sum_{m'}(-1)^{m'}(I\,l\,M\;-m'|I\,l\,I'\,M-m')\\ &\times D_{M-m'}{}^{I'}{}_K(\Theta_i)Y_{l,m}(\theta,\varphi)\\ &=\phi_\Omega(x')\textstyle\sum_{I'}(-1)^{m+I'-I}(I\,l\,K+m\;-m|I\,l\,I'K)\\ &\times\textstyle\sum_{m'}(I'l\,M-m'\;m'|I'l\,I\,M)(2I'+1/8\pi^2)^{\frac{1}{2}}\\ &\times D_{M-m'}{}^{I'}{}_K(\Theta_i)Y_{l,m'}(\theta,\varphi).\quad(10)\end{aligned}$$

In the above formulas, well-known properties[11] of Clebsch-Gordan coefficients under interchange of indexes were employed. From (2) and (3), we can immediately see that

$$\begin{aligned}\Upsilon(I_i,M_i;l,m,K_f,\Omega)&=\sum_{I_f}(-1)^{m+I_f-I_i}(I_il\,K_f+m\;-m|I_il\,I_fK_f)\\ &\times\Phi(I_i,M_i;I_f,l,K_f,\Omega),\quad(11)\end{aligned}$$

and hence that $\Upsilon(I_i,M_i;l,m,K_f,\Omega)$ has the proper transformation properties and generally contains a mixture of rotational states of the daughter nucleus.

To obtain the differential equations for $w_{l,m}(r)$, we substitute (6) in the complete Schrödinger equation, multiply by $\Upsilon^*(I_i,M_i;K_f,l',m')$; and integrate over all the independent variables except r. We get

$$\begin{aligned}\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}(E-E_{\Omega,K_f})-\frac{l'(l'+1)}{r^2}-\frac{4\mu Ze^2}{\hbar^2 r}\right]&w_{l',m'}(r)\\ -\frac{2\mu}{\hbar^2}\sum_{l,m}w_{l,m}(r)(l',m'|H_{\rm rot}|l,m)&\\ -\frac{2\mu}{\hbar^2}\sum_{l,m}w_{l,m}(r)\left(l',m'\left|V-\frac{2Ze^2}{r}\right|l,m\right)&=0,\quad(12)\end{aligned}$$

where E_{Ω,K_f} merely determines the arbitrary zero of energy. Since the states in general contain more than one member of a nuclear rotational band, the nuclear rotational energy operator is not necessarily diagonal in the Υ representation. The matrix elements of $H_{\rm rot}$ may be readily evaluated with the aid of the expansion (11) and the relationship

$$H_{\rm rot}\Phi(I_i,M_i;I_f,l,K_f,\Omega)=\frac{\hbar^2}{2\mathfrak{J}}I_f(I_f+1)\Phi(I_i,M_i;I_f,l,K_f,\Omega),$$

and we find

$$\begin{aligned}(l'm'|H_{\rm rot}|l,m)&=\frac{\hbar^2}{2\mathfrak{J}}\delta_{ll'}\sum_{I_f}(-)^{m'-m}I_f(I_f+1)\\ &\times(I_il'\,K_f+m'\;-m'|I_il'I_fK_f)\\ &\times(I_il'\,K_f+m\;-m|I_il'I_fK_f).\quad(13)\end{aligned}$$

The electrostatic interaction experienced by the alpha particle, though rather complicated in the space-fixed system, is in the body-fixed system merely the charge of the alpha particle times the electrostatic field of the stationary deformed nucleus. If we make the usual multipole expansion, V is given by

$$V=\frac{2Ze^2}{r}+\frac{2e^2Q_0P_2(\cos\theta')}{2r^3}+\cdots,$$

where Q_0 is the "intrinsic" quadrupole moment of the nucleus (the quadrupole moment with respect to the nuclear symmetry axis). With the relation

$$\int Y_{l',m'}{}^*(\theta,\phi)P_k(\cos\theta)Y_{l,m}(\theta,\phi)d\omega=c^{(k)}(lm,l'm'),\quad(14)$$

where the $c^{(k)}(lm,l'm')$ are defined and tabulated by Condon and Shortley,[14,15] the matrix elements of electrostatic energy can be readily evaluated. Equation (12) then becomes

$$\begin{aligned}\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E-E_{\Omega,K_f}-\frac{2Ze^2}{r}\right)-\frac{l(l+1)}{r^2}\right]&w_{l,m}(r)\\ -\frac{\mu}{\mathfrak{J}}\sum_{I_f,m'}w_{l,m'}(r)(-)^{m-m'}I_f(I_f+1)&\\ \times(I_il\,K_f+m'\;-m'|I_ilI_fK_f)(I_il\,K_f+m\;-m|I_ilI_fK_f)&\\ -\frac{2\mu}{\hbar^2}\frac{Q_0e^2}{r^3}\sum_{l'}w_{l',m}(r)c^{(2)}(lm;l'm)&=0.\quad(15)\end{aligned}$$

For the alpha decay of an even-even emitter to the lowest band of the daughter ($K_f=0$), we have $I_i=K_i=K_f=m=m'=0$. The Clebsch-Gordan coefficients appearing in (15) are then nonzero only for the terms $I_f=l$; and (15) reduces to a form derived earlier[9,16]:

$$\begin{aligned}\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E-E_{\Omega,K_f}-\frac{2Ze^2}{r}\right)-\left(\frac{\mu}{\mathfrak{J}}+\frac{1}{r^2}\right)l(l+1)\right]&w_l(r)\\ -\frac{2\mu}{\hbar^2}\frac{Q_0e^2}{r^3}\sum_{l'}w_{l'}(r)c^{(2)}(l0,l'0)&=0.\quad(16)\end{aligned}$$

We now return to the question of the approximate conservation of angular momentum about the symmetry axis and the related question of the axial symmetry of the alpha distribution. As mentioned above, the distribution of alpha particles on the nuclear surface is axially symmetric; then, because of the approximate conservation of the projection of the total angular

[14] E. U. Condon and G. H. Shortley, reference 11, p. 175.

[15] The $C^{(k)}(lm;l'm')$ can be expressed in terms of Clebsch-Gordan coefficients; thus $C^{(k)}(lm;l'm')=(-1)^m(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}}\times(2k+1)^{-1}(ll'00|ll'k0)(ll'm-m'|ll'km-m')$. See G. Racah, Phys. Rev. 62, 438 (1942), Eq. (52).

[16] L. Dresner (unpublished).

momentum on the symmetry axis, the only nonvanishing $w_{l,m}$ on the surface are those having $m=\Delta K = K_i - K_f$. It is to be noted from (15) that the electrostatic potential has no off-diagonal elements in m. The only mixing in of components with $m\neq\Delta K$ results from H_{rot}. This mixing is probably small in the region of large nuclear deformation, so that the $w_{l,m}$ with $m\neq\Delta K$ are not likely to be important in the barrier region. The sum over m in (6) reduces to the single term $m=\Delta K$.

The above considerations with Eq. (11) provide us with a means for determining the branching ratios for decay to the different members of a band. A case of special interest, which was discussed by Bohr, Fröman, and Mottelson,[17] is that of the "favored" transitions in odd-A nuclei. In these transitions the odd-nucleon wave function is thought to remain essentially unaltered.[17,18] Thus, in the case of the favored transitions $\Delta\Omega=0$ and $K_f=K_i$; hence, the decay is similar to the decay of an even-even nucleus to the lowest band of the daughter ($K_i=K_f=0$). From Eq. (11) with $m=0$, we find that, for decay with alpha particles with angular momentum l, the relative reduced transition probabilities to the members of the rotational band with $K_f=K_i$ are

$$B(I_i,K_i;l,I_f)=(I_i\,l\,K_f\,0|I_i\,l\,I_f\,K_f)^2.$$

The relative reduced transition probabilities here may be regarded as the alpha-group intensities expected for the limiting case of infinite nuclear moment of inertia, i.e., degenerate nuclear rotational levels. Bohr, Fröman, and Mottelson,[17] who first derived this relationship, make an approximate correction to the limiting case by applying the ordinary Gamow-type alpha decay rate-energy relations to reduce the relative intensities of alpha groups to the higher rotational states. To obtain the branching ratios of the decay to various rotational states, the reduced probabilities $B(I_i,K_i;l,I_f)$, must be multiplied by the relative probabilities, C_l, of emitting an alpha particle with angular momentum l (i.e., $C_l\sim|w_{l\Delta K}|^2$) and summed over l. Approximate C_l's can be obtained from the decay of neighboring even-even nuclei. The agreement of the intensities computed in this way with experimental values is generally good.[17]

DECAY EQUATIONS IN SPHEROIDAL COORDINATES

There are two procedures that one could follow in treating alpha decay. In the first, the decay equations are integrated outwards starting with nuclear surface boundary values which may be arrived at by a model describing the formation and behavior of alpha particles in nuclear matter. The other procedure is almost the reverse of the first and consists of starting at "infinity" with empirical amplitudes and integrating in to the nuclear surface. (It should be noted here that the alpha-group intensity measurements do not yield information regarding the relative phases.)

In either approach it is necessary to work with the equations in regions close to the nuclear surface, for there the noncentral electrostatic field is most effective in coupling the various partial waves. Since, in the following, we shall assume that the nuclear surface is a prolate spheroid, it is convenient to use prolate spheroidal coordinates.

We take the foci of the spheroids to be at ($x=y=0$; $z=\pm a/2$) and define the spheroidal coordinates of a point in space as

$$\begin{aligned}\xi&=(r_1+r_2)/a, & 1\leq\xi\leq\infty,\\ \eta&=(r_1-r_2)/a, & -1\leq\eta\leq 1,\\ \phi&=\tan^{-1}(y/x), & 0\leq\phi\leq 2\pi,\end{aligned}\tag{17}$$

where r_1 and r_2 are the distances between the point and the two foci. The parameter a is specified by the condition that one of the spheroids, $\xi=\xi_0$, corresponds to the surface of the nuclear spheroid. The connection to spherical polar coordinates is most easily seen in the asymptotic region:

$$\xi\rightarrow(2r/a),\quad \eta\rightarrow\cos\theta\quad \phi=\phi,\text{ as } r\rightarrow\infty.$$

We shall consider the special case of an even-even ($I_i=0$) nucleus. The wave equation in spherical coordinates in the coordinate system fixed in the daughter nucleus is, from (16),

$$\left[-\frac{\hbar^2}{2\mu}\Delta-\frac{\hbar^2}{2\mathfrak{J}}L(\theta,\phi)+V(r,\theta)-E\right]\psi=0,\tag{18}$$

where

$$L(\theta,\phi)=r^2\left[\Delta-\frac{1}{r^2}\frac{1}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right].$$

Since the lowest band of an even-even nucleus is characterized by $K=0$, and since we are considering only decay to the lowest band, the alpha-particle wave function will be axially symmetric ($M=K_i-K_f=0$). In prolate spheroidal coordinates, Eq. (18) becomes, for axially symmetric wave functions,

$$\left\{\frac{4}{a^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}\right]\right.$$
$$+\frac{\mu}{\mathfrak{J}}\frac{(\xi^2-1)}{(\xi^2-\eta^2)}\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}$$
$$\left.+\frac{2\mu}{\hbar^2}(E-V(\xi,\eta))\right\}\psi(\xi,\eta)=0.\tag{19}$$

The rotational energy term is not exactly represented by the second term in (19). This term is an approximation good when $\xi^2\gg\eta^2$. For the deformations expected in actual nuclei the condition is not fulfilled near the surface ($\xi_0=1.5$ for Pu^{238}), but near the surface the

[17] Bohr, Fröman, and Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 29, No. 10 (1955).
[18] J. O. Rasmussen, Arkiv Fysik 7, 185 (1953).

whole rotational term is negligible compared with the potential energy. When the term becomes important (near and beyond the turning point), the approximate expression is accurate.

If we represent the wave function by the expansion

$$\psi(\xi,\eta)=\sum_l(\xi^2-1)^{-\frac{1}{2}}w_l(\xi)Y_{l,0}(\cos^{-1}\eta), \quad (20)$$

we obtain for the equations satisfied by the $w_l(\xi)$:

$$\frac{d^2w_l}{d\xi^2}+\left[\frac{1}{(\xi^2-1)^2}-l(l+1)\left(\frac{1}{\xi^2-1}+\frac{\mu a^2}{4\mathfrak{I}}\right)\right]w_l(\xi)$$
$$+\frac{\mu a^2}{2\hbar^2(\xi^2-1)}\sum_{l'}w_{l'}(\xi)\int Y_{l,0}^*[E-V(\xi,\eta)]$$
$$\times(\xi^2-\eta^2)Y_{l',0}d\phi d\eta=0. \quad (21)$$

It is now necessary for us to evaluate the potential $V(\xi,\eta)$. Since the details of the charge distribution inside the nucleus are not at present known, we shall make the usual and probably reasonable approximation that the charge density is constant throughout the nucleus. The potential can then be found by integrating the Green's function in spheroidal coordinates[19] over the nuclear volume; and we find

$$V(\xi,\eta)=\frac{4Ze^2}{a}[Q_0(\xi)-P_2(\eta)Q_2(\xi)], \quad (22)$$

where $Q_0(\xi)$ and $Q_2(\xi)$ are Legendre functions of the second kind and equal are to

$$Q_0(\xi)=\tfrac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right),$$

$$Q_2(\xi)=P_2(\xi)\tfrac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right)-\tfrac{3}{2}\xi.$$

Asymptotically, $V(\xi,\eta)$ can be expressed in spherical coordinates as

$$V(\xi,\eta)=\frac{2Ze^2}{r}+\frac{Q_0e^2P_2(\cos\theta)}{r^3}+\cdots,$$

where the intrinsic quadrupole moment, Q_0, is given by

$$Q_0=Za^2/10. \quad (23)$$

The above relationship between the intrinsic quadrupole moment, Q_0, and the interfocal distance, a, permits us to determine a from experimental data.

Evaluating the integral in (21) by using (14) and

$$\int Y_{l',0}^*(\eta)P_2(\eta)\eta^2Y_{l,0}(\eta)d\omega$$
$$=\frac{12}{35}c^{(4)}(l0,l'0)+\frac{11}{21}c^{(2)}(l0,l'0)+\frac{2}{15}\delta_{ll'}, \quad (24)$$

we obtain for the Schrödinger equation in the exterior region

$$\frac{d^2}{d\xi^2}w_l-(V_0+V_l)w_l-\sum_{l'}w_{l'}(V_{ll'}^{(2)}+V_{ll'}^{(4)})=0, \quad (25)$$

where

$$V_0(\xi)=\frac{2\mu Ze^2}{\hbar^2(\xi^2-1)}(\xi^2-\tfrac{1}{3})$$
$$\times\left[Q_0(\xi)-\frac{Ea}{4Ze^2}+\frac{2}{15}Q_2(\xi)\right]-\frac{1}{(\xi^2-1)^2},$$

$$V_l(\xi)=l(l+1)\left(\frac{1}{\xi^2-1}+\frac{\mu a^2}{4\mathfrak{I}}\right),\quad l\neq 0,$$

$$V_{ll'}^{(2)}(\xi)=-\frac{2\mu Ze^2a}{\hbar^2(\xi^2-1)}c^{(2)}(l0,l'0)$$
$$\times\left[\frac{2}{3}\left(Q_0(\xi)-\frac{Ea}{4Ze^2}\right)+\left(\xi^2-\frac{11}{21}\right)Q_2(\xi)\right],$$

$$V_{ll'}^{(4)}(\xi)=\frac{2\mu Ze^2a}{\hbar^2(\xi^2-1)}\frac{12}{35}c^{(4)}(l0,l'0)Q_2(\xi).$$

The coupling between various l waves results in part from the noncentral nature of the field and in part from the nature of the spheroidal coordinate system. The Coulomb term is contained in V_0, and $V_{ll'}^{(2)}$, is significantly larger than $V_{ll'}^{(4)}$ throughout the region of interest.

Numerical Work

In all of the regions that must be considered in the treatment of decay through a single barrier (the "barrier" region, the "turning-point" region, and the "far" region), the calculations for the present problem are obviously more difficult than for the corresponding problem with uncoupled waves. However, since the coupling decreases rapidly with distance, we need only give special consideration to the barrier and turning-point regions (the wave functions being very nearly Coulombic in the far region).[20]

In the following, we shall seek approximate solutions to

$$\begin{aligned}w_0''-V_0w_0&=V_{02}w_2+V_{04}w_4,\\ w_2''-(V_0+V_2)w_2&=V_{02}w_0+V_{24}w_4,\\ w_4''-(V_0+V_4)w_4&=V_{04}w_0+V_{24}w_2,\end{aligned} \quad (26)$$

which is the set of equations (25) in which all of the partial waves with $L>4$ are neglected.

Within the barrier region, the wave functions undergo extremely large variations in their magnitudes, making

[19] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, p. 1291.

[20] R. F. Christy, Phys. Rev. 98, 1205(A) (1955); and L. Dresner and J. A. Wheeler (reference 13) have also studied the problem of decay through a nonspherical barrier.

direct calculations with (26) difficult. Instead we have preferred to work with the ratios $y(\xi)=w_2/w_0$ and $z(\xi)=w_4/w_0$ inasmuch as their magnitudes vary within a small range. Solutions for $y(\xi)$ and $z(\xi)$ arising from a WKB-type approximation are

$$\begin{aligned} y(\xi)&=(K_0/K_2)^{\frac{1}{2}}\exp\left\{-\int(K_2-K_0)d\xi\right\},\\ z(\xi)&=(K_0/K_4)^{\frac{1}{2}}\exp\left\{-\int(K_4-K_0)d\xi\right\}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} K_0&=[V_0+V_{02}y+V_{04}z]^{\frac{1}{2}},\\ K_2&=[V_0+V_2+V_{02}/y+(z/y)V_{24}]^{\frac{1}{2}},\\ K_4&=[V_0+V_4+(1/z)V_{04}+(y/z)V_{24}]^{\frac{1}{2}}. \end{aligned}$$

As a result of the coupling, the equations (27) themselves constituted an extremely complicated set of integral equations. We have solved (27) by an iterative procedure, in which $y(\xi)$ and $z(\xi)$ are assumed over a small range of ξ and are used to calculate improved y's and z's.

This procedure, which was continued until self-consistent values were obtained, was found to converge fairly rapidly. To circumvent the difficulty occurring where y or z goes through zero, a change of dependent variable of the type $\bar{y}=(w_2+bw_0)w_0^{-1}=y+b$, with b a constant, was made. Barrier region integrations were made in this manner for both outward and inward integrations.

The solutions (27) are, of course, inapplicable in the turning point region. In this region the wave functions do not vary radically, so that it is feasible to work with the wave equations directly in the form of (26).

Outward Integrations for Cm^{242}

It was decided at first to see whether the simpler pictures of the alpha particle in nuclear matter could lead to the observed ratios of the partial waves. Cm^{242} was selected as an interesting case, as it exhibits a very large $l=4$ hindrance factor. Probably the simplest models are the one-body model in which the alpha particle is thought to move intact in nuclear matter for at least a few traversals of the nucleus and the model in which alpha particles are formed uniformly on the surface of the nucleus. The angular distribution with reference to the nuclear symmetry axis is then altered by the nonuniform barrier.

For the individual alpha-particle model, we assume that the alpha particle is in the lowest state in a spheroidal well (of uniform depth), the depth of which is adjusted so that the emitted alpha particle has the experimentally observed energy. The wave equation for the interior region is separable in spheroidal coordinates and has as its solution an "angular" part which may be expressed as an infinite sum of Legendre polynomials in η and a "radial" part which is a sum of spherical Bessel functions in ξ.[19] From these solutions the boundary values of the alpha-particle wave function on the nuclear spheroid are obtained.

It is appropriate at this point to look into the questions of the size (in this context, the volume) and the shape (i.e., the interfocal distance) of the nuclear spheroid. There is some uncertainty regarding the appropriate nuclear size[21] to be used in these considerations. Thus, our calculations were performed for two "sizes" of nuclei. One had a volume equal to that of a sphere of radius $1.20A^{\frac{1}{3}}\times10^{-13}$ cm, the other to a sphere of radius $1.35A^{\frac{1}{3}}\times10^{-13}$ cm.

To determine the interfocal distance, a, of the spheroidal coordinate system employed and hence the nuclear shape use was made of the relation (23) between Q_0 and a. Since the intrinsic quadrupole moment for Pu^{238} is not empirically known, we used a semiempirical connection[9] between quadrupole moments and energy of the first rotational state, E_2(kev):

$$|Q_0|=1.2ZE_2^{-\frac{1}{2}}\times10^{-24}\text{ cm}^2, \tag{28}$$

where Z is the charge. This formula is based on the experimentally known relation between intrinsic quadrupole moments (from gamma-ray lifetimes and Coulomb excitation cross sections) in the heavy rare earth region. The formula yields a $|Q_0|$ of 17×10^{-24} cm^2 for Pu^{238}. Subsequent to the completion of our calculations, data on the quadrupole moments of a few heavy nuclei, obtained from Coulomb excitation experiments and spectroscopic observations, have become available. It appears that the values of $|Q_0|$ predicted by (28) are somewhat larger than the empirical values[22]; but it is not possible at present to state quantitatively by how much the semiempirical formula overestimates the correct values.[22] In view of this, the quantitative details of our numerical work are subject to some modification, but it seems unlikely that any of our qualitative conclusions will be altered.

[21] On the assumption of spherical shape, various nuclear radii have been obtained by different experiments. For example, the Stanford high-energy electron scattering data [Yenni, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954)] yield what might be called a charge radius of $1.1A^{\frac{1}{3}}\times10^{-13}$ cm. This value is probably small for our purpose, as it neglects the greater extension of the neutrons [M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954)], the finite range of nuclear forces, and the radius of the alpha particle. The radius of the alpha particle is about 2×10^{-13} cm as determined by electron scattering. The alpha-particle scattering measurements of G. W. Farwell and H. E. Wegner, Phys. Rev. **95**, 1212 (1954) indicate a value of $(1.50A^{\frac{1}{3}}+1.4)\times10^{-13}$ cm, but this is probably more a measure of the major axis of the spheroid (because of the lower barrier at the tips) and hence a large value for the spherical shape. See also H. A. Tolhoek and P. J. Brussard, Physica **21**, 449 (1955).

[22] The $|Q_0|$ values for U^{238} and Th^{232} of 8×10^{-24} cm^2 and 9×10^{-24} cm^2 respectively, determined by Coulomb excitation [G. Temmer and N. Heydenburg (private communication)], indicate that (28) may be in error by as much as 50%. However, the recent spectroscopic quadrupole moment determination for Am^{241} and Am^{243} of $Q=+4.9\times10^{-24}$ cm^2 by Manning, Fred, and Tomkins [Spectroscopy Symposium, Argonne National Laboratory, B3, 1956 (unpublished)] leads to an intrinsic quadrupole moment, Q_0, (for $I=5/2$) of $Q_0=+14\times10^{-24}$ cm^2 which is less than 15% below the value predicted by the semiempirical formula.

TABLE I. Results of outward integrations for Cm^{242}.

	ξ_0	Boundary conditions at ξ_0 y_0	z_0	Calculated values at $\xi=5.6$ y	z	Calculated α group intensity (using connection formula approximation) $\alpha_2:\alpha_0$	$\alpha_4:\alpha_0$
Uniform surface distribution	1.406	0	0	+1.20	+0.298	1.13	0.058
	1.514	0	0	+0.91	+0.230	0.68	0.035
One-body model	1.406	−0.277	+0.025	+0.83	+0.202	0.56	0.029
	1.514	−0.232	+0.016	+0.72	+0.156	0.43	0.017
Values needed to match experimental intensities				+0.74	−0.02	0.357	4.8×10^{-4}

The assumption of uniform charge density may be open to question in that it is expected that the protons beyond the closed shell of 82 will fill the lower energy orbitals with maximum concentration in the ends of the prolate spheroid. There is insufficient knowledge to justify any such refinement here.

The interfocal distance, a, corresponding to the Q_0 being used for Pu^{238}, is 1.35×10^{-12} cm. With this value of a, the two surfaces of the spheroids of different volume are given by $\xi_0=1.41$ and $\xi_0=1.51$.

Table I summarizes the results of these calculations. It is seen that there is some suppression of the $l=4$ group, but the agreement with the experimental intensities is not satisfactory. We can conclude only that neither simple picture represents the physical situation.

Dresner[16] has independently drawn the same conclusion from his work.

Inward Integrations for Cm^{242} and Th^{228}

As mentioned earlier, one can exploit the available experimental data directly by integrating the equations for the various partial waves inwards to the nuclear surface and thus obtain information about the alpha probability distribution on the nuclear surface. In the following, we shall restrict ourselves to the study of the two nuclides Cm^{242} and Th^{228}. These nuclides are of interest in that they exhibit opposite extremes in the $l=4$ hindrance factors, Cm^{242} being very strongly hindered and Th^{228} being virtually unhindered. The data and information pertinent to our calculations on these two nuclides are presented in Table II.

The observed intensities, of course, determine only the amplitudes of the various l waves but not their phases. The condition used to determine the possible sets of phases was that the phase factors, $e^{i\delta_l}$, of the waves, which are pure outgoing waves at large separations, be such as to produce an exponential damping of the imaginary part of the partial waves within the barrier. With these phases, one can then obtain the real part of the wave by inward integration.

In carrying out the above procedure of integrating through the turning point and into the barrier region, we have used two different methods. The first, which is very simple but approximate, is based on the circumstance that Eqs. (26) are decoupled at the turning point in that prolate spheroidal coordinate system in which the interfocal distance a, is given by $a^2=6Q_0/Z$. Assuming that the coupling terms remain negligible (compared to the remaining terms in the equation) in a small region about the turning point (an assumption probably reasonable for the Th^{228} case), one can apply the simple WKB connection prescription. A transformation to the spheroidal coordinate system appropriate for the barrier region of the nucleus in question is then performed.

It is important to note that the phase determination procedure yields not one but four sets of relative phases for the three partial waves. This is most readily seen when the coupling can be neglected, as in the abovementioned approximation. If δ_l insures that the part increasing exponentially as $\xi\rightarrow1$ is pure real, so will $\delta_l+\pi$.

Because of the very small magnitude of w_4 as compared to w_2 and w_0 in the case of Cm^{242}, the assumption of negligible coupling in the turning point region in the special coordinate system is not valid. In this case, we resorted to direct numerical solutions to Eqs. (26) (in spherical coordinates) on the UCRL Bush-Type electromechanical differential analyzer.[23]

Table III summarizes the results for the integrations through the turning point. The values of $y(\xi_1)$ and $z(\xi_1)$ contained therein served as initial values for the integrations in the barrier region by the method described in the preceding section. There are two additional phase

TABLE II. Information used in inward integration studies.

Nucleus	Alpha disintegration energy (including electron screening) (Mev)	Energy of 2+ state (Mev)	Energy of 4+ state (Mev)	Relative alpha abundance to 0+	2+	4+	Assumed Q_0 of daughter (10^{-24} cm^2)
Cm^{242}	6.252	0.044	0.146	73.7	26.3	0.035	+17
Th^{228}	5.553	0.084	0.253	71	28	0.2	+11.6

[23] J. Killeen, University of California Radiation Laboratory Report UCRL-2239, 1953 (unpublished).

TABLE III. Initial values for the barrier integrations.

	Cm^{242} ($\xi_1=5.6$)		Th^{228} ($\xi_1=6.0$)			
	Case I	Case II	Case I	Case II	Case III	Case IV
$y(\xi_1)$	0.74	−0.77	0.93	0.94	−0.94	−0.93
$z(\xi_1)$	−0.022	0.025	0.18	−0.23	0.22	−0.17

choices for Cm^{242} which because of the small magnitude of w_4 in the turning point region lead to essentially the same results as the two listed sets.

While the several sets of $y(\xi_1)$ and $z(\xi_1)$ values lead to the observed intensity and proper behavior of the imaginary part, all but one for each nucleus are physically unlikely. Unless the distribution of alpha particles on the nuclear surface is restricted to a narrow band about the equator, we would expect that in the turning point region the distribution will be at least somewhat peaked at the poles because of the lower barrier in those directions. On the basis of these considerations we can select the physically most plausible set, and in Figs. 2 and 3 are shown the results of the integrations for these cases in the two nuclei (Cases I).

The other choices of phases listed in Table III were also studied. Case II for Th^{228} exhibits a somewhat pathological behavior in that y and z increase drastically in the integration inwards. Thus, this choice of phases is unlikely to represent the physical situation.

In Fig. 4 we have plotted the surface distributions for all the above cases. It is to be noted that the

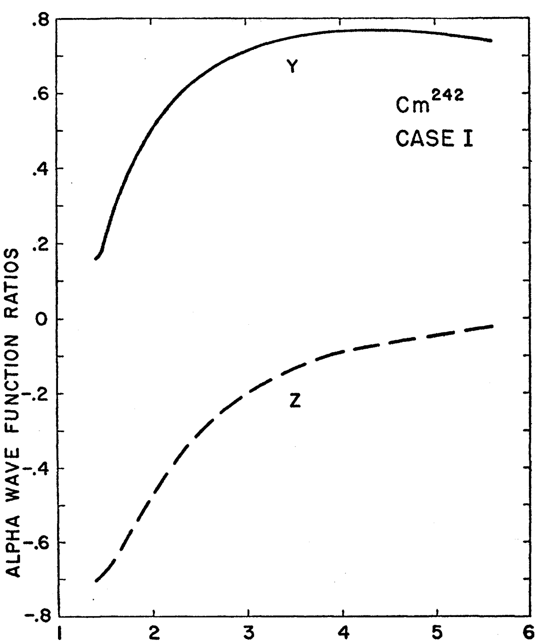

FIG. 2. Results of an inward numerical integration of the alpha-wave equation for Cm^{242} in prolate spheroidal coordinates (Case I). Boundary conditions at $\xi=5.6$ are based on experimental alpha-group intensities. The parameter y goes over asymptotically into the ratio of wave amplitudes of $l=2$ and $l=0$ groups, and z, into the ratio of the $l=4$ to the $l=0$ groups.

$|\psi|^2_{\text{surface}}$ are symmetric about the equator, since only even l values enter in the decay to the lowest band of an even-even nucleus. In each case the distributions are given for the two sizes of nuclei. The results do not differ much with the variation in volume considered here.

As expected, all but the two physically probable sets of initial values at ξ_1 lead to $|\psi|^2_{\text{surface}}$ narrowly restricted about the equator of the spheroids. Granting that cases I probably represent the physically significant cases for both Th^{228} and Cm^{242} we can observe that there is a shift in the surface distribution from a broad peak about the poles in Th^{228} to one more concentrated in the regions midway between poles and the equator in the case of Cm^{242}. That the shift in surface distribution is gradual can be inferred from the continuous growth of the hindrance factor in going from Th^{228} to Cm^{242} (see Fig. 1).

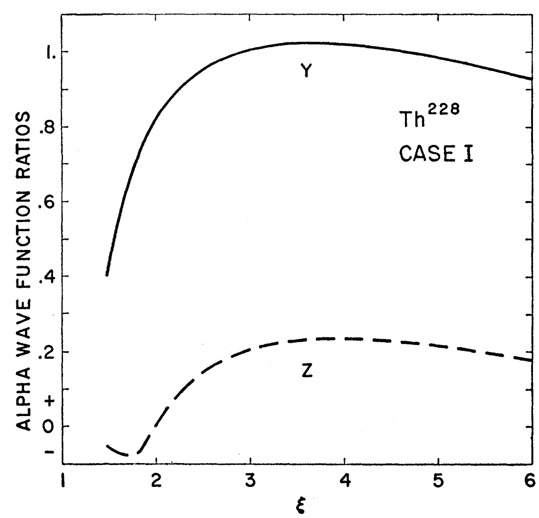

FIG. 3. Results of an inward numerical integration of the alpha-wave equation for Th^{228} in prolate spheroidal coordinates (Case I). Boundary conditions at $\xi=6$ are based on experimental alpha-group intensities.

It was suggested in a preliminary report[9] on the coupled alpha-decay problem that for nuclei of greater atomic number and deformation than Cm^{242} the continuation of the intensity trend might show a reversal of the decreasing behavior of the $l=4$ group. The increasing $l=4$ group of these heaviest nuclei would bear a phase relationship with respect to the $l=0$ group which was the opposite of that of the lower mass alpha emitters. Subsequent to this speculation the study[3] of alpha emitters Cf^{246} and Fm^{254} actually showed the increase in abundance of the $l=4$ group. The discussion of the preliminary report[9] also suggested speculatively that the intensity of the $l=2$ group might begin to decrease for heavier nuclei, and this also was found subsequently for californium and fermium isotopes. The idealized model on which these guesses were based considered a sharp angular alpha distribution of a delta-

function nature. The qualitative success of the guesses certainly does not imply any detailed validity of the delta function picture. Indeed, it seems most probable that the alpha angular distributions, particularly in the turning point region, are fairly broad.

The δ-function model was intended only as a first orientation of the effect of the quadrupole coupling. From the results of the present calculations it does not appear probable that the distribution of alpha particles on the nuclear surface is very sharp. The inclusion of components of the wave function with $l>4$, which are smaller than the ones considered in this work, should not seriously alter the surface distributions of Fig. 4.

A comment on another section of the preliminary report[9] is in order. Uncertainty was expressed as to

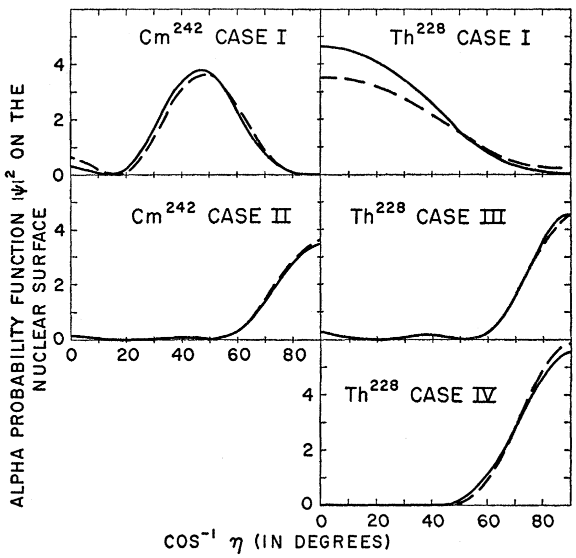

FIG. 4. Calculated alpha-probability distributions on the spheroidal nuclear surface from inward integrations based on experimental alpha-group intensities. The different cases result from different possible choices of relative phases of the various partial waves. Cases I for Cm^{242} and Th^{228} are believed to be the most likely physically significant cases. The solid lines correspond to a choice of the surface to enclose a volume equal to that of a sphere of radius $1.35\times10^{-13}A^{\frac{1}{3}}$ cm, and the dashed lines, that of a sphere of radius $1.20\times10^{-13}A^{\frac{1}{3}}$ cm.

whether the variation in hindrance factors (especially for $l=4$) for different isotopes meant a change in nuclear surface boundary conditions or was simply a consequence of the increase of Q_0 with atomic number. From the work of the present paper we can now definitely say that there must be a shift in nuclear surface boundary conditions between thorium and curium alpha emitters, yet the role of the large quadrupole coupling is an important one in the detailed interpretation of alpha emission in the heavy region.

Total Barrier Penetration

It is of interest to see how the occurrence of a spheroidal deformation affects the total decay constant. Let

TABLE IV. Barrier penetration factors.

Alpha emitter	$\log_e P_C$ (spherical)	$\log_e P_{DI}$ (spheroidal case I (this work)	$\log_e \times(P_{DI}/P_C)$ (this work)	$\log_e (P_D/P_C)$ Hill-Wheeler 1-dimensional WKB formula
Th^{228}	-74.73	-72.58	2.15	4.98
Cm^{242}	-72.79	-70.39	2.40	6.77

us define a generalized penetration factor P as

$$P=\frac{\int_{S_\infty}|\psi|^2dS}{\int_{S_n}|\psi|^2dS},$$

where S_∞ is a surface at a large distance from the nucleus and S_n is the nuclear surface. ψ is the complete wave function, which asymptotically goes over into a pure outgoing wave. For the case of a spherical nucleus, the penetration factor obtained using first order WKB wave functions is

$$P_C=\left(\frac{E}{V(R)_n-E}\right)^{\frac{1}{2}}\exp\left\{-\frac{2}{\hbar}\int_{R_n}^{R_{tp}}\{2\mu[V(r)-E]\}^{\frac{1}{2}}dr\right\},$$

where R_n is the nuclear radius and R_{tp} is the classical turning point.

With y and z known as functions of ξ, it is a simple matter to compute the penetrability for the spheroids taken in the above calculations for Th^{228} and Cm^{242}, with the nuclear volumes in all cases equal to those for spherical nuclei with radii $R=1.20\times10^{-13}A^{\frac{1}{3}}$ cm. The inward integration results, Cases I, are used.

In the last column of Table IV are the values predicted by the Hill-Wheeler[8] one-dimensional WKB formula for the penetration factor, evaluated through the thinnest part of the barrier. While both the Hill-Wheeler formula and the present work lead to larger penetrabilities than in the case of the spherical nucleus (as would be expected), the one-dimensional formula predicts a much larger increase than the detailed treatment and a much larger increase for Cm^{242} than for Th^{228}. This last fact is rather difficult to reconcile with the success of the old correlations of decay-rate data with spherical barrier formulas. The much lower enhancement of penetrability which the present numerical work finds does much to remove the above difficulty.

It appears then that the one-dimensional formula does not give a good estimate for the alpha decay problem. This implies that the alpha wave is unable to take full advantage of the thinner barrier in the vicinity of the poles. The reason for the failure of the one-dimensional estimate may be expressed qualitatively as follows: if the alpha wave function were to be sharply channeled along the most favorable penetration trajectory, the total wave function would contain high

angular momenta components with large amplitudes. The increased centrifugal barrier and higher nuclear rotational energy associated with the wave function would produce a dissipation of the wave function along the trajectory much larger than the one-dimensional WKB integral indicates. The wave function adjusts itself to a compromise involving moderate angular concentration of the wave function along the favorable penetration trajectory.

One can also note from Column 3 of Table IV that the spheroidal shape enhances the penetration factor of Th^{228} almost as much as that of Cm^{242} despite the fact that Cm^{242} has appreciably larger deformation. This is easily understood when we consider that the distribution of alpha particles on the nuclear surface is peaked about the axis in the case of the Th^{228} while the peak for Cm^{242} is closer to the equator.

DISCUSSION

The numerical studies of this paper, which are of an exploratory nature, serve to illuminate some of the general features of alpha decay from strongly deformed nuclei. It appears that though the electric quadrupole field plays an important role in determining the abundances of the various alpha groups, the predominant factor in the shifting of relative alpha group intensities with atomic number is the initial distribution of alpha particles on the nuclear surface. While this fact complicates the picture, it is fortunate in that it makes the study of the relative abundances a tool helpful in the understanding of the formation of alpha particles in nuclear matter. From the failure of the one-body model to yield the proper ratio of intensities in the Cm^{242} case and from the shift of the peaks of the surface density with increasing atomic number, we can conclude that the one-body model does not adequately represent the physical situation. That is, the alpha particle has a transitory existence in the nucleus and it does not move intact for times of the order of a period. We must envision the alpha clusters as continually forming and dissolving with short mean free paths.

Also, the picture of alpha particles being distributed uniformly on the nuclear surface does not appear to represent the physical state of affairs.

We might think of the situation in the following way: the alpha clusters that have any appreciable probability of forming and penetrating the barrier are those which are made up of the most loosely bound neutrons and protons. The distribution of these clusters will then reflect the distribution of the most loosely bound nucleons. If, as we expect, case I is the one that corresponds to the correct picture, we would conclude that the outer nucleons tend to concentrate near the poles in Th^{228} and nearer to the equator in Cm^{242}. In a prolate spheroidal well of appreciable eccentricity, the states concentrated near the poles are expected to be filled first.[24] These are the states with Bohr-Mottelson quantum numbers $\Omega=\pm\frac{1}{2}$ (all other states have nodes at the poles). One might suppose that orbitals with large probability densities at the poles are filled around $A=230$ and that subsequent nucleon pairs tend to fill states with density distributions shifted toward the equator.

ACKNOWLEDGMENTS

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[24] L. Dresner and J. A. Wheeler have explored this aspect of the problem independently (reference 13).