Inertial Parameters for Collective Nuclear Oscillations*

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Inertial parameters for nuclear rotation and quadrupole shape oscillation are known to be several times as large as the values assuming irrotational flow. These parameters may be calculated by using a method due to Inglis and applied in recent work by Bohr and Mottelson. The nucleus is assumed to be mainly in the lowest possible intrinsic state, but slightly perturbed by the time-dependence of the nuclear shape. It is convenient to separate the nuclear Hamiltonian into interactions of the individual nucleons with an average potential of the anisotropic harmonic oscillator type and residual interactions independent of deformation. Then the intrinsic motions of the nucleons must satisfy, at least approximately, self-consistency conditions between the potential and nuclear density. The nucleons outside closed shells are seen to contribute about half of the nuclear deformation. The excess of the inertial parameters over the irrotational values is related directly to the spread in energy of admixed excited states of the intrinsic motion, an essential consequence of the approximately independent motions of the nucleons.

The inertial parameters are determined mainly by excitation of nucleons outside closed shells and they are inversely proportional to the square of the average excitation energy within a major shell. In summary, the nuclear collective motion appears to resemble the shell model situation more than was thought previously, especially as regards the dominant role of the nucleons outside filled shells.

Empirical values of the inertial parameters are discussed in the light of these results, and a comparison is made with the recent analysis of Bohr and Mottelson. Anomalies in the empirical gyromagnetic ratios of rotational motion for odd-A nuclei result from the important role of the unpaired nucleon. With the aid of a simplified model in which the unfilled shell is assumed to be a p shell, it is possible to account qualitatively for a number of the observed features.

I. INTRODUCTION

IN the last few years much progress, both experimental and theoretical, has been made in the interpretation of nuclear states. Thus for a large number of heavy nuclei far removed from closed shells, the level schemes are characterized by very striking rotational bands.[1–3] Detailed evidence for the rotational nature of such states is provided by the empirical ratios of excitation energies and by the branching ratios for transitions leading to several states of a band.[4–6] In even-even nuclei the energies are, apart from small corrections, given by

$$E=(\hbar^2/2\mathfrak{J})I(I+1), \quad I=0,2,4,\cdots, \tag{1}$$

where $\mathfrak{J}$ is the rotational moment of inertia. The intrinsic quadrupole moments of these nuclei, as deduced from electromagnetic transition rates, are an order of magnitude larger than the values expected on basis of a single-particle model.[7–9] Also, even before the discovery of rotational states, the large values of spectroscopic quadrupole moments[10] for nearby odd-A nuclei as well as evidence from isotope shifts[11,12] suggested considerable deviations of the nuclear shape from spherical.[13] Now the nuclear deformation must represent the collective effect of many nucleons. The rotation of such nuclei, i.e., the precession of the nuclear symmetry axis about the angular momentum vector, is therefore a collective motion.[1] The excitation energies within a rotational band are, at least for even-even nuclei, quite small compared to the energies for other forms of excitation, in particular, excitation of nucleons to higher intrinsic quantum states.[2,3] In other words, the nuclear shape rotates slowly compared to the characteristic periods of the nucleonic orbits. The motion of nucleons can then be separated into intrinsic motion (i.e., for a static shape) and collective motion induced by the time-variation of the shape. Thus the total nuclear energy can be written approximately as a sum of an intrinsic energy (i.e., energy for a static field) and a collective kinetic energy, the total kinetic energy of the mass transport induced by the changing nuclear shape.

In early studies of the collective motion it was assumed that as the nucleus rotates, the induced velocity field of the nucleons is irrotational.[1,14–16] This assumption was partially motivated by its inherent simplicity and by the previous success of the semi-

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[5] Alaga, Alder, Bohr, and Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 9 (1955).
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empirical liquid-drop model in accounting for many gross features of nuclear structure. According to this viscosity-free hydrodynamical model, the rotation takes place mainly at the nuclear surface. The resulting moment of inertia is only a small fraction of the moment for rigid rotation of the entire nucleus. It was, however, recognized quite early that the empirical moments of inertia, as deduced from the positions of rotational energy levels, are several times as large as the irrotational values, and appreciable fractions of the rigid values.[7–9,17,18] As a typical example, for the nucleus Yb^{170}, we have[18]:

$$\mathfrak{J}_{\text{emp}} \simeq 5.5\mathfrak{J}_{\text{irrot}} \simeq 0.45\mathfrak{J}_{\text{rig}}. \tag{2}$$

The failure of the hydrodynamical model in predicting details of the collective motion should, however, not be surprising. One of the essential features of a hydrodynamical situation is that the mean free path of the particles involved is small compared to the dimensions of the system itself. Thus in a changing field, the induced current at each point is quickly shared among many particles, and the resulting kinetic energy is small. On the other hand, the striking success of the shell model in accounting for many nuclear properties[19–21] indicates that the nucleons move approximately independently in the average binding potential; i.e., a nucleon spends several periods in a given orbit.[22] Thus some of the nucleons will be able to respond strongly to the time-variations of the potential throughout the entire nucleus. The induced current will be shared with relatively few other nucleons and a large kinetic energy results.

These features may be studied quantitatively by using a method due to Inglis[23] and applied in a recent paper by Bohr and Mottelson.[18] The nucleus is assumed to be essentially in its lowest intrinsic state. However, any time-variation (e.g., rotation) of the potential makes it impossible for the nucleus to be entirely in any stationary state; thus some admixtures of higher intrinsic states are required. The extra energy due to this admixture is just the kinetic energy of the collective motion. For rotation of frequency ω slow compared to nucleonic frequencies, the collective kinetic energy is given by

$$T_{\text{coll}} = \tfrac{1}{2}\mathfrak{J}\omega^2, \tag{3}$$

where the moment of inertia may be expressed in terms of the intrinsic energies and wave functions as follows[18,23]:

$$\mathfrak{J} = 2\hbar^2 \textstyle\sum_K (\Delta E_{KL})^{-1} |\langle K|\partial/\partial\theta|L\rangle|^2. \tag{4}$$

The letters L and K denote lowest and excited intrinsic states, respectively. ΔE_{KL} is the excitation energy of state K, and the Dirac notation is used for the matrix elements. For a nucleus possessing axial symmetry, the axis of rotation (angle of rotation about this axis $=\theta$) is, for the lowest rotational band, perpendicular to the axis of symmetry.

In this method, the nuclear rotation is, in a sense, treated semiclassically, that is, we calculate the moment of inertia for rotation about a prescribed rotation axis with fixed frequency. The same value should, of course, result from a proper quantum-mechanical treatment of the rotation.[24–26]

In their recent paper, Bohr and Mottelson find for several limiting cases, that if the nucleons are assumed to move independently in the average potential, the moment of inertia has the solid-body value. They point out, however, that the expected value will be somewhat less (but still greatly exceed the irrotational value) because of correlations between the motion of individual nucleons induced by the inter-nucleon interactions.

In the present paper the collective motion is discussed further, with the aim of understanding somewhat more fully the origin of the large inertial parameters in the approximately independent motions of the nucleons. Both rotations and vibrations are studied; however, we restrict ourselves to quadrupole oscillations. It is assumed that the nuclear potential energy is made up of two parts: (a) the interaction of each nucleon with an average binding potential of the anisotropic harmonic oscillator type, and (b) residual interactions of various kinds[18]:

$$V = \sum_{\mu}\sum_{i} V_{\text{H.O.}}(\alpha_\mu, \mathbf{r}_i) + \sum_{i<k}\sum V_{\text{res.}}(\mathbf{r}_i, \mathbf{r}_k). \tag{5}$$

The harmonic oscillator (H.O.) is a particularly convenient form of potential to use for calculational reasons. Thus, energy levels, wave functions, and matrix elements are readily obtained in closed form.[27] Yet, the results obtained with this potential should not differ very strikingly from those of a more realistic (but much less tractable) rounded well potential.[28]

The residual interactions include, in principle, all interactions which are not taken into account in the

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[21] E. Feenberg, *Shell Structure of the Nucleus* (Princeton University Press, Princeton, 1955).

[22] This is also strongly supported by the analysis of low-energy neutron-nuclei scattering cross sections using an optical model with a very small absorption term [Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954)].

[23] D. R. Inglis, Phys. Rev. **96**, 1059 (1954); **97**, 701 (1955).

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[26] F. Villars (unpublished).

[27] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955).

[28] The effect of deviations from an H.O. potential and of the spin-orbit coupling on the motions of individual nucleons has been considered by Nilsson.[27] Thus the difference between the "false" magic numbers[23] (2, 8, 20, 40, 70, 112) of the pure H.O. potential and the empirical magic numbers (2, 8, 20, 28, 50, 82, 126) can be accounted for.

H.O. potential, i.e., both one-body and two-body terms; however they are here assumed to be independent of the deformation parameters, i.e., parts of the spin-orbit coupling, Coulomb effect, surface effect, as well as the correlations due to internucleon interactions can be included here. Consequently Eq. (5) represents a good approximation to the nuclear potential energy. Now, certain properties of the nuclear structure (e.g., equilibrium deformations and ground state spin[29,30]) depend sensitively on the detailed character of the residual interactions. On the other hand, some very essential aspects of the intrinsic and collective motions depend only on certain general characteristics of the residual interactions, such as the over-all correlations between the motions of individual nucleons. It is these properties which are treated in the present paper.

As is particularly evident for an isotropic H.O. potential, the motion of nucleons in filled and in unfilled major shells are very different. While most of the nucleons in a heavy nucleus are inside filled shells, the Pauli principle restricts their motion drastically (i.e., no transitions to other states in this shell can occur). On the other hand, the nucleons in unfilled shells constitute a small fraction of the total; yet the Pauli principle is much less effective in restricting their motion because of the availability of low-lying unoccupied excited states in this shell. As will be seen below, it follows from this that a considerable fraction of the quadrupole moment and almost all of the collective kinetic energy is due to the nucleons in unfilled shells. These results suggest that the collective model resembles the shell model more than had been previously thought.

In Sec. II of this paper we discuss certain essential aspects of the intrinsic motion, such as the requirement of self-consistency. In Sec. III, the collective motion is related to the intrinsic motion, in a way independent of details regarding nuclear wave functions and essentially dependent only on the energies of the excited intrinsic states admixed by the time variation of the nuclear shape.

In Secs. IV and V we apply the above results to an analysis of the empirical data on collective excitations. A comparison is made with the recent analysis of Bohr and Mottelson[18] on the rotational moments of inertia.

II. INTRINSIC MOTION

We consider first some general aspects of the motion of nucleons when the potential energy is given by Eq. (5). The H.O. part of the interaction for each particle may be written as follows:

$$V(\boldsymbol{\alpha},\mathbf{r})=V(r')=\tfrac{1}{2}m\omega_0^2r'^2, \tag{6}$$

[29] B. R. Mottelson and S. G. Nilsson, Z. Physik. **141**, 217 (1955); Phys. Rev. **99**, 1615 (1955).

[30] Kurt Gottfried, Ph.D. dissertation, Massachusetts Institute of Technology, June, 1955 (unpublished); Bull. Am. Phys. Soc. Ser. II, **1**, 194 (1956); Phys. Rev. **103**, 1017 (1956).

where $\mathbf{r}'$ is defined in terms of $\mathbf{r}$ by the volume preserving transformation:

$$\mathbf{r}'=[\exp(-\boldsymbol{\alpha})]\cdot\mathbf{r}=[1-\boldsymbol{\alpha}+\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}]\cdot\mathbf{r}. \tag{7}$$

The symmetric spurless tensor $\boldsymbol{\alpha}$ is defined in terms of the five well-known deformation coordinates[2] α_μ by

$$\mathbf{r}\cdot\boldsymbol{\alpha}\cdot\mathbf{r}=\textstyle\sum_\mu r^2Y_{2\mu}\alpha_\mu. \tag{8}$$

The following approximate equality can be easily verified:

$$\mathbf{r}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\cdot\mathbf{r}=(5/8\pi)r^2\beta^2, \tag{9}$$

where

$$\beta^2=\sum_{\mu=-2}^{2}|\alpha_\mu|^2. \tag{10}$$

For small $\boldsymbol{\alpha}$ we have, approximately:

$$V_{\text{H.O.}}(\alpha_\mu,\mathbf{r}_i)=\tfrac{1}{2}m\omega_0^2\{[1+(5/4\pi)\beta^2]r_i^2 -2[\textstyle\sum_\mu \alpha_\mu Y_{2\mu}(\Omega_i)r_i^2]\}, \tag{11}$$

for particle i. The symbol Ω_i denotes angle coordinates. Now the interaction of individual nucleons with this average potential represents the major part of the nuclear potential energy, and almost all of the deviation from sphericity. Yet any deformation is not imposed from the outside, but is due to the nucleons themselves. It is then reasonable to expect that there is a close relation between the deformation coordinates of the potential and the coordinates of the nucleons. Let us first assume that the density distribution follows the potential; e.g., as in the Thomas-Fermi statistical limit of many particles moving independently in the nuclear field. The density is then a function of V and thus of r' alone. For this case it is easy to show that (to first order in α_μ)

$$\alpha_\mu=\frac{4\pi}{5}\int\rho(r')r^2Y_{2\mu}d\mathbf{r}\Big/\int\rho(r')r^2d\mathbf{r}, \tag{12}$$

regardless of how the density depends on r'.

Now the expression on the right hand side of Eq. (12) is just the quadrupole parameter, even if the density does not depend solely on the magnitude r'. A general definition of the quadrupole parameter is[2]

$$\alpha_\mu{}^\rho=\frac{4\pi}{5}\frac{\langle L|\sum_i r_i^2Y_{2\mu}^*|L\rangle}{\sum r_i^2}, \tag{13}$$

where

$$\textstyle\sum r_i^2=\langle L|\sum_i r_i^2|L\rangle. \tag{14}$$

By means of the relations

$$\langle L|\partial/\partial\alpha_\mu|L\rangle=0, \tag{15}$$

$$\frac{\partial E}{\partial\alpha_\mu}=\left\langle L\left|\frac{\partial H}{\partial\alpha_\mu}\right|L\right\rangle=\left\langle L\left|\sum_i\frac{\partial}{\partial\alpha_\mu}V_{\text{H.O.}}(\boldsymbol{\alpha},\mathbf{r}_i)\right|L\right\rangle, \tag{16}$$

it is then readily shown that

$$\alpha_\mu{}^\rho=\alpha_\mu-\frac{4\pi}{5m\omega_0{}^2\sum r_i{}^2}\frac{\partial E_L}{\partial\alpha_\mu{}^*},\tag{17}$$

regardless of the nature of the residual interactions (except that they must be independent of the α_μ). As a special but important case, for any configuration at equilibrium, we have simply

$$\alpha_\mu{}^\rho=\alpha_\mu\tag{18}$$

for all five pairs of parameters, even though in general the density is not a function of r' alone. Equation (18) expresses the approximate self-consistency of the intrinsic motion. (In the statistical limit it is, of course, satisfied automatically.) This equality appears indeed reasonable since the nuclear potential is expected to follow the density quite closely[31] (apart from the effect of the finite range of the nuclear forces).

Let us now consider the way in which the ground state nucleonic wave function adjusts adiabatically to a change in the deformation. In this section we neglect any effect of the finite rate of change in the shape. The response of the nucleons to the changing potential is twofold: First, the wave function of each nucleon undergoes a simple distortion so as partially adjust to the changed potential. However, in addition more complicated changes in the wave functions occur. These can be described as mixing with close-lying states. We study these two effects in turn.

For a potential of the H.O. type, the quadrupole parameters $\alpha_\mu{}^\rho$ change upon distortion, but only half as fast as the corresponding deformation parameters α_μ. To show this more explicitly, we define a double primed coordinate system as follows:

$$\mathbf{r}''=[\exp(-\tfrac{1}{2}\boldsymbol{\alpha})]\cdot\mathbf{r}=[\exp(\tfrac{1}{2}\boldsymbol{\alpha})]\cdot\mathbf{r}'.\tag{19}$$

A configuration is here specified by $\psi(\mathbf{r}'')$. Distortion may then be defined as a change of the wave function $\psi(\mathbf{r})$ without a change in the configuration. In particular, we have

$$(\delta\alpha_\mu{}^\rho)_{\text{dist}}=\tfrac{1}{2}\delta\alpha_\mu.\tag{20}$$

The configurations are often conveniently characterized by one of the following sets of quantum numbers (for each nucleon): rectangular: n_1, n_2, n_3; cylindrical: N, n_3, Λ; spherical: N, l, Λ, where 1, 2, 3, denote principal axes, $N=n_1+n_2+n_3$. Here n_i is the number of nodes along axis i, l is the orbital angular momentum, and Λ is the component of orbital angular momentum along the symmetry axis.

For each configuration, there exists a unique set of equilibrium deformation parameters $\alpha_\mu(\text{eq})$. Furthermore, it is easily seen from Eqs. (18) and (19) that

$$\alpha_\mu{}^\rho=\tfrac{1}{2}[\alpha_\mu+\alpha_\mu(\text{eq})]\tag{21}$$

for any given configuration and deformation. We also note that

$$\alpha_\mu{}^\rho(\text{eq})=\alpha_\mu(\text{eq})=2\alpha_\mu{}^\rho(0),\tag{22}$$

where $\alpha_\mu{}^\rho(0)$ refers to the value of the corresponding quadrupole parameter when the potential is spherically symmetric. As a simple example, a closed-shell configuration has zero equilibrium deformation. For any configuration, in the H.O. model fully half of the quadrupole moment at equilibrium deformation is due due to the intrinsic density anisotropy of the nucleons in unfilled shells, and only half is due to distortion, mainly of the core of filled shells.[32]

It is also of interest to write down the expression for the total intrinsic energy for a given configuration and deformation. This is readily done with the aid of Eqs. (17) and (21) and the assumption that $\sum r_i{}^2$ does not depend on the α_μ to first order. The result is

$$E_L(\alpha_\mu)=E_L(\text{eq})+(5/16\pi)m\omega_0{}^2\sum r_i{}^2|\alpha_\mu-\alpha_\mu(\text{eq})|^2.\tag{23}$$

We now assume, and justify later, that the parameters α_μ and $\alpha_\mu{}^\rho$ are equal not only at equilibrium, but at least approximately so for the lowest intrinsic state at any deformation. Thus it is required that

$$\delta\alpha_\mu{}^\rho\simeq\delta\alpha_\mu.\tag{24}$$

On the other hand, from Eq. (17) we find immediately that

$$\frac{\partial\alpha_\mu{}^\rho}{\partial\alpha_\mu}=1-\frac{4\pi}{5m\omega_0{}^2\sum r_i{}^2}\frac{\partial^2E_L}{\partial\alpha_\mu\partial\alpha_\mu{}^*}.\tag{25}$$

In view of the results contained in these equations, we see that the rigidity of the nucleus with respect to deformation, defined by

$$C=\partial^2E/\partial\alpha_\mu\partial\alpha_\mu{}^*,\tag{26}$$

must be small compared to the value C_0 which would result if only distortion were allowed.[33]

$$C\ll C_0,\tag{27}$$

$$C_0=(5/8\pi)m\omega_0{}^2\sum r_i{}^2.\tag{28}$$

The validity of Eq. (24) requires that as the deformation changes not only distortion occurs, but also another kind of adjustment which involves a change in the configuration. This adjustment consists of mixing between close-lying intrinsic states, and for small deformations only nucleons outside closed shells can participate in it.

[31] It is emphasized that his self-consistency refers only to the *angular* shape and not to the *radial* shape. In fact, it appears very strongly that the radius of the density distribution is about 10^{-13} cm smaller than the radius of the potential. [For a summary of these results see, for example, the review article by K. Ford and D. Hill, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), Vol. 5, p. 25.]

[32] It is expected that regardless of the detailed form of the average potential (as long as it resembles the nuclear potential), the outer particles contribute significantly to the quadrupole moment. The fraction contributed may differ somewhat from the H.O. value $\frac{1}{2}$, though probably not by very much.

[33] This is evidently realized in both the statistical limit and the liquid drop model. For both cases, whatever small rigidity there is arises from the surface energy and is partially canceled by the Coulomb energy.

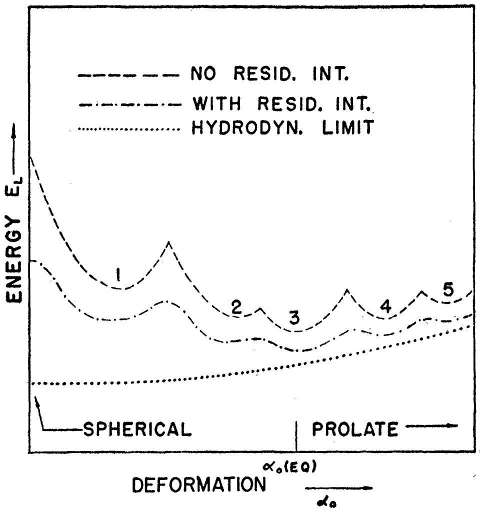

FIG. 1. Ground state energies *vs* deformation for various strengths of the residual interactions. This diagram is illustrative only. It is assumed that α_0 is positive and that the other four quadrupole deformation coordinates vanish. The curve labeled "With Resid. Int." indicates qualitatively the expected behavior of the ground state energy as function of deformation in nuclei. The equilibrium deformation giving minimum energy is shown. In this example, it has practically the same value for the two upper curves.

If there are no residual interactions, the change of configuration associated with a changing shape can occur in jumps. Thus, as is seen in Fig. 1, the curve representing the lowest energy as function of deformation (for a spheroidal deformation characterized by α_0) is the envelope of a number of intersecting parabolas, each one corresponding to a different configuration. While the local curvature of the envelope varies between C_0 (the average curvature of each parabola) and $-\infty$ (at the jumps), the average curvature of the envelope is only a small fraction of C_0. For this case the quantity $\alpha_\mu{}^\rho$ fluctuates rather strongly about α_μ. Similarly, the slope of the dashed curve shown in Fig. 2 is $\frac{1}{2}$ (pure distortion) between crossover points, and ∞ at jumps, since the configuration and $\alpha_\mu{}^\rho$ change discontinuously at these points. For this case the self-consistency condition is satisfied only on the average.

On the other hand, for pure rotation without change of shape, the energy is independent of orientation and Eq. (24) must be satisfied even without residual interactions, provided that the nucleus is at equilibrium deformation. For this case the mixing proceeds continuously rather than in jumps. In any case, the presence of residual interactions, especially those representing internucleon couplings, smooth out considerably any local deviations from self-consistency. Thus even for rather weak residual internucleon interactions, such as expected in nuclei, the minima in Fig. 1 are much shallower than without such interactions. Note that the residual interactions are expected to have an especially large effect near spherical shapes.[18]

The role of residual interactions near a crossover point is illustrated in more detail in Fig. 3. In the absence of residual interactions, the energy levels corresponding to configurations 1 and 2 (each of which is assumed to have one node) cross at (*c*). For deformations between (*a*) and (*c*), the system will be in configuration 1. As the deformation changes, say from (*a*) to (*b*), the wave function, and thus the node, is distorted without change of configuration. When the deformation crosses the value (*c*), the nucleus suddenly jumps into configuration 2 with a quite different wave function. The presence of residual interactions prevents the two energy levels from crossing and the configuration changes gradually from 1 to 2, a process of mixing. The range of deformations over which this change occurs is essentially proportional to the strength of the interaction.

It is seen that the velocity of the node when mixing occurs is much larger than it is for pure distortion. In the absence of residual interactions, the velocity would in fact, be infinite at the crossover point. The presence of residual interactions, by virtue of spreading out the region of the crossover, decreases the nodal velocity. In anticipation of the results in the next section, the collective motion corresponding to distortion is irrotational. The motion representing mixing involves large nodal velocities, and also large kinetic energy of the transported matter. It is motions of this kind which result in the large values of the inertial parameters for collective oscillations.

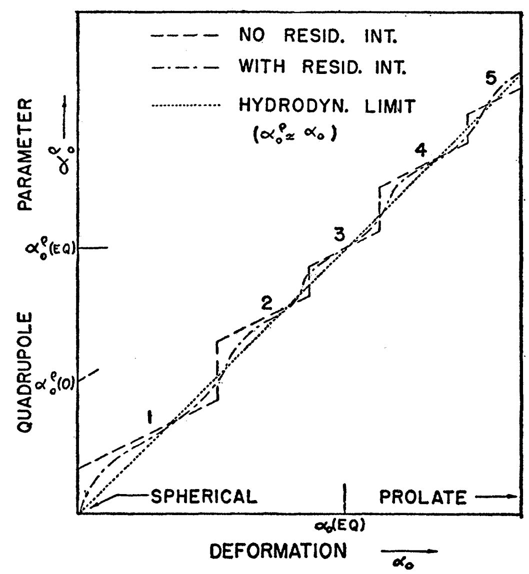

FIG. 2. Variation of quadrupole parameter as function of deformation for various strengths of the residual interactions. In this figure the quadrupole parameter $\alpha_0{}^\rho$ [defined by Eq. (13) of the text] is plotted against deformation for the same ground state configuration as in Fig. 1. We illustrate here the fact that of the quadrupole parameter at equilibrium, fully half is due to the intrinsic anisotropy of the density distribution in unfilled shells [denoted by $\alpha_\mu{}^\rho(0)$] and only half is due to core distortion.

It is seen from detailed work,[1,15] and illustrated in Fig. 1, that there are, as a rule, several equilibrium deformations corresponding to different configurations. Now, for any given configuration, the deformation giving the lowest energy is uniquely defined. [Equation (22) must be satisfied at equilibrium.] Conversely, for any particular deformation, the configuration which gives minimum energy is also, in principle, uniquely determined.[34] On the other hand, the value of the deformation (and associated configuration) giving the absolute minimum energy depends on the details of the intrinsic motion. The strong sensitivity of the absolute equilibrium deformation may be seen in Fig. 1. Thus a slight "tilting" of the energy curve could throw the absolute minimum from configuration 3 to configuration 2. An effect of this type has been postulated to be the cause of the striking difference between the empirical deformations of Eu^{151} and Eu^{153}.[29,30] Further, even when it is known which of the several minima is the absolute one, the value of the equilibrium deformation still depends somewhat on the detailed structure. Thus it is, in general, expected to decrease with increasing strength of the internucleon coupling. If this interaction were strong enough, the shell structure would be erased, and we would have the hydrodynamical limit for which the energy has the characteristic liquid-drop behavior

$$E_{\text{liq. drop}}=E_L(0)[1+C_2\beta^2-\tfrac{1}{3}C_3\beta^3\cos 3\gamma], \qquad (29)$$

where C_2 and C_3 are positive constants. The fractional elongations along the three principal axes are given by[14,15]

$$(\Delta R_n/R_0)=(5/4\pi)^{\frac{1}{2}}\beta\cos(\gamma-\tfrac{2}{3}\pi n). \qquad (30)$$

This curve is very shallow, gives a minimum for spherical shape, and discriminates less against prolate shapes ($\gamma=0$, $2\pi/3$, $4\pi/3$) than against oblate shapes ($\gamma=\pi/3$, π, $5\pi/3$) of the same β. As far as is known, all strongly deformed nuclei have axial symmetry[3] and positive quadrupole moments,[35] i.e., essentially prolate spheroidal shapes, apart from the influence of higher multipoles. The quadrupole moments are known to reach maximum values approximately midway between closed major shells. To calculates these equilibrium deformations from first principles, it would seem necessary to correct for the effect of deformation on the Coulomb and surface energies [e.g., as in Eq. (29)].

We note again that the large empirical equilibrium deformations imply that the configurations have strong intrinsic anisotropy. The values of empirical quadrupole moments require about ten unpaired nucleons add their intrinsic quadrupole moments. In a spheroidal field, neither the angular momentum of each nucleon, nor the total intrinsic angular momentum are good quantum numbers (only their components along the axis of symmetry are), since the coupling of the nucleons to the deformed nuclear shape dominates their direct coupling to each other.[1,36] In terms of an independent-particle description we must have, for these cases, strong configuration interaction.

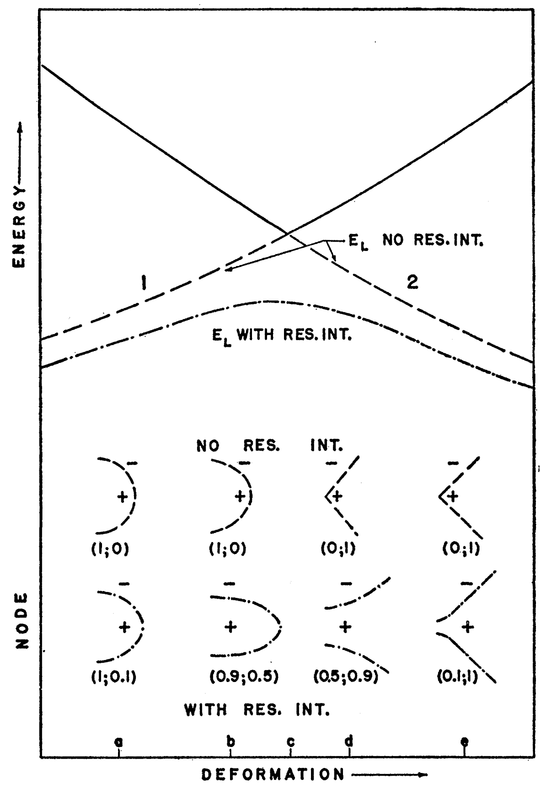

FIG. 3. Effect of residual interactions on the behavior of energy levels, configurations and nodes near a crossover point. As the deformation of the nucleus goes through the crossover point (c), the configuration is assumed to change from "1," a wave function with one radial node, to "2," a wave function with one angular node. For each configuration we indicate its node and the coefficients c_1, c_2, the approximate amplitudes of the two limiting configurations.

It may be instructive to compare the conclusions of the present work with those of Rainwater.[13] In Rainwater's paper, it was assumed that only a few unpaired nucleons provide the centrifugal force which causes the deformation, and that the rigidity of the nucleus against deformation has essentially the liquid-drop value. On the other hand, the configuration is not assumed to change with deformation. Thus on this model the outer nucleons determine the value of the quadrupole moment by virtue of their deforming influence, but most of this quadrupole moment arises from distortion of the core of filled shells. However, as

[34] There may be some ambiguity because of uncertainty regarding the residual interactions. However, Mottelson and Nilsson,[29] and independently Gottfried,[30] have been able to give a classification of nucleonic states in strongly deformed nuclei which agrees very well with experimental evidence.

[35] S. A. Moszkowski and C. H. Townes, Phys. Rev. **93**, 306 (1954).

[36] For configurations in which the direct coupling of the nucleons to each other dominates over the effect of deformation (e.g., L-S and j-j coupling) it seems difficult to obtain quadrupole moments larger than the single-particle values.

is emphasized in the present paper, large deformations can result only from the combined action of many nucleons. A small average rigidity results only if we permit the configuration to change with deformation.[37] Such changes of configuration are necessary anyway to maintain self-consistency. As will be seen in the next section, these changes are also at the root of the large inertial parameters.

Finally, it may be of interest to look more closely at the dependence of the nuclear density on the deformation. Since the deformation is assumed to be volume-preserving, no over-all change of the density should occur. Now, for a system of uniform density confined within an ellipsoid of arbitrary shape but fixed volume, it is readily shown that

$$\sum r_i^2(\alpha_\mu)=\sum r_i^2(0)[1+(5/4\pi)\sum_\mu|\alpha_\mu^\rho|^2]. \quad (31)$$

For an arbitrary density distribution, it follows from the virial theorem that

$$\sum r_i^2(\alpha_\mu)=(m\omega_0^2)^{-1}E_L(-\alpha_\mu). \quad (32)$$

Thus from Eqs. (21) and (23) we find, for any given configuration:

$$\sum r_i^2(\alpha_\mu)=[(m\omega_0^2)^{-1}E_L(\text{eq})] +[(5/4\pi)(\alpha_\mu^\rho)^2\sum r_i^2(0)]. \quad (33)$$

Making the plausible assumption (less restrictive than the self-consistency requirement) that $E_L(\text{eq})$ depends only weakly on α_μ, we see that (31) is valid for an arbitrary density distribution.

III. COLLECTIVE MOTION

In this section we study the effect of a slow time variation of the potential on the nuclear wave functions and energies. Such effects (i.e., the collective motions) depend very strongly on certain properties of the intrinsic states and especially on the behavior of the nuclear ground state as function of deformation, which was treated in the previous section.

To begin with, in view of the time-dependence of the potential, the nuclear system cannot be in any stationary state. The wave function corresponding to lowest mean energy is readily shown to be[23,38]

$$\psi=u_L+\hbar i\dot{\alpha}\sum_K[(\Delta E_{KL})^{-1}\langle K|\partial/\partial\alpha|L\rangle u_K], \quad (34)$$

to first order in $\dot{\alpha}$. Here α refers to any deformation coordinate (e.g., α_μ, θ, or β). For simplicity, we restrict ourselves to a consideration of a single deformation parameter, but the results may be easily generalized to take into account all five quadrupole deformation parameters. Associated with the complex wave function ψ is a current

$$\mathbf{j}=(\hbar^2/m)\dot{\alpha}\sum_K[(\Delta E_{KL})^{-1}\langle K|\partial/\partial\alpha|L\rangle \times(u_L\nabla u_K-u_K\nabla u_L)], \quad (35)$$

and an extra energy (above E_L), the kinetic energy of collective motion[18,23]

$$T_{\text{coll}}=\tfrac{1}{2}B\dot{\alpha}^2, \quad (36)$$

where

$$B=2\hbar^2\sum_K(\Delta E_{KL})^{-1}|\langle K|\partial/\partial\alpha|L\rangle|^2. \quad (37)$$

The quantity B is designated as the inertial parameter corresponding to the deformation variable α. If α is one of the quantities α_μ, then B is described more specifically as the mass parameter.[1,14] The density (and α_μ^ρ) adjusts itself adiabatically to the changing deformation, apart from terms of order $\dot{\alpha}^2$.

It is convenient to relate changes in the α_μ^ρ to changes in the potential with the following expression:

$$\dot{\alpha}_\mu^\rho=\dot{\alpha}_\mu[\sum_K w_K], \quad (38)$$

where w_K is defined to be the "weight" of an admixed excited intrinsic state.

With aid of definition (13) and of the following equation:

$$\langle K|\partial/\partial\alpha_\mu|L\rangle=(\Delta E_{KL})^{-1}m\omega_0^2\langle K|\sum_i r_i^2Y_{2\mu}^*|L\rangle, \quad (39)$$

it is readily verified that

$$w_k=[(8\pi/(5m\omega_0^2\sum r_i^2)]\Delta E_{KL}|\langle K|\partial/\partial\alpha_\mu|L\rangle|^2. \quad (40)$$

The self-consistency condition Eq. (24) requires that

$$\sum_K w_K=1. \quad (41)$$

It proves very convenient to express the mass parameter in terms of the weight of each admixed state, rather than in terms of the matrix elements as in Eq. (37). Then we obtain the simple result

$$B/B_I=2\sum_K(\hbar\omega_0/\Delta E_{KL})^2w_K, \quad (42)$$

where B_I refers to the value assuming irrotational flow, with maintainance of self-consistency (as for a hydrodynamical model).

$$B_I=(5/8\pi)m\sum r_i^2. \quad (43)$$

The equation for the mass parameter may also be written in more symmetrical form

$$(B/B_I)=[\sum_K(\Delta E_{KL})^2w_K][\sum_K(\Delta E_{KL})^{-2}w_K], \quad (44)$$

as is verified by use of the sum rule

$$\sum_K(\Delta E_{KL})^2w_K=2(\hbar\omega_0)^2. \quad (45)$$

Equation (43) holds provided only the potential energy is of the form (5) and independently of any details of the residual interactions.

It is seen immediately that the mass parameter can never be smaller than B_I if it is required that self-consistency be satisfied. The irrotational value is attained only if all admixed excited states are degenerate. This is indeed the case for a strict hydrodynamical model, where no distinction between the particles can be made. The excess of the mass parameter over the irrota-

[37] This was previously pointed out by S. Gallone and C. Salvetti, Nuovo cimento 10, 145 (1953), and by Hill and Wheeler.[15]

[38] Apart from a time-dependent phase factor of modulus unity.

tional value is a consequence of the spread of the excitation energies. This feature is a necessary consequence of the approximately independent motions of the nucleons. Thus the average excitation energy as defined by Eq. (43) is of order $\hbar\omega_0$, but for independent particle motion there are some excited states at much lower energies.

Let us now consider the nature of the collective oscillations allowing only distortion to occur. Of course, the resulting imperfect adjustment of the nucleonic density to the changing deformation expressed by (20) and by

$$(\textstyle\sum_K w_K)_{\text{dist}}=\tfrac{1}{2} \tag{46}$$

violates the self-consistency condition. However, this case, which has been considered by Inglis,[23] is quite instructive.

For pure distortion, the collective current is given by

$$(\mathbf{j})_{\text{dist}}=\rho\dot{\boldsymbol{\alpha}}^{\rho}\cdot\mathbf{r}=\rho\textstyle\sum_\mu \tfrac{1}{2}\dot{\alpha}_\mu{}^{\rho}\nabla(r^2Y_{2\mu}), \tag{47}$$

where ρ denotes the density, and the tensor $\boldsymbol{\alpha}^{\rho}$ is defined as in (8). Equation (47) may be readily verified by making a transformation to the double-primed coordinate system. In this system the wave functions are static, apart from a harmonic time factor. Thus the collective motion resulting from pure distortion is irrotational. Note that all nucleons (and thus mainly those in filled shells because of their greater number) participate in the flow. The mass parameter is given by

$$B_{\text{dist}}=\tfrac{1}{4}B_I. \tag{48}$$

The occurrence of the factor $\frac{1}{4}$ in this equation is a result of the breakdown of self-consistency. Thus the collective kinetic energy for this case is given by

$$T_{\text{coll}}=\tfrac{1}{2}B_I\textstyle\sum_\mu|\dot{\alpha}_\mu{}^{\rho}|^2=\tfrac{1}{2}B_{\text{dist}}\sum_\mu|\dot{\alpha}_\mu|^2. \tag{49}$$

In terms of Eq. (37), in the limit of small deformation, the nucleonic states admixed by the distortion are at an excitation energy $2\hbar\omega_0$, i.e., in the next major shell of the same parity.

The rigidity of the nucleus with respect to deformation (allowing only distortion) is given as follows [see Eqs. (28) and (43)]:

$$C_0=B_I\omega_0{}^2. \tag{50}$$

Thus the characteristic vibrational quantum energy is

$$(\hbar\omega_{\text{vib}})_{\text{dist}}=2\hbar\omega_0, \tag{51}$$

clearly violating the original assumption that the collective kinetic energy is small compared to the spacing of nucleonic energy levels.[23] Thus collective motion which consists entirely of distortion, e.g., oscillations of a closed shell configuration, cannot be adiabatic. Nonadiabatic oscillations of this type seem, however, to occur, and they result in enhanced probabilities for $E2$ transitions, even for light nuclei near magic numbers.[3,8]

Let us now consider the role of mixing. Clearly, specific effects of mixing depend on the details of the nucleonic level scheme. However, a general property of mixing must be that when it occurs simultaneously with distortion, the changing nuclear density satisfies the self-consistency condition. In view of Eqs. (41) and (46) it is then seen that

$$(\textstyle\sum_K w_K)_{\text{mix}}=\tfrac{1}{2}. \tag{52}$$

Thus the rigidity of the nucleus is reduced to a small fraction of C_0, as was indicated in Eq. (27). Perhaps an even more important effect of mixing is its dominant contribution to the mass parameter. This results from the fact, expressed by Eq. (42), that the contribution of an excited state to the mass parameter varies inversely as the square of its excitation energy. More specifically, the mass parameter is given, approximately, by the simple expression

$$(B/B_I)=\mathcal{E}^{-2}, \tag{53}$$

where $\mathcal{E}$ is an appropriately defined average excitation energy [Eq. (42)] in $\hbar\omega_0$ units, of admixed states with nucleons in the same major shells as the ground state. Thus we see that the dominant role of the few nucleons in unfilled shells on the collective motion results from the small excitation energies of some of the admixed states.[39] The large mass parameter and small rigidity resulting from mixing combine to give vibrational energies smaller than nucleonic excitation energies. Thus the adiabatic condition is usually satisfied.[40]

The role of distortion and mixing for the various collective properties of interest is shown in Fig. 4. Also in this figure, these results are compared with those based on a model assuming irrotational flow but self-consistency, e.g., the hydrodynamic limit.

The inertial parameters depend considerably on the strength of the internucleon interactions. As was pointed

	Q_0	$\dot{\alpha}_\mu^{\rho}$	ΔE_{KL}	IRROT.?	B	$\frac{\partial^2 E_L}{\partial\alpha_\mu^2}$	$\hbar\omega_{\text{vib}}$
DISTORTION (FILLED SHELLS)	$\frac{Q_0}{2}$	$\frac{\dot{\alpha}_\mu}{2}$	$2\hbar\omega_0$	YES	$\frac{B_I}{4}$	C_0	$2\hbar\omega_0$
MIXING (UNFILLED SHELLS)	$\frac{Q_0}{2}$	$\frac{\dot{\alpha}_\mu}{2}$	$\mathcal{E}\hbar\omega_0$ $[\mathcal{E}\ll 1]$	NO	$\frac{B_I}{\mathcal{E}^2}$	C_0	$\mathcal{E}\hbar\omega_0$
DISTORTION AND MIXING	Q_0	$\dot{\alpha}_\mu$	$\mathcal{E}\hbar\omega_0$	NO	$\frac{B_I}{\mathcal{E}^2}$	$C\,[\ll C_0]$	$\sqrt{\frac{C}{C_0}}\,\mathcal{E}\hbar\omega_0$
HYDRODYNAMIC LIMIT	Q_0	$\dot{\alpha}_\mu$	$\sqrt{2}\hbar\omega_0$	YES	B_I	C	$\sqrt{\frac{C}{C_0}}\,\hbar\omega_0$

FIG. 4. The role of nucleons in filled and unfilled shells for various collective properties.

[39] It is emphasized again that the simple numerical factors which appear in the present treatment result from the use of the H.O. potential. However, it is expected that regardless of the form of the average binding potential, the nucleons outside closed shells contribute (by virtue of mixing) the major share of the inertial parameters, and much more than the irrotational values.

[40] Unless, as occurs for some odd-A nuclei, there exists a nucleonic excited state of very low energy, much smaller than $\hbar\omega_0\mathcal{E}$.

out by Bohr and Mottelson,[18] the correlations induced by these slow down the collective flow below the values for independent particle motion and thus decrease the inertial parameters. In the language of Eqs. (42) and (43), the residual interactions increase the average energy denominator. If the residual interactions are strong enough to destroy the shell structure (i.e., if the mean free path for nucleons is small compared to the nuclear diameter), then we have essentially the hydrodynamical situation with irrotational flow.

The inertial parameters may also be expressed in a slightly different form. From Eq. (25) we see that the admixture of an excited nucleonic state decreases the rigidity of the nucleus against deformation by an amount proportional to its weight. Thus we obtain

$$C=2C_0-\textstyle\sum_K \Delta C_K, \tag{54}$$

where

$$\Delta C_K=2C_0 w_K. \tag{55}$$

The magnitude of the inertial parameter is simply related to the rigidity as shown by the following equation:

$$\left(\frac{B}{B_I}\right)=\sum_K\left(\frac{\hbar\omega_0}{\Delta E_{KL}}\right)^2\frac{\Delta C_K}{C_0}. \tag{56}$$

To obtain further insight into the role of energy differences for determining the essential features of the collective motion, we consider the simple case that the excited states admixed by a time-dependence of the potential are at only two different energies relative to the ground state. The energies are denoted by $\mathcal{E}_M$ and $\mathcal{E}_D$ (in units of $\hbar\omega_0$) for mixing and distortion, respectively. The weights of all states at each of these two energies are assumed to total $\frac{1}{2}$. We now define a new variable ζ as follows:

$$\tan\tfrac{1}{2}\zeta=(\mathcal{E}_M/\mathcal{E}_D). \tag{57}$$

It is then seen from Eq. (45) that

$$\mathcal{E}_M=2\sin\tfrac{1}{2}\zeta, \tag{58}$$

$$\mathcal{E}_D=2\cos\tfrac{1}{2}\zeta. \tag{59}$$

The mass parameter is given by

$$(B/B_I)=\csc^2\zeta. \tag{60}$$

The parameter ζ is evidently related to the strength of the residual interactions. If ζ is small, so is the energy of some of the admixed states, i.e., the spread of excited state energies is large. This situation, resulting from independent particle motion, leads to a large value of the mass parameter.

As ζ increases, the excitation energies move close together, and the inertial parameter decreases. When ζ reaches the limiting value $\frac{1}{2}\pi$, the excited states are degenerate, and the inertial parameter has the irrotational value, just as for the hydrodynamical model.

It is also of interest to study the behavior of the induced collective current for this case. Equation (35) may, after some manipulation, be rewritten (apart from sums over particles) as

$$\mathbf{j}=u_L{}^2\sum_\mu \dot{\alpha}_\mu\left[\frac{1}{\mathcal{E}_M\mathcal{E}_D}\nabla\left(\frac{r_2Y_{2\mu}u_L}{u_L}\right)+\frac{\mathcal{E}_D-\mathcal{E}_M}{(\mathcal{E}_M\mathcal{E}_D)^{\frac{3}{2}}}\nabla\left(\frac{Q_\mu u_2}{u_L}\right)\right]. \tag{61}$$

The terms inside the sum are the components of the collective velocity; only the first of these is irrotational. The function $Q_\mu u_L$ is defined as follows:

$$Q_\mu u_L=c_1(\partial u_L/\partial\alpha_\mu)+c_2(r^2Y_{2\mu}u_L),\quad (c_1>0), \tag{62}$$

where the coefficients are chosen so that $Q_\mu\mu_L$ has the same normalization as $r^2Y_{2\mu}u_L$ and is orthogonal to it. If the two levels are degenerate, the flow is evidently irrotational. As the two levels separate, the irrotational part of the flow increases in magnitude. However, another kind of flow now appears and in fact dominates for $\mathcal{E}_M\ll\mathcal{E}_D$, i.e., in the independent-particle limit.

The difference between these two motions can be understood as follows: Both flows are irrotational away from the nodes of the wave function.[41] Also whenever u_L vanishes, so does $r^2Y_{2\mu}u_L$; on the other hand, since Q_μ is a differential operator $Q_\mu u_L$ remains finite. Thus at the nodes, the velocity of the irrotational flow remains finite, but the velocity for the other kind of flow diverges, giving rise to vorticity of the collective flow.

IV. INTERPRETATION OF THE INERTIAL PARAMETERS

To make rough estimates for the magnitudes of the inertial parameters, we may apply Eq. (53). Consider first the rotational moments of inertia, about which the most is known. The nucleus is assumed to have prolate spheroidal shape and to be rotating slowly about an axis perpendicular to the symmetry axis. If the nucleons are assumed to move independently in the harmonic oscillator potential (i.e., if no residual interactions of any kind are present), then the moment of inertia has the rigid value provided we are at an equilibrium deformation.[18,42] The rigid moment is also attained in the statistical limit of independent particle motion; i.e., if the nucleus is a rotating Fermi-Thomas gas of arbitrary shape.[18,42] In fact, for this limiting case, there is no current at all with respect to the rotating system—the collective motion is simply rigid rotation.

For a nucleus of the shape considered here, the rigid and irrotational moments are related by the well-known equation

$$\mathfrak{J}_{\mathrm{irrot}}=\mathfrak{J}_{\mathrm{rig}}\epsilon^2, \tag{63}$$

[41] As was previously pointed out by G. C. Wick, Phys. Rev. **73**, 51 (1948), the induced flow is rigorously irrotational for a single particle in the (nodeless) ground state of an arbitrary potential.

[42] V. F. Weisskopf, Reported at Ottawa Conference on Theoretical Physics, June, 1955 (unpublished).

where ϵ is defined in terms of the shape parameters β by[14]

$$\epsilon=[45/(16\pi)]^{\frac{1}{2}}\beta\simeq 0.95\beta, \qquad (64)$$

and for small deformations

$$\epsilon=(R_{\rm maj}-R_{\rm min})/R_0, \qquad (65)$$

the fractional difference between major and minor axes. Thus, in the language of Eq. (53), the average excitation energy of states in the same major shell admixed by the rotation is given by

$$\mathcal{E}=\epsilon. \qquad (66)$$

The nucleonic ground state has Ω (the component of intrinsic angular momentum about the symmetry axis) equal to zero.[1] The rotation, by virtue of the Coriolis force, induces a small admixture of excited states with $\Omega=\pm 1$. Mixing represents coupling to close-lying states of this type. It is easily seen that whenever all unpaired nucleons are in the same major shell, then the close-lying excited $\Omega=\pm 1$ states are also in the same major shell as the ground state. Further, these states are all degenerate, at an excitation energy given, for small deformations, by $\hbar\omega_0\epsilon$. For this case, Eq. (53) is then true exactly, rather than in an "average" sense.

The use of an average excitation energy is useful for estimating the effect of residual interactions on the inertial parameters. The most important residual interaction is the pairing effect[18] resulting in a lowering of the ground state (all nucleons in paired orbits) relative to other states (e.g., $\Omega=\pm 1$) in which at least two nucleons are not paired. Let us say that the average excitation energy is thus increased by an amount $\hbar\omega_0 p$.

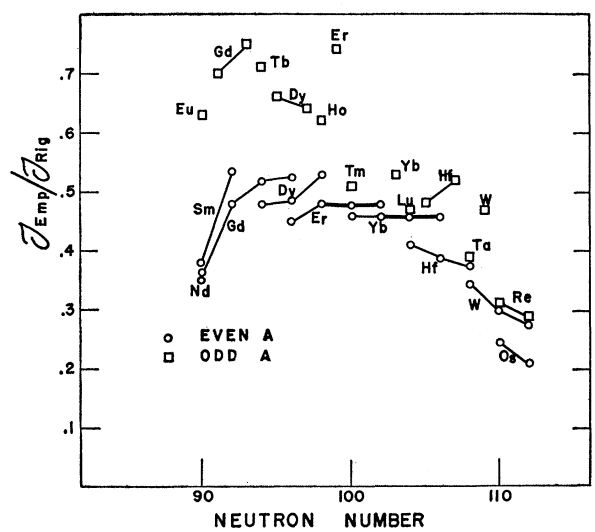

FIG. 5. Empirical moments of inertia for nuclei with mass numbers between 150 and 190 plotted against neutron number. Empirical moments of inertia are deduced from the position of the excited states. The rigid moment of inertia is calculated using Eq. (70), assuming a nuclear radius of $1.2\times 10^{-13}A^{\frac{1}{3}}$ cm. The numbers have been taken from the compilation of reference 18.

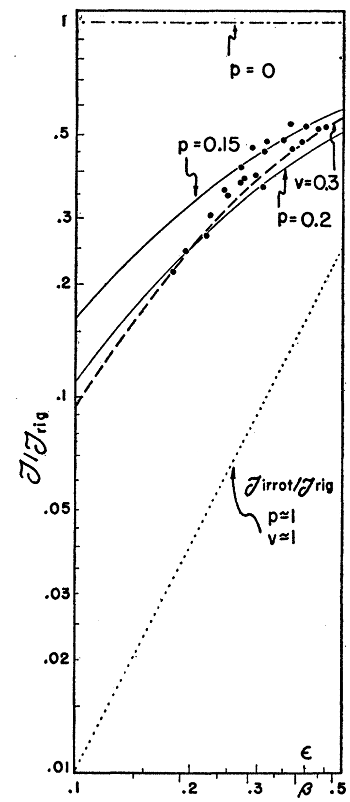

FIG. 6. Empirical moments of inertia for even-A nuclei with mass numbers between 150 and 190 plotted against deformation. Values of deformation (ϵ,β) are deduced from $E2$ transition probabilities. $(Q_0=\frac{4}{5}ZeR^2\epsilon)$. Numbers are taken from the compilation of reference 18. The curves indicate values of $\mathfrak{I}/\mathfrak{I}_{\rm rig}$, for several assumed values of the pairing term p [Eq. (77)]. The curve $v=0.3$ [Eq. (82)] is very similar to the corresponding curve in reference 18.

Then the moment of inertia is given as follows:

$$\mathfrak{I}=\mathfrak{I}_{\rm rig}\left(\frac{\epsilon}{\epsilon+p}\right)^2=\mathfrak{I}_{\rm irrot}\left(\frac{1}{\epsilon+p}\right)^2. \qquad (67)$$

For weak residual interactions, such as expected in nuclei, the resulting moment of inertia is smaller than the rigid value but larger than the irrotational value.

Empirical moments of inertia for strongly deformed nuclei with mass numbers between 150 and 190 are compared with the rigid values in Fig. 5. For even-even nuclei, the empirical moments are given in terms of the position of the first excited state by[1]

$$E_{0+\to 2+}=3\hbar^2/\mathfrak{I}. \qquad (68)$$

For odd-A nuclei (except when $I_0=\pm\frac{1}{2}$), they are given by

$$E_{I_0\to I_0+1}=(I_0+1)\hbar^2\mathfrak{I}. \qquad (69)$$

The rigid moment of inertia is calculated on the assumption that the nucleus is of uniform density. Then[18]

$$\mathfrak{I}_{\rm rig}=m\sum_i(y_i^2+z_i^2)=\tfrac{2}{5}MAR_0^2(1+\tfrac{1}{3}\epsilon\cdots), \qquad (70)$$

where R_0 is the nuclear radius averaged over directions.

In Fig. 6, it is shown that the empirical moments for even-A nuclei can be fitted quite well by Eq. (67) with values of p ranging from 0.15 for the most strongly deformed nuclei ($\epsilon\sim 0.4$) to 0.2 for the least deformed ones (but still exhibiting rotational spectra, $\epsilon\sim 0.2$).

It is also interesting to note that $\mathfrak{J}$ seems to be approximately proportional to the deformation. Thus we have shown, for the even-even nuclei,

$$\mathfrak{J}/\mathfrak{J}_{\mathrm{rig}} \simeq 1.3\epsilon. \tag{71}$$

This feature can be interpreted as resulting from a semirigid rotation of the nucleus[43]; i.e., rigid rotation of the outer nucleons and slow (irrotational) motion of the core. On the basis of such a model, the moment is easily shown to be given by

$$\mathfrak{J}/\mathfrak{J}_{\mathrm{rig}} = (Gf\epsilon) + O(\epsilon^2), \tag{72}$$

where G is a geometrical factor which depends only on the *angular* shape of the outer particle density distribution. For the reasonable distribution

$$\rho_{\mathrm{outer}} = \rho(r) \cos^2\theta, \tag{73}$$

we have $G=2$. The symbol f denotes the fraction of the quadrupole moment due to the outer particles; on basis of the H.O. model it is $\frac{1}{2}$. The coefficient of ϵ in Eq. (72) is then calculated to be unity, in good agreement with the experimental value 1.3, considering the crudity of the model.

Elements with mass numbers above 225, mainly of the actinide series, also exhibit striking rotational spectra.[6,44] For the even-even nuclei in this group, two features stand out. First, the rotational excitation energies and thus the moments of inertia, are roughly constant, and are about half the rigid values between mass numbers 234 and 250. The energy of the first excited state is close to 44 kev for all these nuclei.[6,44,45] As Blin-Stoyle recently pointed out,[46] this may indicate that while correlations are important for the nucleonic motions, these motions can in these limiting cases be described as approximately independent, but with an effective nucleon mass only about half of the value for a free nucleon.[47,48] This "saturation limit" is apparently reached more readily in the actinide series than for the rare earth elements. Another feature of interest concerns the relation of moments of inertia to deformation. Only for two nuclei, Th^{232} and U^{238}, are the intrinsic quadrupole moments known.[8] In both of these cases the moments of inertia are about 50% larger than the values calculated using Eq. (67) and with $p \simeq 0.15$–0.20. This large difference suggests the existence of higher multipoles, e.g., $E4$ moments, which are expected to become significant only for the heaviest elements. In fact, it is difficult to explain detailed features of the α-decay fine structure, e.g., the branching ratios for the $L=4$ mode as function of nucleon number, without invoking an $E4$ moment.[49,50] In other words, one may assume that the nuclear radius varies somewhat more with angle in the region of the least bound nucleon than would be the case for a spheroidal potential. No change in the intrinsic quadrupole moment results. On the other hand, the average excitation energy is increased, and consequently the residual interactions will be relatively less effective and a moment closer to the rigid value than for a spheroidal shape is expected. In addition to this, the parameter p is expected to be slightly smaller for the actinides than for the rare earths because of the larger nuclear size.[18]

Moments of inertia for odd-A nuclei are in general somewhat (10–40%) larger than those for neighboring even-even nuclei,[3,18] as is also shown in Fig. 5. In some cases (e.g., W^{183},[51] Np^{237} [52]), part of this difference can be attributed to the existence of a low-lying nucleonic state of the same parity as the ground state and with Ω differing from the ground state value by 1. While such states may have a small weight [Eq. (40)] because of their small excitation energy, so that they do not greatly affect the contribution of the other states, they may nevertheless contribute significantly to the inertial parameters. In addition, it is expected (see next section) that the effect of residual interactions is somewhat less for odd-A nuclei than for even-even nuclei. Both of these effects will enable the moments for odd-A nuclei to attain values closer to the rigid values than is the case for even-A nuclei. This is also seen especially for the actinide elements where the quantity $\hbar^2/2\mathfrak{J}$ is roughly constant at 6.2 kev for odd-A nuclei as compared to 7.3 for even-even nuclei. In any case, the larger moments of inertia for odd-A nuclei can be attributed to a comparatively large contribution of the unpaired nucleon to the collective motion.

Thus in a typical heavy nucleus, perhaps 10 to 20% of the nucleons are outside filled major shells. Yet these few nucleons are expected to contribute half of the quadrupole moment and fully 80 to 90% of the collective kinetic energy. Furthermore, for an odd-A nucleus, the unpaired nucleon contributes perhaps about 10 to 20 percent to the collective kinetic energy, as much as the entire core of filled shells!

The important role of the unpaired nucleon in odd-A nuclei is expected to show up in the empirical values of g_R, the gyromagnetic ratio of the collective motion. Thus it is easily shown that

$$g_R \simeq (Z/A) + [g_0 - (Z/A)](\Delta\mathfrak{J}/\mathfrak{J}), \tag{74}$$

where g_0 is the intrinsic gyromagnetic ratio of the unpaired nucleon, and $\Delta\mathfrak{J}$ is the contribution of this

[43] K. Ford and D. L. Hill, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), Vol. 5, p. 25.
[44] I. Perlman and F. Asaro, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 157.
[45] J. M. Hollander (to be published).
[46] R. J. Blin-Stoyle (to be published).
[47] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[48] M. H. Johnson and E. Teller, Phys. Rev. **98**, 783 (1955).
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[50] J. O. Rasmussen and B. Segall, University of California Radiation Laboratory Report UCRL-3040 (unpublished).
[51] A. Kerman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 15 (1956).
[52] Hollander, Smith, and Rasmussen, University of California Radiation Laboratory Report UCRL-3239 (unpublished).

nucleon to the moment of inertia (i.e., the difference between $\mathfrak{J}$ for the nucleus under consideration and the average value of $\mathfrak{J}$ for the neighboring even-even nuclei). The gyromagnetic ratio of the paired nucleons outside filled shells is usually close to Z/A.[18] While the value of g_0 depends on the state of the unpaired nucleon, this quantity is, in general, expected to be larger than Z/A for protons and smaller than Z/A for neutrons. A recent analysis of empirical $M1$ transition probabilities in odd-proton nuclei indicates that g_R exceeds $Z/A \simeq 0.4$ in four out of five cases,[53] averaging about 0.55. This result is consistent with the value expected from Eq. (74) if $g_0 \simeq 1$, a reasonable average value for an odd proton and if $\Delta\mathfrak{J} \simeq 0.25\mathfrak{J}$. For Np^{237}, the quantity g_R seems to be significently larger than Z/A.[52]†

While much is known about rotational moments of inertia, information regarding inertial parameters for shape oscillations is much more limited. Some conclusions can, however, be drawn from a study of the excitation spectra. Thus for even-even nuclei with mass numbers between 66 and 150 but away from magic numbers, both the first and second excited state have spin 2 and a ratio of excitation energies between 1.8 and 2.5.[54,55] This behavior has been qualitatively explained on the basis of the following assumptions[56]:

(1) The energy of the intrinsic ground state is taken for these nuclei, independent not only of the nuclear orientation, but also of the shape parameter[14] γ.

(2) The mass parameter has essentially the same values for oscillations in γ as for rotations.‡

Some information is also available regarding the characteristic of shape oscillations about a stable equilibrium shape, both from a study of vibrational excitation spectra and from a careful analysis of the very small deviations of the rotational spectra from the $I(I+1)$ law.[3,14] Both of these features involve the quantum frequencies for vibrations, which in turn depend not only on the values of the inertial parameters, but also on the average rigidity of the nucleus against changes in the shape. Thus, it is likely that the rigidity of the nucleus against changes in β is, in general, larger than its irigidity against changes in γ. This conclusion is suggested empirically by the appearance of the γ-oscillation spectra, and by the fact that the lowest known nonrotational excited even-parity states in even-even nuclei (Gd^{154},[57] W^{182},[58] Pu^{238} [59]) have spins 2. It is suggested theoretically by the hydrodynamic result, and by more detailed calculations of nucleonic energies as function of β and γ.[15,60] It is also expected and consistent with the data that the rigidity against β deformations is much smaller than the value C_0 [Eq. (28)] which would result from pure distortion, but somewhat larger than the hydrodynamic value.§

V. *p*-SHELL NUCLEON MODEL

To interpret some of the features discussed in the last section, it is useful to consider a highly simplified model of the intrinsic nuclear motion. In this paper we assume that the nucleus contains only one partially filled shell; a p shell.[61] The deforming force and the main features of the intrinsic and collective motion are then determined by the motion of the small number of nucleons of one kind in this shell. Also, we consider only nucleons, e.g., neutrons; thus it requires but 6 nucleons to fill the p shell. Further, we neglect any spin-orbit coupling. The resulting nuclear Hamiltonian consists of the following four terms: (*a*) The energy of all nucleons (core and p shell) moving independently in a spherical well; this energy is normalized to zero. (*b*) The interaction of each p-shell nucleon with the nonspherical part of the potential; this term is proportional to the deformation. (*c*) The interaction of the p-shell nucleons with each other; the internucleon force is taken to be a two-body attractive δ-function interaction. (*d*) The effect of deformation on the core of filled shells; this gives a restoring term proportional to the square of the deformation.

Let us first consider the level schemes for the configurations p^m, with n taking all possible values from 0 to 6. These are shown in Fig. 7. Only the interactions of particles in the p shell (*b*) and (*c*) are considered here, the restoring term (*d*) is not shown; for a given deformation it shifts all energies upward by the same amount. The deformation, taken to be prolate spheroidal, is denoted by ϵ [Eq. (64)], and the strength of the residual interactions is measured by v, the splitting

[53] G. M. Temmer and N. P. Heydenburg, Bull. Am. Phys. Soc. Ser. II, **1**, 43 (1956).

† *Note added in proof.*—A forthcoming review article[4] lists 13 odd-A nuclei for which information on the gyromagnetic ratios is available. For at least 9 of these, g_R appears to deviate significantly from Z/A in the expected direction. It is emphasized, however, that this trend is expected only on the average, and the fact that there are exceptions to it (e.g., the well-explored case of Ta^{181}) should not be surprising.

[54] G. Scharff-Goldhaber and J. Weneser, Phys. Rev. **98**, 212 (1955).

[55] M. Nagasaki and T. Tamura, Progr. Theoret. Phys. (Japan) **12**, 248 (1954).

[56] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).

‡ *Note added in proof.*—More detailed analysis of the empirical data[4] suggests that the effective mass parameters for shape oscillations (assuming they are adiabatic) are somewhat larger than for rotations. The ratio B/B_I seems to be largest for nuclei near closed shells. On the other hand, nonadiabatic effects are expected to be of considerable importance here and they tend to increase the inertial parameters above the value calculated by time-dependent perturbation theory.

[57] F. S. Stephens, Jr, University of California Radiation Laboratory Report UCRL-2970 (unpublished).

[58] Murray, Boehm, Marmier, and Du Mond, Phys. Rev. **97**, 1007 (1955).

[59] Rasmussen, Stephens, Strominger, and Aström, Phys. Rev. **99**, 47 (1955).

[60] M. Gursky, Phys. Rev. **98**, 1205(A) (1955).

§ *Note added in proof.*—Recent investigations of Pu^{238} [discussed in I. Perlman and J. O. Rasmussen, "Alpha radioactivity," S. Flügge, editor, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 42] show that there is a 0^+ state slightly below the nonrotational 2^+ state. This suggests that for heavy nuclei the surface rigidity with respect to β oscillations drops since we are approaching the saddle point for fission.

[61] Such a model has been employed by Bohr and Mottelson[1] who treated the p^2 configuration.

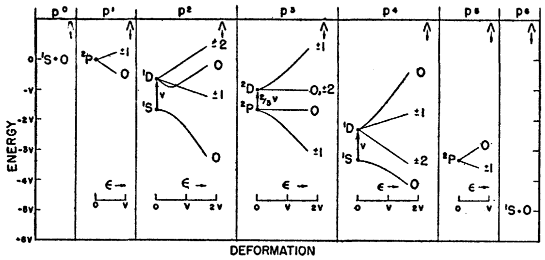

FIG. 7. Energy levels (in units of $\hbar\omega_0$) as function of deformation for configurations of interacting p-shell nucleons. Spin-orbit coupling is neglected. The component of orbital angular momenta about the symmetry axis is denoted by Λ, and its value indicated for each configuration. In this figure, the energy does not include any restoring terms.

between ^1S and ^1D states for p^2 and p^4 configurations at zero deformation.[18]

Only configurations of the lowest possible intrinsic spin (0 for even n and $\frac{1}{2}$ for odd n) are considered. States of higher spin are at considerable higher energy. For the spheroidal potential, the intrinsic orbital angular momentum is no longer a good quantum number, but its component Δ along the symmetry axis is. For the p^3 configuration, the ground state is 4-fold degenerate. This degeneracy is partially lifted when a spin-orbit coupling term is introduced, resulting in a low-lying nucleonic excited state. On the other hand, for all even-n configurations, the $\Lambda=0$ ground state is nondegenerate and is well separated from all excited states.

Note that as a result of the residual interparticle interactions, the level scheme for the p^3 configuration differs considerably from the schemes for p^1 and p^5. We may consider the p^3 configuration as a very highly simplified model of a deformed odd-A nucleus. It is also seen from Fig. 7 that the schemes for the con-

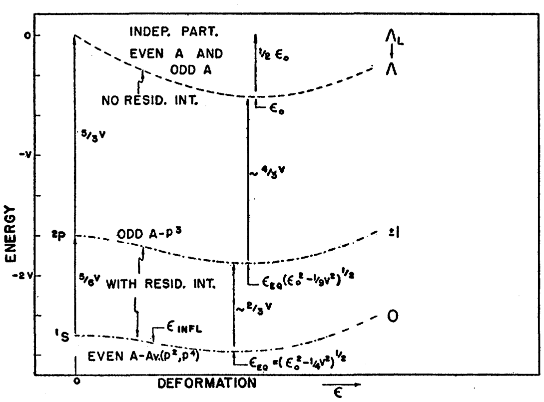

FIG. 8. Ground state energies as function of deformation for "typical" configurations of interacting p-shell nucleons. The energy includes a restoring term given by $\frac{1}{2}\epsilon^2/\epsilon_0$. For the lower two curves, the strength of the residual interactions is taken to be $v=0.3$. Values of the equilibrium deformation ϵ_{eq} (for which $\partial E_L/\partial\epsilon=0$) are indicated for all three cases. The inflection point ϵ_{infl} (here $\partial^2E_L/\partial\epsilon^2=0$) is indicated for the even-A case.

figurations p^2 and p^4 are quite similar. For calculational reasons, it is useful to take, as typical for an even-even nucleus, essentially the average of the p^2 and p^4 level schemes.

In Fig. 8 we compare the energies of the nucleonic ground state as function of deformation for the cases "even-A" (the average of p^2 and p^4), "odd-A" (p^3), and "indep. part.," the scheme which would be obtained for either of the above cases in the absence of residual interactions. A restoring term given by $\frac{1}{2}\epsilon^2/\epsilon_0$ has been included so as to give a value ϵ_0, for the equilibrium deformation in absence of residual interactions. The progression of deformations in heavy nuclei as function of degree of shell filling can be simulated by an appropriate variation of this coefficient.[18] In Fig. 9 we compare the intrinsic level scheme for these independent-particle, odd-A, and even-A configurations. As in Fig. 7, the restoring term is not included. Since the

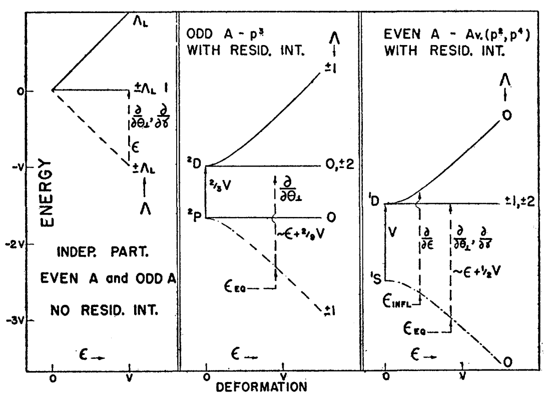

FIG. 9. Intrinsic energy levels as function of deformation for "typical" configurations of interacting p-shell nucleons. No restoring term has been included in the energy. Dashed arrows indicate average excitation energies of intrinsic states admixed by the particular kind of quadrupole oscillation (θ,γ,ϵ) indicated.

intrinsic spin is not coupled to the deformation in absence of spin-orbit coupling, it will not follow the nuclear rotation or any other change of the nuclear shape. On the other hand, the orbital angular momentum is strongly coupled to the nuclear shape and follows a slow rotation adiabatically. Thus for the present model, the Coriolis interaction, given in general by $i\hbar\omega(\partial/\partial\theta_\perp)$, is proportional to $L_\perp$, rather than to $J_\perp$ (where the $\perp$ indicates either direction perpendicular to the nuclear symmetry axis).

For the odd-A configuration, the energy of the $\Lambda=\pm1$ ground state is given by

$$E_L=-(\epsilon^2+\tfrac{1}{9}v^2)^{\frac{1}{2}}-(4/3)v, \tag{75}$$

and its wave function is

$$\psi_L=\cos\phi(^2P)-\sin\phi(^2D), \tag{76}$$

where

$$\cot2\phi=v/3\epsilon. \tag{77}$$

The Coriolis interaction couples the ground state to the $^2P(\Lambda=0)$ states at $-(5/3)v$ and to the $^2D(\Lambda=0,\pm2)$ states at $-v$. The appropriately weighted average energy difference $\mathcal{E}$ is $\epsilon+(2/9)v$ for large deformations.

The even-A level scheme is essentially the average of the rather similar schemes corresponding to p^2 and p^4 configurations shown in Fig. 7. The energy of the $\Lambda=0$ ground state is

$$E_L=-(\epsilon^2+\tfrac{1}{4}v^2)^{\frac{1}{2}}-2v, \tag{78}$$

and its wave function is

$$\psi_L=\cos\phi(^1S)-\sin\phi(^1D), \tag{79}$$

where now

$$\cot2\phi=v/2\epsilon. \tag{80}$$

The $\Lambda=0$ excited state (coupled to the ground state by oscillations in ϵ) is at energy $(\epsilon^2+\frac{1}{4}v^2)^{\frac{1}{2}}-2v$. Rotations and γ oscillations couple the ground state to states of $\Lambda=\pm1$ and $\Lambda=\pm2$, respectively. Note that all states with $\Lambda\neq0$ are pure ^1D regardless of deformation and have energy $-\frac{3}{2}v$. Thus the average excitation energy for the even-A case is given by

$$\mathcal{E}=(\epsilon^2+\tfrac{1}{4}v^2)^{\frac{1}{2}}+\tfrac{1}{2}v. \tag{81}$$

Let us now compare the predictions of this simple model with the empirical results. First, consider moments of inertia of even-A nuclei with mass numbers between 150 and 190. These can be fitted quite well by Eq. (67) where, using Eq. (81), we have

$$p=(\epsilon^2+\tfrac{1}{4}v^2)^{\frac{1}{2}}+\tfrac{1}{2}v-\epsilon, \tag{82}$$

and the value of v is taken as 0.3. (See Fig. 6.)

According to this model and also empirically, note that p decreases slightly with increasing deformation. Thus we have, for large deformations

$$p\rightarrow\tfrac{1}{2}v+[v^2/(8\epsilon)]. \tag{83}$$

Turning our attention now to the differences between odd-A and even-A nuclei, we note first of all (Figs. 8 and 9) that according to the p-shell model, the odd-A nuclei are expected to have properties intermediate between those for independent particle motion and for even-A nuclei. Indeed moments of inertia for odd-A nuclei are somewhat larger (5 to 40%) than for even-even nuclei, as was mentioned in Sec. IV, but not by as much (50 to 100%) as expected on the simple model given here.

The simple p-shell model also relates the strength of the residual interactions to the average binding energy difference between ground states of odd-A and even-A nuclei. According to this model, the difference should amount to about $\frac{2}{3}\hbar\omega_0 v$; i.e., $\simeq1$ to 1.5 Mev for nuclei of mass numbers considered here,[62] and somewhat larger than the empirical values. The fact that these features are qualitatively but not quantitatively reproduced with the p-shell model suggests that this model exaggerates somewhat (but not very much) the role of each individual nucleon. This behavior is not unexpected as it requires only 6 nucleons to fill the p shell but 76 nucleons to fill the last proton and neutron major shells before $_{82}Pb_{126}{}^{208}$.[63]

It is also instructive to compare the strength of the pairing forces deduced from moments of inertia on basis of the p-shell model with the empirical values for nearly spherical nuclei. Thus for Pb^{206}, lacking just two neutrons from being doubly magic, the energy of the first excited state is 803 kev, and the detailed analysis of the level scheme suggests a pairing energy (lowering of the ground state) of 0.77 Mev for this case.[64] According to the p-shell model, however, we should have found for both quantities a value $\hbar\omega_0 v \simeq 2$ Mev.

It may be of interest to remark on the possible even-odd staggering of equilibrium deformations. According to the p-shell model (Fig. 8) the odd-A nuclei are expected to be very slightly (5%) more deformed than neighboring even-A nuclei; i.e., the residual interaction decreases the equilibrium deformation. In general, the equilibrium deformation will be increased or decreased depending on whether the states admixed to the ground state by the residual interaction have larger or smaller quadrupole parameters than the ground state. For the p shell the admixed states are spherically symmetrical so that the deformation is always decreased; it is however, far from clear that this occurs also for heavy nuclei. At present, intrinsic quadrupole moments are not known accurately enough to draw reliable conslusions.[65]

Finally, let us briefly consider the inertial parameters for shape-oscillations in even-A nuclei according to the p-shell model. For the even-A configuration (but not for p^2 or p^4) the level scheme is easily shown to be independent of the shape parameter γ. Thus there is no rigidity with respect to γ oscillations, and the self-consistency condition [Eq. (24)] is satisfied. Such rigidity is expected to be comparatively small for most nuclei. As indicated in Fig. 9, the average energy denominators for γ oscillations ($\Delta\Lambda=\pm2$) and for rotations ($\Delta\Lambda=\pm1$) are equal and given by Eq. (81).

On the other hand, at equilibrium deformation, the ground state energy has an appreciable rigidity with respect to ϵ; [and from Eq. (25) we see that $\delta\alpha_0{}^p<\delta\alpha_0$], while in an actual nucleus it is expected that this rigidity will be much less. Now the inertial parameters are very essentially related to the decrease in the surface rigidity resulting from admixture of higher nucleonic states. It is, therefore, perhaps more appropriate to

[62] Assuming $\hbar\omega_0=41A^{-1/3}$ as was done by Nilsson.[27]

[63] Somewhat better agreement with experimental results could presumably be obtained by use of a model in which the unfilled nucleons are in the $2s$-$1d$ major shell of the harmonic oscillator potential.

[64] M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).

[65] However, an analysis of isotope shifts in optical spectra [Wilets, Hill, and Ford, Phys. Rev. **91**, 1488 (1953)] suggests that odd-A nuclei have appreciably smaller deformations than neighboring even-even nuclei.

evaluate the average energy difference for ϵ at the point of inflection of the ground state energy shown in Fig. 8. Although this deformation does not represent an equilibrium state, at least the quadrupole parameter and deformation change at the same rate, i.e., Eq. (24) is satisfied. The energy difference for oscillations in $\epsilon(\Delta\Lambda=0)$ is now nearly equal to the value for rotations and γ oscillations.

From the above arguments it is therefore plausible, and in agreement with experimental evidence (Sec. IV) that the mass parameter has approximately the same value for the various modes of quadrupole oscillations.

VI. CONCLUSIONS

The main conclusion of this paper is that the nucleons outside filled shells appear, in spite of their relatively small numbers, to be of greater significance in contributing to collective properties than was sometimes believed. According to the oscillator model used here, the outer particles contribute fully half of the static quadrupole moment; thus core distortion is important here but not dominant. Even more strikingly, the outer particles contribute the major share of the collective motion so that the nucleons inside filled shells play only a minor role for the dynamic properties.

This behavior resembles the situation in the usual shell model where the nucleons in filled shells do not, in general, enter in the properties of the low-lying states. The evident similarity of the shell model and collective model in this important respect is very encouraging for the prospect of constructing a unified model of nuclear structure.

ACKNOWLEDGMENTS

The author is indebted to Professor K. Ford, Dr. D. Inglis, Dr. G. Temmer, Professor J. A. Wheeler, and Dr. L. Wilets for a number of very stimulating discussions. He is also very grateful to Professor A. Bohr and Dr. B. R. Mottelson for informing him of the the results of their work before publication.

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Neutron Resonance Parameters of U^{235}†

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The neutron total cross section of $_{92}U^{235}$ has been studied for neutron energies between 1.5 and 60 ev by using the Brookhaven fast chopper. The transmission data were analyzed by the "area" method and the Breit-Wigner parameters obtained for the resonances up to 35 ev. The observed level spacing of 0.65 ev is the smallest yet reported for any isotope. The best available fission data have been combined with total cross-section data to obtain Γ_γ. The values of Γ_γ are approximately constant and are consistent with those of neighboring heavy nuclei, whereas the Γ_n^0's have a very broad distribution similar to that observed in a number of other elements. The distribution of Γ_f's is also rather broad and resembles the neutron widths much more than the radiation widths in this respect. Average values of these parameters are presented, together with a discussion of the implications of these findings for current theories of fission.

I. INTRODUCTION

In spite of the great importance of the low-energy neutron cross section of U^{235} for the design of various types of reactors, the data available in 1953 were still incomplete. Although it had been established that sharp resonances were present and that the ratio of capture to fission varied from resonance to resonance, the cross-section data were not accurate enough to give parameters of many resonances. Since an active program of the measurement of the parameters of resonances in nonfissionable nuclei was in progress with the Brookhaven fast chopper, it was decided to include U^{235} because of its fundamental nuclear physics interest and the value of the results for reactor design.

It is of interest to review the results that were available at the time the present work was begun. Very early measurements in 1939 by the Columbia group under Fermi,[1] using cadmium and boron filters, showed that the number of fission bursts in a fission counter approximately followed a $1/v$ law. In a theoretical paper in 1939 on the mechanism of nuclear fission, Bohr and Wheeler[2] concluded that fission widths of the resonances in U^{235} must be greater than 10 ev. Since radiation widths were estimated to be $\sim$0.1 ev, it was believed that the absorption of neutrons in U^{235} almost always led to fission, and that, since the level spacing was expected to be less than 10 ev, it was not expected that any sharp resonances would be found.

† Work performed under contract with U. S. Atomic Energy Commission.
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[1] Anderson, Booth, Dunning, Fermi, Glasoe, and Slack, Phys. Rev. **55**, 511 (1939).
[2] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).