Solving for S_1 and S_2, we find

$$S_1 = A_1[J' - (ab)^{\frac{1}{2}}]^2 - B_1[J' + (ab)^{\frac{1}{2}}]^2,$$
$$S_2 = -A_2[J' - (ab)^{\frac{1}{2}}]^2 + B_2[J' + (ab)^{\frac{1}{2}}]^2.$$

Using the relation $\lambda_1\lambda_2 = 1$, we have

$$A_1 = B_2 = (1-\lambda_2)/(\lambda_1-\lambda_2) > 0,$$
$$A_2 = B_1 = (1-\lambda_1)/(\lambda_1-\lambda_2) > 0.$$

Now if we take

$$t = \{A_2/A_1\}^{\frac{1}{2}}[J' - (ab)^{\frac{1}{2}}]/[J' + (ab)^{\frac{1}{2}}],$$

we find

$$x = \pm[5/4A_2(ab)^{\frac{1}{2}}] \int [(1-k^2t^2)(1-t^2)]^{-\frac{1}{2}} dt, \qquad k = A_1/A_2,$$

or

$$t = \mathrm{sn}(\pm 4A_2(ab)^{\frac{1}{2}}x, k); \quad J = (ab)^{\frac{1}{2}}(1+tk^{\frac{1}{2}})/(1-tk^{\frac{1}{2}}) + 1.$$

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Nonclassical Transformation in Special Relativity

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The "nonclassical" complex Lorentz transformation, recently introduced as a mathematical convenience by Reulos, is related to the unique factoring of any 4×4 orthogonal transformation. This factoring also shows, in 4×4 form, the well known possibility of introducing a 2×2 complex spin space, in which unimodular transformations are isomorphic to the proper future-preserving Lorentz group.

REULOS has proposed a new transformation as convenient for calculations in special relativity.[1] We may write its matrix in Minkowski space as

$$P = \begin{bmatrix} P_4 & -P_3 & P_2 & -P_1 \\ P_3 & P_4 & -P_1 & -P_2 \\ -P_2 & P_1 & P_4 & -P_3 \\ P_1 & P_2 & P_3 & P_4 \end{bmatrix}, \tag{1}$$

where

$$P_1^2 + P_2^2 + P_3^2 + P_4^2 = 1. \tag{2}$$

Another such transformation, of opposite chirality, also exists:

$$Q = \begin{bmatrix} Q_4 & -Q_3 & Q_2 & -Q_1 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ -Q_2 & -Q_1 & Q_4 & Q_3 \\ Q_1 & -Q_2 & -Q_3 & Q_4 \end{bmatrix}, \tag{3}$$

where

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = 1. \tag{4}$$

These two matrices are each orthogonal by inspection; they commute; they are unique in form (up to changes of sign of row or column); all P-type matrices form a group under matrix multiplication, and similarly for all Q type. Their multiplication rules are those of quaternions: for the Q type, however, the order of the quaternion factors must be reversed. Thus if we define the unimodular matrices

$$\mathbf{Q} = \begin{pmatrix} Q_4 - iQ_3 & -Q_2 + iQ_1 \\ Q_2 + iQ_1 & Q_4 + iQ_3 \end{pmatrix},$$
$$\mathbf{P} = \begin{pmatrix} P_4 + iP_3 & P_2 + iP_1 \\ -P_2 + iP_1 & P_4 - iP_3 \end{pmatrix}, \tag{5}$$

the multiplication rules of the 4×4 orthogonal matrices can now be written in 2×2 form:

$$\mathbf{Q}'' = \mathbf{Q}'\mathbf{Q}, \quad \mathbf{P}'' = \mathbf{P}\mathbf{P}'. \tag{6}$$

Now the remarkable fact is that *any* proper orthogonal transformation in Euclidean 4-space can be uniquely factored into a product of these two types:

$$A = QP. \tag{7}$$

Finally, if to the 4-vector (x_1,x_2,x_3,x_4) we associate a 2×2 matrix

$$\mathbf{x} = \begin{pmatrix} x_3 + ix_4 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 + ix_4 \end{pmatrix}, \tag{8}$$

it is immediately found that the determinant of $\mathbf{x}$ is the square of the length of x, and that the orthogonal transformation

$$x' = Ax \tag{9}$$

can also be written in 2×2 matrix form:

$$\mathbf{x}' = \mathbf{Q}\mathbf{x}\mathbf{P}. \tag{10}$$

[1] René Reulos, Phys. Rev. **102**, 535 (1956).

We note the law of composition of successive transformations:

$$\mathbf{x}''=\mathbf{Q}'\mathbf{x}'\mathbf{P}'=\mathbf{Q}'\{\mathbf{QxP}\}\mathbf{P}'=\{\mathbf{Q}'\mathbf{Q}\}\mathbf{x}\{\mathbf{PP}'\}=\mathbf{Q}''\mathbf{x}\mathbf{P}''. \quad (11)$$

The above relations seem to have been known for some time. The author discovered them in 1952, working from the parametric formulation of Cayley. Subsequently, a letter from S. R. Milner gave several earlier references,[2,3] and stated "I think [P and Q] must have been known from the early days of relativity, and quite likely from the early days of the classical quaternionists. But nobody seems to have been interested enough to pursue the matter further." The above results are also implicit in Milner's published work.[4]

To complete the resumé, in all coordinate frames of special relativity we wish x_4 to be intrinsically imaginary, and x_1, x_2, and x_3 real. This means that $\mathbf{Q}$ becomes the conjugate transpose of $\mathbf{P}$, say $\mathbf{P}^\dagger$, and (10) becomes

$$\mathbf{x}'=\mathbf{P}^\dagger\mathbf{x}\mathbf{P}. \quad (12)$$

This is now in the form of a generalization of the usual Cayley-Klein formalism for 3 dimensions to the 3+1 dimensions of space-time. The coordinate matrix $\mathbf{x}$ is still Hermitean, but in generalizing we have allowed its trace to be nonzero. The transformation matrix $\mathbf{P}$ still has unit determinant, but in generalizing we have given up its unitary character. Also, now, $A_{44}=P_1P_1^*+P_2P_2^*+P_3P_3^*+P_4P_4^*>0$, so the proper Lorentz transformation (12) is future-preserving.

[2] B. Qual, Bull. Sci. Math. **59**, 328 (1935).
[3] H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants* (Blackie and Son, Limited, London, 1929), p. 167.
[4] S. R. Milner, Proc. Roy. Soc. (London) **A214**, 292 (1952).

Equation (12) is, in fact, exactly the tensor transformation of the spinor calculus. It appears that our above results are now usually known in this modern algebraic guise, which followed the work of Pauli and Dirac. The often seen remark that "spin-space is a square-root space of space-time," and the advantages of linearization that occur by taking square roots in this manner,[1] can, we see, be as easily demonstrated by the factoring of any future-preserving proper Lorentz transformation, Eq. (7) subject to $\mathbf{Q}=\mathbf{P}^\dagger$:

$$A=\begin{pmatrix} P_4^* & -P_3^* & P_2^* & P_1^* \\ P_3^* & P_4^* & -P_1^* & P_2^* \\ -P_2^* & P_1^* & P_4^* & P_3^* \\ -P_1^* & -P_2^* & -P_3^* & P_4^* \end{pmatrix} \times \begin{pmatrix} P_4 & -P_3 & P_2 & -P_1 \\ P_3 & P_4 & -P_1 & -P_2 \\ -P_2 & P_1 & P_4 & -P_3 \\ P_1 & P_2 & P_3 & P_4 \end{pmatrix}; \quad (13)$$

the P's are complex, subject only to

$$P_1^2+P_2^2+P_3^2+P_4^2=1. \quad (14)$$

The transformation P, acting on a *vector* $\mathbf{P}=(P_1,P_2,P_3,P_4)$ gives the purely time-like vector (0,0,0,1). The full transformation A, acting on $\mathbf{P}$, gives just $\mathbf{P}^*$. In Reulos' case (P_1, P_2, P_3 pure imaginary and P_4 real), $\mathbf{P}$ is interpretable as a real relative 4-velocity of two coordinate frames; A then transforms the description to a frame in which the corresponding vector 3-velocity appears oppositely directed.