It is worthwhile to comment that the agreement of measurements of $B(E2)$ between different laboratories and from conversion electron or gamma-ray observations is still no better than a factor of 2. It is noted that results for Ta differ from those of Stelson and McGowan[1] by about 20%, while the results on Re and Ir are in fair agreement with those of Huus[1] but vary as much as 50% from those of Bernstein and Lewis[28] and Fagg *et al.*[29] The Hg results are consistently lower than those of the Saclay group.[34] Some of the discrepancy can be traced to choice of conversion coefficients, but there appears to be a need for more accurate measurements, especially when one attempts to determine mixing ratios from the intensity ratio of crossover to cascade radiations.

ACKNOWLEDGMENTS

We are indebted to Dr. C. P. Keim of the Isotopes Division of Oak Ridge National Laboratory for making the separated isotopes available and to Dr. H. Wilhelm of the Ames Laboratory for supplying the purified thorium metal. Dr. J. J. van Loef, Burton Gasten, L. C. Northcliffe, and Richard Greiner gave considerable service in the experimental measurements.

PHYSICAL REVIEW VOLUME 103, NUMBER 6 SEPTEMBER 15, 1956

Bremsstrahlung and Pair Production in Condensed Media at High Energies

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(Received May 11, 1956)

The effect of multiple scattering on bremsstrahlung and pair production is considered. The probability of these processes decreases considerably at energies $\gtrsim 10^{13}$ ev.

The calculations are carried out with the aid of the density matrix. The formulas thus obtained yield the probability of pair production and bremsstrahlung for arbitrary electron and photon energies.

I. INTRODUCTION

At high energies, when the directions of the particles participating in pair production and bremsstrahlung almost coincide, large longitudinal distances begin to play an important role. Thus, if a photon of wavelength $\lambdabar$ is emitted during bremsstrahlung, a certain length $l\sim\lambdabar/(1-v/c)$ is found to be essential, v being the electron velocity. Landau and Pomeranchuk[1,2] have shown that multiple scattering over this length leads to a significant decrease of the probability of the aforementioned processes. An estimate of the cross sections for bremsstrahlung and pair production in the limiting case of ultra-high energies ($E\gg 10^{13}$ ev) is given in reference 2.

The intensity of emission of soft photons by electrons of arbitrary energy has been computed previously.[3] In that paper the classical formula for intensity of emission by an electron moving along a given trajectory was averaged over all possible trajectories. This procedure was carried out by means of the distribution function which was averaged over the positions of the atoms of the scattering medium and which satisfies the usual kinetic equation.

The aim of the present paper is the deduction of formulas for the probability of bremsstrahlung (formula 61) and pair production (formula 63) per unit path in a condensed medium for arbitrary photon and electron energies. This is done by connecting the transition probability with the density matrix and then using the equation for the density matrix averaged over the scattering atom coordinates deduced previously.[4,5] At low energies formulas (61) and (63) transform into the Bethe-Heitler formula[6]; in the limiting case of ultra-high energies the formulas confirm the estimation obtained in reference 2. At photon energies much lower than that of the electron, formula (61) changes into the expression obtained in reference 3. Finally, for very soft photons, when the deviation of the dielectric constant from unity is important, formula (56) of the present paper yields in the limiting case the same results as those of Ter-Mikaelyan.[7]

Formulas (61) and (63) can be used to construct a theory of shower production in condensed materials at energies exceeding 10^{13} ev.

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II. RELATION BETWEEN TRANSITION PROBABILITY AND AVERAGED DENSITY MATRIX

The probabilities for bremsstrahlung and pair production must be averaged over all possible distributions of the atoms of the scattering material. We first express the radiation transition rate through the density matrix and then make use of the equation for the averaged density matrix obtained in references 4 and 5.

Restricting our treatment to the first approximation with respect to the electron-radiation field interaction and denoting the electron proper functions in the scattering medium by ψ_s and the initial electron wave function by ψ_0, we get

$$i\dot{c}_s^{(1)}=\sum_{s'}(\psi_s|e^{iHt}Ae^{ikt}e^{-iHt}|\psi_{s'})c_{s'}^0 =(\psi_s|e^{iHt}Ae^{ikt}e^{-iHt}|\psi_0),$$

where A is the radiative transition operator

$$A=\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu e^{-i\mathbf{k}\cdot\mathbf{r}}e(2\pi/k)^{\frac{1}{2}}.$$

$\boldsymbol{\varepsilon}_\nu$ is the polarization vector, $\mathbf{k}$ the photon wave vector, H the electron Hamiltonian which includes the potential of all the scatterers

$$H=H_0+\sum_m V(\mathbf{r}-\mathbf{r}_m);\quad H\psi_s=E_s\psi_s.$$

The system of units in which $m=\hbar=c=1$ has been chosen. The electron and photon ψ functions have been normalized per unit volume.

The radiative transition rate is then given by

$$Q_s=\frac{d}{dt}|c_s|^2=2\,\mathrm{Re}\dot{c}_s^*c_s=2\,\mathrm{Re}\int_0^t(\psi_0|e^{iHt_1}A^\dagger e^{-iHt_1}|\psi_s) \times(\psi_s|e^{iHt}Ae^{-iHt}|\psi_0)e^{ik(t-t_1)}dt_1. \quad (1)$$

We now determine the rate of transition to all final states of the electron. For bremsstrahlung, which we first consider, one must sum over all values of s corresponding to positive electron energies. Introducing an energy sign operator (projection operator) $K=[H+|E(\mathbf{p})|]/2|E(\mathbf{p})|$, where $\mathbf{p}$ is the electron momentum operator, and applying the relation

$$\sum_s\psi_s^*(x)\psi_s(x')=\delta(x-x'),$$

we obtain

$$Q=\sum_{E_s>0}Q_s=2\,\mathrm{Re}\int_0^t(\psi_0|e^{iHt_1}A^\dagger Ke^{iH(t-t_1)}Ae^{-iHt}|\psi_0) \times e^{ik(t-t_1)}dt_1. \quad (2)$$

At high energies the operator K in (2) is essentially the same as the free-electron operator $K_0=(H_0+|E_p^0|)/2|E_p^0|$. Replacing K by K_0, we see that the coordinates of the scattering centers enter (2) only through factors of the form $e^{\pm iHt}$.

We shall now show that it is possible to average independently over the scatterer coordinates entering through the factors $e^{\pm iHt_1}$ and $e^{\pm iH(t-t_1)}$. Indeed, the nuclear coordinates in the first expression correspond to collisions which take place in a time interval $0-t_1$, whereas in the second expression the coordinates correspond to scatterers which undergo collisions at a later period t_1-t.

Suppose that a large number of collisions takes place during a time t_1; in this case, after one averages over the first collisions, the factors of the type $e^{\pm iHt_1}$ will practically cease to be dependent on the collisions taking place at times close to t_1, and this is just why independent averaging was found to be possible.

We now write the integrand of (2) as the matrix element of the product of the operators in the representation of the free-electron wave functions

$$\varphi_p^\lambda=u_p^\lambda e^{i\mathbf{p}\cdot\mathbf{r}}.$$

Assuming $\psi_0=\varphi_{p_0}^{\lambda_0}$ (where $\mathbf{p}_0$ is the initial electron momentum) and designating the average by the sign $\langle\ \rangle$, we obtain from (2)

$$\langle Q\rangle=2\,\mathrm{Re}\int_0^t d\tau e^{ik\tau}I, \quad (3)$$

where

$$I=\langle(\psi_0|e^{iHt_1}A^\dagger K_0e^{iH\tau}Ae^{-iH\tau}e^{-iHt_1}|\psi_0)\rangle =\sum_{\mathbf{p}_1\lambda_1\mathbf{p}_2\lambda_2}\langle(\mathbf{p}_0\lambda_0|e^{iHt_1}|\mathbf{p}_1\lambda_1)(\mathbf{p}_1\lambda_1|A^\dagger K_0e^{iH\tau}Ae^{-iH\tau}|\mathbf{p}_2\lambda_2) \times(\mathbf{p}_2\lambda_2|e^{-iHt_1}|\mathbf{p}_0\lambda_0)\rangle,$$

and $\tau=t-t_1$. As the factors $e^{\pm iHt_1}$ and $e^{\pm iH\tau}$ are statistically independent, we find

$$I=\sum_{p_1\lambda_1p_2\lambda_2}\langle(p_2\lambda_2|e^{-iHt_1}|p_0\lambda_0)(p_0\lambda_0|e^{iHt_1}|p_1\lambda_1)\rangle \times\langle(p_1\lambda_1|A^\dagger K_0e^{iH\tau}Ae^{-iH\tau}|p_2\lambda_2)\rangle.$$

It should be noted that at high energies, where small relative changes of the electron momentum are important, scattering does not change the spinor state, i.e.,

$$(\mathbf{p}\lambda|e^{\pm iHt}|\mathbf{p}'\lambda')=\delta_{\lambda,\lambda'}(\mathbf{p}\lambda|e^{\pm iHt}|\mathbf{p}'\lambda),$$

with an error of the order $|\mathbf{p}'-\mathbf{p}|/p$; and in this case

$$I=\sum_{\mathbf{p}_1\mathbf{p}_2}\langle(\mathbf{p}_2\lambda_0|e^{-iHt_1}|\mathbf{p}_0\lambda_0)(\mathbf{p}_0\lambda_0|e^{iHt_1}|\mathbf{p}_1\lambda_0)\rangle \times(\mathbf{p}_1\lambda_0|A^\dagger K_0e^{iH\tau}Ae^{-iH\tau}|\mathbf{p}_2\lambda_0\rangle. \quad (4)$$

The first factor in (4), considered as a function of the variables $\mathbf{p}_1$, $\mathbf{p}_2$, and t_1, satisfies the same equation as the averaged density matrix

$$\langle(\mathbf{p}_1\lambda_0|\rho|\mathbf{p}_2\lambda_0)\rangle=\langle(\mathbf{p}_1\lambda_0|e^{-iHt_1}\rho_0e^{iHt_1}|\mathbf{p}_2\lambda_0)\rangle.$$

It follows from this equation[4,5] that the difference $\mathbf{p}_2-\mathbf{p}_1$ remains constant during scattering (this is a result of the uniformity of the scattering medium). As

the first factor in (4) equals

$$\delta_{\mathbf{p}_1,\mathbf{p}_0}\delta_{\mathbf{p}_2,\mathbf{p}_0} \quad \text{for} \quad t_1=0,$$

it may be written as follows:

$$\langle(\mathbf{p}_2\lambda_0|e^{-iHt_1}|\mathbf{p}_0\lambda_0)(\mathbf{p}_0\lambda_0|e^{iHt_1}|\mathbf{p}_1\lambda_0)\rangle = f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)\delta_{\mathbf{p}_2,\mathbf{p}_1}. \quad (5)$$

The second factor in (4) may be written as the sum of the operator products in the momentum representation. Using (5) and the expression for the operator A, we obtain

$$\begin{aligned}\langle(\mathbf{p}_1\lambda_0|A^\dagger K_0 e^{iH\tau}Ae^{-iH\tau}|\mathbf{p}_1\lambda_0)\rangle = &\frac{2\pi e^2}{k}\Big\langle \sum_{\substack{\mathbf{p}\lambda_1\\ E^{\lambda_1}>0}} (\mathbf{p}_1\lambda_0|\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu|\mathbf{p}_1-\mathbf{k}\,\lambda_1)\\ &\times(\mathbf{p}_1-\mathbf{k}\,\lambda_1|e^{iH\tau}|\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1)\\ &\times(\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1|\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu|\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0)\\ &\times(\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0|e^{-iH\tau}|\mathbf{p}_1\lambda_0)\Big\rangle. \quad (6)\end{aligned}$$

Here

$$(\mathbf{g}_1\mu_1|\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu|\mathbf{g}_2\mu_2)=(u_{\mathbf{g}_1}^{\mu_1},\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu u_{\mathbf{g}_2}^{\mu_2}),$$

where $u_{\mathbf{g}1,2}^{\mu1,2}$ are spinor functions.

Let

$$\langle(\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0|e^{-iH\tau}|\mathbf{p}_1\lambda_0)(\mathbf{p}_1-\mathbf{k}\,\lambda_1|e^{iH\tau}|\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1)\rangle \equiv f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p},\tau).$$

The equation coefficients and initial conditions (10) and (12) for $f_{\mathbf{k}}^{\lambda_0,\lambda_1}(\mathbf{p},\tau)$ and $f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)$ are independent on the spin orientation and therefore, on summing over λ_0 and λ_1 for a fixed energy sign, one may drop the spinor indices in these functions.

Inserting (5) and (6) in (4), summing over the photon polarization, and averaging over the initial state spins, we get

$$I_1=\tfrac{1}{2}\sum_{\substack{\lambda_0,\,\nu\\ E^{\lambda_0}>0}} I = \frac{\pi e^2}{k}\int \mathfrak{L}(\mathbf{p}_1,\mathbf{p})f_0(\mathbf{p}_1,t_1)f_{\mathbf{k}}(\mathbf{p},\tau)\frac{d\mathbf{p}_1}{(2\pi)^3}\frac{d\mathbf{p}}{(2\pi)^3}, \quad (7)$$

where

$$\mathfrak{L}(\mathbf{p}_1,\mathbf{p})=\sum_{\substack{\lambda_0\lambda_1\nu\\ E^{\lambda_0},\,E^{\lambda_1}>0}} (\mathbf{p}_1\lambda_0|\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu|\mathbf{p}_1-\mathbf{k}\,\lambda_1) \times(\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1|\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_\nu|\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0). \quad (8)$$

It may be noted that the quantity

$$\begin{aligned}(\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0|\rho|\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1) &= (\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0|e^{-iH\tau}|\mathbf{p}_1\lambda_0)(\mathbf{p}_1-\mathbf{k}\,\lambda_1|e^{iH\tau}|\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1)\\ &= (\mathbf{p}+\tfrac{1}{2}\mathbf{k}\,\lambda_0|e^{-iH\tau}\rho_0 e^{iH\tau}|\mathbf{p}-\tfrac{1}{2}\mathbf{k}\,\lambda_1)\end{aligned}$$

satisfies the equation $\partial\rho/\partial\tau=-i[H,\rho]$. Moreover,

$$\begin{aligned}\mathrm{Sp}\rho &= \sum_{\mathbf{g}_1\mu_1}(\mathbf{g}_1\mu_1|\rho|\mathbf{g}_1\mu_1)\\ &= \sum_{p\lambda_1}(p_1\lambda_0|e^{iH\tau}|p\lambda_1)(p\lambda_1|e^{-iH\tau}|p_1\lambda_0)\\ &=1,\end{aligned}$$

i.e., ρ is an element of the density matrix in the momentum representation. We shall call the quantity $f_{\mathbf{k}}(\mathbf{p},\tau)$ the averaged density matrix.

The problem of averaging the transition rate reduces to determination of the averaged density matrix and to evaluation of the sum (8) and the integral (7).

III. EQUATION FOR AVERAGED DENSITY MATRIX

As was shown in[4] the averaged density matrix satisfies the equation

$$\begin{aligned}\frac{\partial f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p},\tau)}{\partial\tau}&+i(E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}^{\lambda_0}-E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}^{\lambda_1})f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p},\tau)\\ &= n\pi\int\frac{d\mathbf{p}'}{(2\pi)^3}|V_{\mathbf{p}'-\mathbf{p}}|^2\{\delta(E_{\mathbf{p}'+\frac{1}{2}\mathbf{k}}^{\lambda_0}-E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}^{\lambda_0})\\ &\quad+\delta(E_{\mathbf{p}'-\frac{1}{2}\mathbf{k}}^{\lambda_1}-E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}^{\lambda_1})\}\\ &\quad\times[f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p}',\tau)-f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p},\tau)], \quad (9)\end{aligned}$$

together with the initial condition which follows from the definition of $f_{\mathbf{k}}(\mathbf{p},\tau)$:

$$f_{\mathbf{k}}^{\lambda_0\lambda_1}(\mathbf{p},\tau)|_{\tau=0}=\delta_{\mathbf{p},\mathbf{p}_1-\mathbf{k}/2}. \quad (10)$$

This equation differs from the classical kinetic equation in that the difference $E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}^{\lambda_0}-E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}^{\lambda_1}$ enters the left-hand side instead of $(\partial E/\partial\mathbf{p})\cdot\mathbf{k}$ and the collision term is the half-sum of the usual collision terms for the momentum $\mathbf{p}+\frac{1}{2}\mathbf{k}$ and energy $E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}^{\lambda_0}$ and for the momentum $\mathbf{p}-\frac{1}{2}\mathbf{k}$ and energy $E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}^{\lambda_1}$. For $k\ll p$, (9) changes into the classical kinetic equation for the $\mathbf{k}$th Fourier component of the distribution function. The function $f_0(\mathbf{p}_1,t_1)$ satisfies the equation

$$\begin{aligned}\frac{\partial f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)}{\partial t_1}&=2\pi r\int\frac{d\mathbf{p}'}{(2\pi)^3}|V_{\mathbf{p}'-\mathbf{p}_1}|^2\delta(E_{\mathbf{p}_1}^{\lambda_0}-E_{\mathbf{p}'}^{\lambda_0})\\ &\quad\times[f_0^{\lambda_0\lambda_0}(\mathbf{p}',t_1)-f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)], \quad (11)\end{aligned}$$

and initial condition

$$f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)|_{t_1=0}=\delta_{\mathbf{p}_1,\mathbf{p}_0}. \quad (12)$$

From (11) and (12), we find

$$\int f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)\frac{d\mathbf{p}_1}{(2\pi)^3}=1. \quad (13)$$

It follows from (11) and (12) that $f_0^{\lambda_0\lambda_0}$ differs from zero only for $p_1=p_0$, and therefore a function $v_0(\boldsymbol{\theta},t_1)$

can be introduced, in accord with the formula

$$f_0^{\lambda_0\lambda_0}(\mathbf{p}_1,t_1)\frac{d\mathbf{p}_1}{(2\pi)^3}=\delta(p_1-p_0)v_0(\boldsymbol{\theta},t_1)dp_1d\boldsymbol{\theta},\qquad v_0(\boldsymbol{\theta},t_1)|_{t_1=0}=\delta(\boldsymbol{\theta}), \tag{14}$$

where $\boldsymbol{\theta}=(\mathbf{p}_1-\mathbf{p}_0)/p_0$ is a vector in the direction of the difference between $\mathbf{p}_0$ and $\mathbf{p}_1$. From (13), we obtain

$$\int v_0(\boldsymbol{\theta},t_1)d\boldsymbol{\theta}=1. \tag{14$'$}$$

It is easy to see that $f_{\mathbf{k}}(\mathbf{p},\tau)$ differs from zero only for values of p which are close to $g=p_0-\frac{1}{2}k$. For $\tau=0$, it follows from (10) that

$$p^2=p_0^2+\frac{k^2}{4}-p_0k+p_0k(1-\cos(\mathbf{p}_1,\mathbf{k}))=g^2\left(1+\frac{p_0k}{2g^2}\eta_1^2\right),$$

where η_1 is a small angle between $\mathbf{p}_1$ and $\mathbf{k}$.

We now introduce vectors corresponding to the angles between $\mathbf{p}$ and $\mathbf{k}$ and between $\mathbf{p}'$ and $\mathbf{k}$:

$$\boldsymbol{\eta}=\mathbf{p}_\perp/g;\quad \boldsymbol{\eta}'=\mathbf{p}_\perp'/g;\quad g=p_0-\tfrac{1}{2}k. \tag{15}$$

Here $\mathbf{p}_\perp$ and $\mathbf{p}_\perp'$ designate the projections of $\mathbf{p}$ and $\mathbf{p}'$ on the plane perpendicular to $\mathbf{k}$. The δ functions in the right-hand part of (9) may be rewritten as follows

$$\delta(E_{\mathbf{p}'\pm\frac{1}{2}\mathbf{k}}-E_{\mathbf{p}\pm\frac{1}{2}\mathbf{k}})\cong\delta\left(p'-p\pm\frac{gk(\eta'^2-\eta^2)}{4(g\pm\frac{1}{2}k)}\right)\cong\delta(p'-p).$$

Thus during collisions the modulus of $\mathbf{p}$ remains approximately constant, with an accuracy $\delta p/p\sim\eta^2$. One may use the approximate constancy of p to determine the function $v(\boldsymbol{\eta},\tau)$ from the formula

$$f_{\mathbf{k}}(\mathbf{p},\tau)\frac{d\mathbf{p}}{(2\pi)^3}=\delta(p-g)v(\boldsymbol{\eta},\tau)dpd\boldsymbol{\eta}. \tag{16}$$

From condition (10), we obtain

$$v(\boldsymbol{\eta},\tau)|_{\tau=0}=\delta(\boldsymbol{\eta}-\boldsymbol{\eta}_0);\quad \boldsymbol{\eta}_0=\frac{(\mathbf{p}_1-\frac{1}{2}\mathbf{k})_\perp}{g}=\frac{\mathbf{p}_{1\perp}}{g}, \tag{17}$$

where $\boldsymbol{\eta}_0$ is the vector of the angle between $\mathbf{p}_1-\frac{1}{2}\mathbf{k}$ and $\mathbf{k}$. Vector $\boldsymbol{\eta}_0$ is related to vector $\boldsymbol{\theta}$ introduced in the foregoing by the relation

$$\boldsymbol{\theta}=\frac{\mathbf{p}_1}{p_0}-\frac{\mathbf{p}_0}{p_0}=\frac{\mathbf{p}_1}{p_0}-\mathbf{n}+\mathbf{n}-\frac{\mathbf{p}_0}{p_0}=\frac{\mathbf{p}_{1\perp}}{p_0}+\boldsymbol{\vartheta}=\frac{g}{p_0}\boldsymbol{\eta}_0+\boldsymbol{\vartheta}. \tag{18}$$

The vector of the angle between $\mathbf{n}=\mathbf{k}/k$ and the initial direction of the electron has been denoted by $\boldsymbol{\vartheta}$. From definition (15), we obtain

$$\begin{aligned}&\mathbf{p}=(\mathbf{pn})\mathbf{n}+\mathbf{p}_\perp\cong g\mathbf{n}+g\boldsymbol{\eta};\\ &\mathbf{p}+\tfrac{1}{2}\mathbf{k}\cong p_0\mathbf{n}+g\boldsymbol{\eta};\quad \mathbf{p}-\tfrac{1}{2}\mathbf{k}\cong(p_0-k)\mathbf{n}+g\boldsymbol{\eta}.\end{aligned} \tag{19}$$

The difference $E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}{}^{\lambda_0}-E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}{}^{\lambda_1}$ in (9) takes the form

$$\begin{aligned}E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}{}^{\lambda_0}-E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}{}^{\lambda_1}&=[1+(p_0\mathbf{n}+g\boldsymbol{\eta})^2]^{\frac{1}{2}}-[1+[(p_0-k)\mathbf{n}+g\boldsymbol{\eta}]^2]^{\frac{1}{2}}\\&=k\left[1-\frac{1}{2p_0(p_0-k)}-\frac{g^2}{2p_0(p_0-k)}\eta^2\right].\end{aligned} \tag{20}$$

Using (16) and integrating (9) over p, we get

$$\frac{\partial v(\boldsymbol{\eta},\tau)}{\partial\tau}+i(a-\tfrac{1}{2}b\eta^2)v(\boldsymbol{\eta},\tau)=\frac{ng^2}{(2\pi)^2}\int|V(g(\boldsymbol{\eta}'-\boldsymbol{\eta}))|^2\{v(\boldsymbol{\eta}',\tau)-v(\boldsymbol{\eta},\tau)\}d\boldsymbol{\eta}, \tag{21}$$

where

$$a=k\left(1-\frac{1}{2p_0(p_0-k)}\right);\quad b=\frac{g^2k}{p_0(p_0-k)};\quad \boldsymbol{\eta}'=\frac{\mathbf{p}_\perp'}{g}. \tag{21$'$}$$

For $V(\mathbf{f})$ we adopt the expression

$$V(\mathbf{f})=\frac{4\pi Ze^2}{f^2+\kappa^2},\quad \kappa\sim\frac{1}{a}, \tag{22}$$

where a is the Thomas-Fermi radius $a\sim137/Z^{\frac{1}{3}}$.

Inserting this in (21), we get

$$\frac{\partial v}{\partial\tau}+i(a-\tfrac{1}{2}b\eta^2)v=\frac{4nZ^2e^4}{g^2}\int\frac{d\boldsymbol{\eta}'}{[(\boldsymbol{\eta}'-\boldsymbol{\eta})^2+\theta_1^2]^2}[v(\boldsymbol{\eta}',\tau)-v(\boldsymbol{\eta},\tau)],\qquad \theta_1=\kappa/g. \tag{23}$$

Expanding $v(\boldsymbol{\eta}',\tau)$ into a power series of $\boldsymbol{\eta}'-\boldsymbol{\eta}$, we obtain from (23) the Fokker-Planck differential equation:

$$\frac{\partial v}{\partial\tau}+i(a-\tfrac{1}{2}b\eta^2)v=q\Delta_\eta v,\qquad q=\frac{2\pi nZ^2e^4}{g^2}\ln\left(\frac{\theta_2}{\theta_1}\right)=\frac{B}{g^2}. \tag{24}$$

The quantity θ_2 may be determined from the condition of applicability of the Fokker-Planck expansion. The first term of the series expansion is

$$\int_{\theta_1}^{\theta_2}\frac{\theta d\theta}{\theta^4}\theta^2\tfrac{1}{4}\Delta_\eta v=\tfrac{1}{4}\ln\left(\frac{\theta_2}{\theta_1}\right)\Delta_\eta v.$$

The next term is of the order

$$\int_{\theta_1}^{\theta_2}\frac{\theta d\theta}{\theta^4}\theta^4\frac{\partial^4 v}{\partial\eta^4}\sim\theta^2\frac{1}{\eta^2}\Delta_\eta v.$$

The quantities

$$\int_0^\infty ve^{ik\tau}d\tau=\int_0^\infty v'd\tau \quad \text{and} \quad v'=e^{ik\tau}v$$

will be needed in further calculations. The equation for v' can be obtained from Eq. (24) by replacing a by $a'=a-k$.

The significant values of η^2 are determined by the relation

$$(a'-\tfrac{1}{2}b\eta^2)v'\sim b\eta^2v'\sim q\Delta_\eta v'; \quad \eta^2\sim(q/b)^{\frac{1}{2}}. \qquad (25)$$

This estimation will be confirmed below.

Thus, the condition for expansion of v into a series has the form

$$\theta_2^2(b/q)^{\frac{1}{2}}\sim\ln(\theta_2/\theta_1); \quad \theta_2\sim(q/b)^{\frac{1}{4}}L^{\frac{1}{2}}; \quad L=\ln(\theta_2/\theta_1). \qquad (26)$$

At sufficiently high energies, θ_2 may be of the same order of magnitude as the angle of diffraction by the nucleus which is equal to $1/gR$, and in this case the upper limit of integration with respect to $|\boldsymbol{\eta}'-\boldsymbol{\eta}|$ is determined by the quantity $1/gR$. Putting $R\cong 0.5r_0Z^{\frac{1}{3}}$, we obtain, for $\theta_2>1/gR$,

$$L=\ln(137^2/0.5Z^{\frac{2}{3}})=2\ln(190/Z^{\frac{1}{3}}). \qquad (27)$$

IV. SUMMATION OVER ELECTRON SPIN AND PHOTON POLARIZATION

It is not possible to carry out the summation over λ_0 and λ_1 in expression (8) in the usual manner since the momentum subscripts in the spinor functions are different.

The summation in (8) can be reduced to the determination of the trace of two-row matrices. The spinor functions are taken in the form

$$u_g^\mu=\begin{Bmatrix} v_\mu \\ [\boldsymbol{\sigma}\mathbf{g}/(E_g+1)]v_\mu \end{Bmatrix}N_g; \quad v_1=\begin{Bmatrix}1\\0\end{Bmatrix};$$
$$v_2=\begin{Bmatrix}0\\1\end{Bmatrix}; \quad N_g^2=\frac{1}{1+[g^2/(E_g+1)^2]}\cong\frac{1}{2}, \qquad (28)$$

where σ_1, σ_2, and σ_3 are the Pauli matrices. The "3" axis is oriented along $\mathbf{n}$.

Substitution in (8) gives

$$\mathfrak{L}(\mathbf{p}_1,\mathbf{p})=\tfrac{1}{4}\sum_{i=1,2}\mathrm{Sp}\left\{\left[\frac{\sigma_i\boldsymbol{\sigma}(\mathbf{p}_1-\mathbf{k})}{E_{\mathbf{p}_1-\mathbf{k}}+1}+\frac{\boldsymbol{\sigma}\mathbf{p}_1\sigma_i}{E_{\mathbf{p}_1}+1}\right]\times\left[\frac{\sigma_i\boldsymbol{\sigma}(\mathbf{p}+\frac{1}{2}\mathbf{k})}{E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}+1}+\frac{\boldsymbol{\sigma}(\mathbf{p}-\frac{1}{2}\mathbf{k})\sigma_i}{E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}+1}\right]\right\}. \qquad (29)$$

We introduce the notation

$$\mathbf{A}=\frac{\mathbf{p}_1}{E_{\mathbf{p}_1}+1}; \quad \mathbf{C}=\frac{\mathbf{p}+\frac{1}{2}\mathbf{k}}{E_{\mathbf{p}+\frac{1}{2}\mathbf{k}}+1};$$
$$\mathbf{B}=\frac{\mathbf{p}_1-\mathbf{k}}{E_{\mathbf{p}_1-\mathbf{k}}+1}; \quad \mathbf{D}=\frac{\mathbf{p}-\frac{1}{2}\mathbf{k}}{E_{\mathbf{p}-\frac{1}{2}\mathbf{k}}+1}. \qquad (30)$$

From (29) we then get

$$\mathfrak{L}(\mathbf{p}_1,\mathbf{p})=\tfrac{1}{4}\sum_{i=1,2}\mathrm{Sp}(\sigma_i\mathbf{B}\boldsymbol{\sigma}+\mathbf{A}\boldsymbol{\sigma}\sigma_i)(\sigma_i\boldsymbol{\sigma}\mathbf{C}+\mathbf{D}\boldsymbol{\sigma}\sigma_i)$$
$$=\mathbf{BD}+\mathbf{AC}-(\mathbf{Bn})(\mathbf{Cn})-(\mathbf{An})(\mathbf{Dn}). \qquad (31)$$

Each of the terms in (31) is close to unity. However, as further calculations show, the complete expression is of order η^2. We now express (31) as a sum of small terms, in each of which only the first term of the expansion in powers of $1/p_0$ is retained.

Each of the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ can be expressed as the sum of two terms, one of these being parallel and the other perpendicular to $\mathbf{n}$: $\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2$; $\mathbf{A}_1\|\mathbf{n}$; $\mathbf{A}_2\perp\mathbf{n}$ and similarly for $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$.

From (31), we then obtain

$$L=(D_1-C_1)(B_1-A_1)+\mathbf{B}_2\mathbf{D}_2+\mathbf{A}_2\mathbf{C}_2. \qquad (32)$$

The magnitude of each term in (32) is of order

$$\sim 1/p_0^2 \quad \text{or} \quad \eta^2.$$

Using (17), (19), and (30), we find with the required accuracy

$$A_1=C_1=1-\frac{1}{p_0}; \quad B_1=D_1=1-\frac{1}{p_0-k};$$
$$\mathbf{A}_2=\frac{g\boldsymbol{\eta}_0}{p_0}; \quad \mathbf{B}_2=\frac{g\boldsymbol{\eta}_0}{p_0-k}; \quad \mathbf{C}_2=\frac{g\boldsymbol{\eta}}{p_0}; \quad \mathbf{D}_2=\frac{g\boldsymbol{\eta}}{p_0-k}. \qquad (33)$$

Insertion in (32) gives

$$\mathfrak{L}=K_1+K_2\boldsymbol{\eta}\boldsymbol{\eta}_0,$$
$$K_1=\frac{k^2}{p_0^2(p_0-k)^2}; \quad K_2=\frac{g^2[p_0^2+(p_0-k)^2]}{p_0^2(p_0-k)^2}. \qquad (34)$$

V. PROBABILITY OF BREMSSTRAHLUNG

Let $W_r(p_0,k)dk$ designate the probability of emission per unit length of a photon having an energy lying between k and $k+dk$. The initial electron ψ function is normalized per unit volume or, for $c=1$, per unit flux; thus from formulas (3) and (7), we obtain

$$W_r=\tfrac{1}{2}\sum_{\substack{\lambda_0,\nu\\ E^{\lambda_0}>0}}\int\langle Q\rangle d\boldsymbol{\vartheta}\frac{k^2}{(2\pi)^3}$$
$$=\frac{k^2}{(2\pi)^3}2\,\mathrm{Re}\int_0^t d\tau e^{ik\tau}\int I_1 d\boldsymbol{\vartheta}. \qquad (35)$$

Inserting into (7) the expressions for v_0, v, and $\mathfrak{L}$ defined by formulas (14), (16), and (34), we obtain

$$\int I_1 d\boldsymbol{\vartheta}=\frac{\pi e^2}{k}\int v_0(\boldsymbol{\theta},t_1)v(\boldsymbol{\eta},\tau)[K_1+K_2\boldsymbol{\eta}\boldsymbol{\eta}_0]d\boldsymbol{\theta}d\boldsymbol{\eta}d\boldsymbol{\vartheta}.$$

Expressing $\boldsymbol{\theta}$ through $\boldsymbol{\eta}_0$ and $\boldsymbol{\vartheta}$ by formula (18), we get

$$\int I_1 d\boldsymbol{\vartheta} = \frac{\pi e^2}{k}\int v_0\left(\frac{g}{p_0}\boldsymbol{\eta}_0+\boldsymbol{\vartheta},\, t_1\right)v(\boldsymbol{\eta},\tau) \times (K_1+K_2\boldsymbol{\eta}\boldsymbol{\eta}_0)\frac{g^2}{p_0^2}d\boldsymbol{\eta}_0 d\boldsymbol{\eta} d\boldsymbol{\vartheta}.$$

Using the normalization condition (14′) for v_0, we obtain

$$W_r = \frac{e^2g^2k}{(2\pi)^2p_0^2}\,\mathrm{Re}\int_0^t d\tau e^{ik\tau}\int (K_1+K_2\boldsymbol{\eta}\boldsymbol{\eta}_0) \times v(\boldsymbol{\eta},\tau)d\boldsymbol{\eta}d\boldsymbol{\eta}_0. \quad (36)$$

We denote the significant value of τ in this integral by τ_0. From (24) and (25), we obtain the estimates

$$\eta^4\sim q/b,\quad \tau_0\sim 1/b\eta^2\sim 1/(bq)^{\frac{1}{2}},\quad \eta^2\sim q\tau_0. \quad (37)$$

If the time of motion of the electron in the medium, t, is much greater than τ_0, the upper limit of the integral in (35) with respect to τ can be replaced by infinity and W_r ceases to depend on t.

Only the case $t\gg\tau_0$ or $l\gg l_k$ will be considered below; l is the thickness of the scattering material and $l_k=c\tau_0$.

For a condensed medium ($n=3\times10^{22}$ cm^{-3}), one obtains from (37):

$$l_k\sim\frac{p_0}{\sqrt{k}}\frac{1}{Z}\times10^{-5}\text{ cm}. \quad (38)$$

For $Z=10$, $E_0=10^{10}$ ev ($p_0=10^{10}/5\times10^5=2\times10^{10}$), and $k=\frac{1}{2}p_0$, one obtains $l_k\sim0.2$ cm.

For $t\gg\tau_0$, the angular distribution of the photons can easily be found. The width of the angular distribution defined by the function $v_0(\boldsymbol{\theta},t_1)$ is of the order $\langle\theta^2\rangle_{\text{Av}}\sim qt_1$. However, $\eta^2\sim\eta_0^2\sim q\tau$; therefore the function $v_0(g\boldsymbol{\eta}_0/p_0+\boldsymbol{\vartheta},\, t_1)$ can be replaced by $v_0(\boldsymbol{\vartheta},t)$ and the photon angular distribution is given by the expression

$$W_r'(p_0,k,\boldsymbol{\vartheta})d\boldsymbol{\vartheta} = v_0(\boldsymbol{\vartheta},t)W_r(p_0,k)d\boldsymbol{\vartheta}. \quad (39)$$

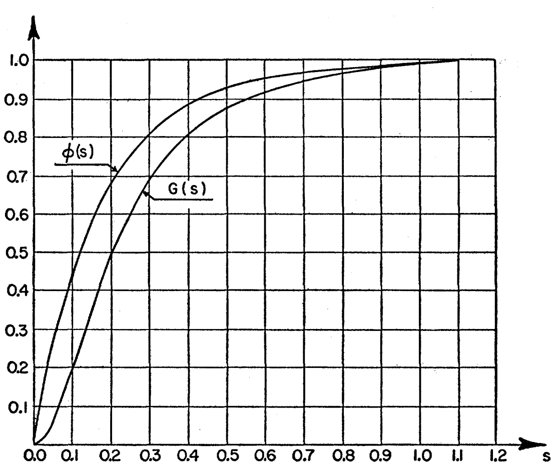

FIG. 1. Values of the functions $\phi(s)$ and $G(s)$ of Eq. (47).

Thus, the photon angular distribution is the same as that for multiple scattered electrons with an energy p_0.

Let

$$\int v(\boldsymbol{\eta},\tau;\boldsymbol{\eta}_0)d\boldsymbol{\eta}_0\equiv h_1(\boldsymbol{\eta},\tau);\quad \int v(\boldsymbol{\eta},\tau;\boldsymbol{\eta}_0)\boldsymbol{\eta}_0 d\boldsymbol{\eta}_0\equiv \mathbf{R}(\boldsymbol{\eta},\tau).$$

Since the coefficients in Eq. (24) do not contain $\boldsymbol{\eta}_0$, the equations for h_1 and $\mathbf{R}$ will coincide with (24) and the initial conditions $h_1(\boldsymbol{\eta},\tau)|_{\tau=0}=1$; $\mathbf{R}(\boldsymbol{\eta},\tau)|_{\tau=0}=\boldsymbol{\eta}$ follow directly from (17).

The coefficients in Eq. (24) contain only η^2, and the solution can therefore be written in the form $h_1(\boldsymbol{\eta},\tau)=h(z,\tau)$; $\mathbf{R}(\boldsymbol{\eta},\tau)=\boldsymbol{\eta}g(z,\tau)$, where $z=\frac{1}{2}\eta^2$; h and g satisfy the equations

$$\frac{\partial h}{\partial\tau}+i(a-bz)h=2zq\left(\frac{\partial^2 h}{\partial z^2}+\frac{1}{z}\frac{\partial h}{\partial z}\right),$$
$$\frac{\partial g}{\partial\tau}+i(a-bz)g=2zq\left(\frac{\partial^2 g}{\partial z^2}+\frac{2}{z}\frac{\partial g}{\partial z}\right). \quad (40)$$

We introduce the functions

$$\varphi_1(z)=\int_0^\infty e^{ik\tau}h(z,\tau)d\tau,$$
$$\varphi_2(z)=z\int_0^\infty e^{ik\tau}g(z,\tau)d\tau. \quad (41)$$

Then, according to (36),

$$W_r=\frac{e^2kg^2}{2\pi p_0^2}\,\mathrm{Re}\left\{K_1\int_0^\infty\varphi_1 dz+2K_2\int_0^\infty\varphi_2 dz\right\}. \quad (42)$$

The equations for φ_1 and φ_2 can be obtained by integrating (40) over τ and using the initial conditions for h and g:

$$z\varphi_1''+\varphi_1'+i(\alpha+\beta z)\varphi_1=-1/2q, \quad (43)$$

$$z\varphi_2''+i(\alpha+\beta z)\varphi_2=-z/2q, \quad (44)$$

TABLE I. Values of the functions $\phi(s)$ and $G(s)$ of Eq. (47).

s	$\phi(s)$	$G(s)$
0	0	0
0.05	0.258	0.094
0.1	0.446	0.206
0.2	0.686	0.475
0.3	0.805	0.695
0.4	0.880	0.800
0.5	0.931	0.875
0.6	0.954	0.917
0.7	0.965	0.945
0.8	0.975	0.963
0.9	0.985	0.975
1.0	0.990	0.985
1.5	0.998	0.994
2.0	0.999	0.998

where

$$\alpha=\frac{k-a}{2q}=\frac{k}{4p_0(p_0-k)q}>0;\quad \beta=\frac{kg^2}{2p_0(p_0-k)q}>0. \quad (45)$$

The solution of Eqs. (43) and (44) and the computation of the integrals in (42) are carried out in the Appendix. The following results are obtained:

$$\mathrm{Re}\int_0^\infty \varphi_1 dz=\frac{1}{12q\alpha^2}G(s);\quad \mathrm{Re}\int_0^\infty \varphi_2 dz=\frac{1}{6\alpha\beta q}\phi(s), \quad (46)$$

$$G(s)=48s^2\left(\frac{\pi}{4}-\frac{1}{2}\int_0^\infty e^{-st}\frac{\sin st}{\sinh(t/2)}dt\right),$$

$$\phi(s)=12s^2\int_0^\infty \coth\frac{x}{2}e^{-sx}\sin sx dx-6\pi s^2, \quad (47)$$

$$s=\frac{1}{8g}\left(\frac{k}{p_0(p_0-k)q}\right)^{\frac{1}{2}}.$$

The function $\phi(s)$ was introduced in reference 3. The values of $\phi(s)$ and $G(s)$ are presented in Table I, and are plotted in Fig. 1.

The asymptotic behavior of ϕ and G is given by the formulas

$$\phi_{s\to 0}\to 6s;\qquad \phi_{s\to\infty}\to 1-\frac{0.012}{s^4};$$

$$G_{s\to 0}\to 12\pi s^2;\quad G_{s\to\infty}\to 1-\frac{0.022}{s^4}. \quad (48)$$

For $s\lesssim 1$, one obtains

$$\eta_0^2\sim\eta^2\sim\frac{\mathrm{Re}\int\varphi_2 dz}{\mathrm{Re}\int\varphi_1 dz}\sim\frac{\alpha}{\beta s}\sim\frac{1}{\sqrt{\beta}}=\sqrt{\left(\frac{2q}{b}\right)},$$

which confirms the estimate of the significant values of η^2, η_0^2 given above.

Substitution of (24), (34), (45), and (46) into (42) yields

$$W_r=\frac{e^2kg^2}{2\pi p_0^2}\left\{K_1\frac{G}{12q\alpha^2}+2K_2\frac{\phi}{6\alpha\beta q}\right\}$$

$$=\frac{2e^2}{3\pi p_0^2 k}B\{k^2G(s)+2[p_0^2+(p_0-k)^2]\phi(s)\}; \quad (49)$$

$$B=2\pi z^2e^4n\ln(\theta_2/\theta_1).$$

The estimate (26) for θ_2 can be expressed in a more convenient form:

$$\theta_2\sim(q/b)^{\frac{1}{4}}L^{\frac{1}{4}}=(1/2\beta)^{\frac{1}{4}}L^{\frac{1}{4}}\sim L^{\frac{1}{4}}/gs^{\frac{1}{2}}. \quad (50)$$

Put $s\gtrsim 1$; then

$$W_r=\frac{2e^2}{3\pi p_0^2 k}B\{k^2+2[p_0^2+(p_0-k)^2]\}.$$

This expression differs from the familiar formula (see, for example reference 6) only by a factor of the order of unity under the logarithm sign. The uncertainty of the factor under the logarithm sign is a result of application of the Fokker-Planck method. Solution of the integral equation (23), a difficult task, should yield more precise formulas.

Since the functions ϕ and G are close to unity for $s=1$, a convenient formula can be obtained by defining the numerical factor under the logarithm sign in the following manner:

$$L=\ln(\theta_2/\theta_1)=\ln(190/Z^{\frac{1}{3}}s^{\frac{1}{2}}). \quad (51)$$

In this case, W_r is defined for $s\leqslant 1$, and for $s=1$ it coincides with the usual expression.

For $s\ll 1$, Eqs. (48) and (49) yield

$$W_r=\frac{8e^2}{\pi p_0^2 k}Bs[p_0^2+(p_0-k)^2]$$

$$=\frac{e^2}{\pi p_0^2}\left(\frac{B}{kp_0(p_0-k)}\right)^{\frac{1}{2}}[p_0^2+(p_0-k)^2]. \quad (52)$$

In this case, the emission probability is proportional to the square root of the density.

For $k\ll p_0$, one obtains from (49) (see reference 3)

$$W_r=\frac{8e^2}{3\pi k}B\phi(s). \quad (53)$$

At very small photon energies the deviation of the dielectric constant from unity must be taken into account. The dielectric constant ϵ enters the initial formulas through normalization of operator A and also enters the integral over τ in (3), where $e^{ik\tau}$ should be replaced by $e^{i\omega\tau}$; $\omega=k/\sqrt{\epsilon}$.

By considering the frequencies $\omega\gg\omega_a=(4\pi nZe^2)^{\frac{1}{2}}$, one obtains

$$\epsilon\cong 1-\frac{\omega_a^2}{\omega^2};\quad \omega=\frac{k}{\sqrt{\epsilon}}\cong k\left(1+\frac{1}{2}\frac{\omega_a^2}{\omega^2}\right). \quad (54)$$

Taking account of ϵ in the normalization factor of A is equivalent to multiplication of W_r by a factor which is close to unity and which can therefore be dropped.

Thus, the effect of the dielectric constant on the calculations is equivalent to substitution of the quantity α defined in (45) by α':

$$\alpha'=\frac{\omega-a}{2q}=\alpha+\frac{\omega-k}{2q}\cong\alpha\left(1+p_0^2\frac{\omega_a^2}{\omega^2}\right). \quad (55)$$

Using (49) and (47) one obtains for small k instead of (53) a more general formula

$$W_r=\frac{8e^2}{3\pi k}B'\phi(s\gamma)\frac{1}{\gamma},\quad \gamma=1+p_0^2\frac{\omega_a^2}{\omega^2}, \quad (56)$$

where B' differs from B in that under the logarithm sign s is replaced by $s\gamma$. At $s\gamma>1$, $\gamma\gg1$ one finds

$$W_r=\frac{4}{3\pi}Ze^4L\frac{k}{p_0^2}, \quad (57)$$

which is in agreement with the result obtained in reference 7 for this limiting case. Thus, radiation in a medium is not attended by any infrared catastrophe difficulties.

In order to introduce the shower unit of length, it will be convenient to define a function $\xi(s)$ which takes into account the variation of L with energy:

$$\begin{aligned}\xi(s)&=1+(\ln s/\ln s_1), & 1\geqslant s\geqslant s_1\\ &=1, & s>1\\ &=2, & s<s_1\\ s_1^{\frac{1}{2}}&=Z^{\frac{1}{3}}/190.\end{aligned} \quad (58)$$

Here, s_1 is the value of s for which $L=2\ln(190/Z^{\frac{1}{3}})$. Formulas (51) and (58) yield

$$B=2\pi ne^4Z^2\xi(s)\ln(190/Z^{\frac{1}{3}})=\tfrac{1}{2}\pi\times137\xi(s)/t_0',$$

$$\frac{1}{t_0'}=\frac{4ne^4Z^2}{137}\ln\left(\frac{190}{Z^{\frac{1}{3}}}\right); \quad (59)$$

$$t_0'=t_0(mc/\hbar)=2.59\times10^{10}t_0,$$

where t_0 is the shower unit expressed in centimeters. From expression (47) for s, one finds

$$s=\tfrac{1}{8}\left(\frac{2kt_0'}{p_0(p_0-k)137\pi\xi(s)}\right)^{\frac{1}{2}}=1.37\times10^3\left(\frac{kt_0}{p_0(p_0-k)}\right)^{\frac{1}{2}}. \quad (60)$$

The probability of emission per shower unit length is

$$W_rt_0'=\frac{\xi(s)}{3p_0^2k}\{k^2G(s)+2[p_0^2+(p_0-k)^2]\phi(s)\}. \quad (61)$$

In lead ($t_0\cong0.5$ cm), for $k=\frac{1}{2}p_0$, we obtain for $s=1$ the values $p_0=2\times10^6t_0$, $E_0=5\times10^{11}$ ev; for $s=0.2$ which corresponds to a 30% deviation of (67) from the Bethe-Heitler formula, we obtain $E_0=1.25\times10^{13}$ ev; the value $s=s_1$ corresponds to an energy $E_0\cong2\times10^{18}$ ev.

VI. PROBABILITY FOR PAIR PRODUCTION

Let $W_p(k,p_0)$ denote the probability per unit length of production of a pair, the electron of which possesses an energy lying between p_0 and p_0+dp_0 (W_p is summed over all possible positron states).

The probability for the inverse process $\tilde{W}_r$ may be found by summing over negative-energy states in (8) and by changing the sign of E^{λ_1} in (9). The final formulas are obtained from those given above by replacing p_0-k with $k-p_0$. For example, the quantity $g=\frac{1}{2}(p_0+p_0-k)$ changes into $\tilde{g}=\frac{1}{2}(p_0-p_0+k)=\frac{1}{2}k$. Thus, the probability $\tilde{W}_r$, which differs from W_p only in statistical weight, can be obtained from (49) by replacing p_0-k by $k-p_0$.

$$W_p=\frac{p_0^2}{k^2}\tilde{W}_r=\frac{2e^2}{3\pi k}B(\tilde{s})\left\{G(\tilde{s})+2\left[\frac{p_0^2}{k^2}+\left(1-\frac{p_0}{k}\right)^2\right]\phi(\tilde{s})\right\}. \quad (62)$$

Here $\tilde{s}$ differs from s only in that p_0-k has been replaced by $k-p_0$. The probability of pair production per shower unit of length is

$$W_pt_0'=\frac{\xi(\tilde{s})}{3k}\left\{G(\tilde{s})+2\left[\frac{p_0^2}{k^2}+\left(1-\frac{p_0}{k}\right)^2\right]\phi(\tilde{s})\right\}. \quad (63)$$

At $\tilde{s}=1$ this expression transforms into the familiar formula for pair production.[6] For $\tilde{s}\ll1$, we get

$$W_pt_0'=\frac{4\xi(\tilde{s})}{k}\left\{\frac{p_0^2}{k^2}+\left(1-\frac{p_0}{k}\right)^2\right\}\tilde{s}. \quad (64)$$

Formulas (61) and (63) are solutions of the bremsstrahlung and pair production problem for high energies in condensed media.

ACKNOWLEDGMENTS

The author expresses his appreciation to V. M. Galitsky for discussion of the results and to S. A. Heifetz for carrying out the numerical calculations.

APPENDIX

Equations (43) and (44) can be solved by the Laplace method. Assuming

$$u(\lambda)=\int_0^\infty \varphi_1(z)e^{-\lambda z}dz,$$

one obtains from (43)

$$u'+\frac{\lambda-i\alpha}{\lambda^2+i\beta}u=\frac{1}{2q\lambda(\lambda^2+i\beta)}. \tag{1A}$$

Hence

$$u(\lambda)=\frac{1}{2q}\frac{1}{(\lambda_1^2-\lambda^2)^{\frac{1}{2}}}\left(\frac{\lambda_1+\lambda}{\lambda_1-\lambda}\right)^\mu \times\int_\lambda^{\xi_1}\frac{d\xi}{\xi(\lambda_1^2-\xi^2)^{\frac{1}{2}}}\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu, \tag{2A}$$

where

$$\lambda_1=\beta^{\frac{1}{2}}e^{-i\pi/4},\quad \mu=\frac{\alpha}{2\beta^{\frac{1}{2}}}e^{-i\pi/4}.$$

The arbitrary constant ξ_1 is determined from the condition of finiteness of the function $\varphi_1(z)$ at $z\to\infty$. For this it is necessary that the function $u(\lambda)$ does not possess any singularities in the right semiplane, i.e., $\xi_1=\lambda_1$.

Expression (42) contains

$$\mathrm{Re}\int_0^\infty \varphi(z)dz=\mathrm{Re}_{\lambda\to 0}u(\lambda).$$

At $\lambda=0$, $u(\lambda)$ has a logarithmic singularity and therefore $\lim_{\lambda\to 0}u(\lambda)$ depends on how λ approaches zero. It follows from the expression

$$\mathrm{Re}u(\lambda)=\int_0^\infty \exp(-\lambda^0 z)(\mathrm{Re}\varphi_1\cos\lambda' z-\mathrm{Im}\varphi_1\sin\lambda' z)dz;$$

$$\lambda=\lambda^0+i\lambda',$$

that it is sufficient for λ to approach zero along the real axis in order that the equality

$$\int_0^\infty \mathrm{Re}\varphi_1 dz=\mathrm{Re}u(0)$$

be satisfied.

Separating from integral (2A) the divergence at $\xi=0$, we obtain

$$\underset{\lambda\to 0,\ \arg\lambda=0}{\mathrm{Re}}\ u(\lambda)=\underset{\lambda\to 0,\ \arg\lambda=0}{\mathrm{Re}}\ \frac{1}{2q}\frac{1}{\lambda_1}\left\{\int_\lambda^{\lambda_1}\frac{d\xi}{\xi}\left[\frac{1}{(\lambda_1^2-\xi^2)^{\frac{1}{2}}}\times\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu-\frac{1}{\lambda_1}\right]+\frac{1}{\lambda_1}\ln\frac{\lambda_1}{\lambda}\right\},$$

as

$$\underset{\lambda\to 0,\ \arg\lambda=0}{\mathrm{Re}}\ \frac{1}{\lambda_1^2}\ln\left(\frac{\lambda_1}{\lambda}\right)=\frac{\pi}{4\beta},$$

$$\underset{\lambda\to 0,\ \arg\lambda=0}{\mathrm{Re}}\ u(\lambda) =\frac{\pi}{8\beta q}-\frac{1}{2q\beta}\,\mathrm{Im}\int_0^1\frac{dx}{x}\left\{\frac{1}{(1-x^2)^{\frac{1}{2}}}\left(\frac{1-x}{1+x}\right)^\mu-1\right\}.$$

Inserting $x=\tanh(t/2)$ and separating the imaginary part, we find

$$\int_0^\infty \mathrm{Re}\varphi_1 dz=\frac{1}{12q\alpha^2}G(s);$$

$$G(s)=48s^2\left(\frac{\pi}{4}-\frac{1}{2}\int_0^\infty e^{-st}\frac{\sin st}{\sinh(t/2)}dt\right); \tag{3A}$$

$$s=\frac{\alpha}{2(2\beta)^{\frac{1}{2}}}=\frac{1}{8g}\left(\frac{k}{p_0(p_0-k)q}\right)^{\frac{1}{2}}.$$

In order to solve Eq. (44), it will be convenient to introduce $f=\varphi_2-(i/2q\beta)$,

$$zf''+i(\alpha+\beta z)f=\alpha/2q\beta. \tag{4A}$$

Inserting $v(\lambda)=\int e^{-\lambda z}fdz$, we obtain

$$v'+\frac{2\lambda-i\alpha}{\lambda^2+i\beta}v=\frac{f(0)-(\alpha/2q\beta\lambda)}{\lambda^2+i\beta}. \tag{5A}$$

$\varphi_2(0)=0$ and therefore $f(0)=-i/2q\beta$. The solution of Eq. (5A) is

$$v(\lambda)=-\frac{i}{2q\beta}\cdot\frac{1}{\lambda_1^2-\lambda^2}\left(\frac{\lambda_1+\lambda}{\lambda_1-\lambda}\right)^\mu \times\int_\lambda^{\lambda_1}\left(1-i\frac{\alpha}{\xi}\right)\left(\frac{\lambda_1-\xi}{\lambda_1+\xi}\right)^\mu d\xi.$$

$\mathrm{Re}v(0)$ is calculated in exactly the same manner as $\mathrm{Re}u(0)$.

Performing once again the substitution $\xi/\lambda_1=x$ and separating the divergent part from the integral, we obtain

$$\operatorname*{Re}_{\lambda\to0,\,\arg\lambda=0} v(\lambda)=\frac{1}{2q\beta^2}\operatorname*{Re}_{\lambda\to0,\,\arg\lambda=0}\left\{\lambda_1\int_0^1\left(\frac{1-x}{1+x}\right)^\mu dx -i\alpha\int_0^1\frac{dx}{x}\left[\left(\frac{1-x}{1+x}\right)^\mu-1\right]-i\alpha\ln\left(\frac{\lambda_1}{\lambda}\right)\right\}.$$

Inserting $x=\tanh(t/2)$ and separating the real part, we get

$$\operatorname*{Re}_{\lambda\to0,\,\arg\lambda=0} v(\lambda)=\frac{1}{2q\beta^2}\left\{-\frac{\pi\alpha}{4}+\alpha\int_0^\infty e^{-st}\frac{\sin st}{\sinh t}dt +\frac{1}{2}\left(\frac{\beta}{2}\right)^{\frac{1}{2}}\int_0^\infty e^{-st}\frac{\cos st+\sin st}{\sinh(t/2)}dt\right\}.$$

Thus,

$$\int_0^\infty \operatorname{Re}\varphi_2 dz=\int_0^\infty \operatorname{Re} f dz=\frac{1}{6\alpha\beta q}\phi(s), \tag{6A}$$

where

$$\phi=3s\int_0^\infty e^{-st}\frac{\cos st+\sin st}{\cosh^2(t/2)}dt+24s^2\int_0^\infty e^{-st}\frac{\sin st}{\sinh t}dt-6\pi s^2$$

$$=12s^2\int_0^\infty \coth(t/2)e^{-st}\sin st\,dt-6\pi s^2. \tag{7A}$$

The functions $\phi(s)$ and $G(s)$ can be expressed through the logarithmic derivatives of the Γ function.[8]

$$\begin{aligned}\phi(s)&=12s^2\{-\operatorname{Im}[\Psi(s-is)+\Psi(s+1-is)]-\tfrac{1}{2}\pi\}\\&=6s-6\pi s^2+24s^3\sum_{k=1}^\infty\frac{1}{(k+s)^2+s^2},\\G(s)&=48s^2[\tfrac{1}{4}\pi+\operatorname{Im}\Psi(s+\tfrac{1}{2}-is)]\\&=12\pi s^2-48s^3\sum_{k=0}^\infty\frac{1}{(k+s+\frac{1}{2})^2+s^2}.\end{aligned} \tag{8A}$$

These formulas are useful for the tabulations of $\phi(s)$ and $G(s)$.

[8] Relations (8A) were obtained by S. A. Heifetz.

PHYSICAL REVIEW VOLUME 103, NUMBER 6 SEPTEMBER 15, 1956

Time Variation of Primary Heavy Nuclei in Cosmic Radiation*

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(Received June 13, 1956)

The time variation of heavy nuclei in the primary cosmic radiation was investigated by using the method of a moving-plate mechanism which was flown to an altitude of 100 000 feet by a Skyhook balloon. The results obtained clearly indicate a time variation of primary heavy nuclei $Z\gtrsim10$. The variation is characterized by its maximum at around 9:00 A.M., having an amplitude of $34\pm7\%$ at the maximum. Comparisons are made with other experimental data on the same subject and also with the neutron intensity variation on the same day at Climax, Colorado. Possible consequences of this rather large fluctuation of the primary heavy nuclei are discussed.

I. INTRODUCTION

THE primary cosmic radiation has long been studied as to the intensity, the energy spectrum, the chemical or isotopical composition.[1] The investigation of the intensity variation with time, among others, is of importance in order to understand the problem of where and how the primary cosmic radiation is accelerated or modulated. Some information on this subject has been obtained from the observations at sea level or at mountain altitudes using counters, ionization-chambers, and neutron detectors. For example, from these observations we know approximately the type of intensity variations that exist in the cosmic radiation, the energy dependence of the intensity variation of a certain type, etc.[1]

These investigations, however, are based on the observations of secondary effects which were generated in the atmosphere by the interactions of the primary radiation; thus implying, among others: (1) that it is, in general, impossible to detect the intensity fluctuations of very low-energy primary particles which do not give rise to observable effects in detectors deep in the atmosphere, and (2) that at the present time the variations of heavy nuclei which constitute only a small fraction of the primary cosmic radiation cannot

* Supported in part by a joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

[1] Summaries on these subjects are given, for example, in J. G. Wilson, *Progress in Cosmic-Ray Physics* (North-Holland Publishing Company, Amsterdam, 1952); W. Heisenberg, *Kosmische Strahlung* (Springer-Verlag, Berlin, 1953).