evaluated. This analysis indicates the real part of the nuclear scattering amplitude at 10° is rather small and positive.

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No-Recoil Approximations to Charged Scalar Meson Scattering

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Feynman's method of functional integration is applied to meson-nucleon scattering. The scattering amplitude is obtained exactly in charged meson pair theory with neglect of nucleon recoil, and it is obtained in an approximation to linear charged meson theory with neglect of recoil. The results are the same as one obtains using the Chew-Low method.

1. INTRODUCTION

FEYNMAN'S[1] method of summing over histories (or functional integration) is applied to charged meson pair theory, with neglect of recoil. It appears very clearly from this approach that this interaction after mass and nucleon field normalization represents nothing but the scattering of mesons in a δ-function potential. Since the mesons are treated relativistically, they may be scattered (virtually) "backwards in time" (i.e., virtual pair creation), which gives rise to the "crossing" symmetry of the result. However, scattering is the only possible real process. These are the two typical features of scattering in a local static potential.

The method is then applied to the linear interaction in what we shall refer to as the "potential" approximation. This gives the same result as the Chew-Low[2] "one-meson" approximation.

The one-nucleon propagator and the one-nucleon–one-meson propagator, according to Feynman are, respectively,[3]

$$G(x_1,x_2)=\int G(x_1,x_2,\phi)N(\phi)\exp[I_m]\delta\phi\delta\phi^* \Big/ \int N(\phi)\exp[I_m]\delta\phi\delta\phi^*, \quad (1)$$

$$\Delta(x,y;x_1,x_2)=\int G(x_1,x_2,\phi)\phi(x)\phi^*(y)N(\phi) \times\exp[I_m]\delta\phi\delta\phi^* \Big/ \int N(\phi)\exp[I_m]\delta\phi\delta\phi^*, \quad (2)$$

where I_m is the uncoupled meson action, $G(x_1,x_2,\phi)$ is the one-nucleon propagator, and $N(\phi)$ is the vacuum expectation value of the S matrix in an external field ϕ. For nonrecoil theories, $N(\phi)=1$. The whole problem thus reduces to finding $G(\phi)$ and performing the functional integral, as accurately as possible.

It is clear that (2) is closely related to (1). It will be shown that the renormalized scattering amplitude can be obtained from (1) by parametric (functional) differentiation. It turns out that in order to determine the renormalized phase shift, it is only necessary to derive a formal expression for (1) for an infinite time interval. This is a special feature of no-recoil theories, which greatly simplifies the problem.

2. CHARGED PAIR THEORY

Consider the Lagrangian

$$L=\psi^*\left(i\frac{\partial}{\partial t}-m_0\right)\psi(t)-\phi^*(-\Box+\mu^2)\phi(x) -g^2\psi^*(t_x)\psi(t_x)\phi^*(x)\phi(x)\delta(\mathbf{x}). \quad (3)$$

Then $G(t,t',\phi)$ is determined by

$$\left[i\frac{\partial}{\partial t}-m_0-g^2\phi^*\phi(t)\right]G(t,t',\phi)=i\delta(t-t'). \quad (4)$$

Thus

$$G(t,t',\phi)=\theta(t-t')e^{-im_0(t-t')}\exp\left[-ig^2\int_{t'}^{t}\phi^*\phi(\tau)d\tau\right], \quad (5)$$

where $\theta(t)=1$ for $t>0$ and $\theta(t)=0$ for $t<0$.

Substituting (5) into (1) we get for the nucleon

[1] R. P. Feynman, Revs. Modern Phys. **20**, 367 (1948).
[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[3] R. P. Feynman, Phys. Rev. **84**, 108 (1951); P. T. Matthews and A. Salam, Nuovo cimento **11**, 120 (1955).

propagator, for an infinite time interval,

$$G=\lim_{T\to\infty} e^{-im_0T}F, \tag{6}$$

where

$$F=\lim F(T)$$
$$=N^{-1}\int \exp\left[\int \phi^*(\xi)(K+\Lambda)_{\xi\xi'}\phi(\xi')d\xi d\xi'\right]\delta\phi\delta\phi^*, \tag{7}$$

and

$$K_{\xi\xi'}=-i\delta(\xi-\xi')(-\Box_{\xi'}+\mu^2), \tag{8}$$

$$\Lambda_{\xi\xi'}=-ig^2\delta(\xi-\xi')\delta(\xi), \tag{9}$$

$$N=\int \exp[\phi^*K\phi]\delta\phi\delta\phi^*. \tag{10}$$

Therefore we obtain[4] the following formal expression for F:

$$F=\left[\det\left(\frac{K+\Lambda}{K}\right)\right]^{-1}, \tag{11}$$

where the integration is performed and F interpreted as if ξ, ξ' were discrete variables. But the expectation value of G between nucleon states of the observed mass,

$$m=m_0+\delta m, \tag{12}$$

is just the nucleon field renormalization constant Z_2. Thus

$$G=Z_2 \lim e^{-imT}. \tag{13}$$

Combining (6), (11), and (13),

$$\left[\det\left(\frac{K+\Lambda}{K}\right)\right]^{-1}=Z_2 \lim e^{-i\delta mT}. \tag{14}$$

By (2), the meson-nucleon propagator for an infinite nucleon time interval is

$$\Delta(x,y)=e^{-im_0T}N^{-1}\int \phi(x)\phi^*(y) \times\exp[\phi^*(K+\Lambda)\phi]\delta\phi\delta\phi^*. \tag{15}$$

This is just the functional derivative of G [defined by (6) and (7)] with respect to Λ_{yx}. Therefore

$$\Delta(x,y)=-e^{-im_0T}\left[\det\frac{K+\Lambda}{K}\right]^{-1}\left(\frac{1}{K+\Lambda}\right)_{xy} \tag{16}$$

$$=-e^{-imT}Z_2\left(\frac{1}{K+\Lambda}\right)_{xy}. \tag{17}$$

It is now necessary to solve for

$$D(x,y)=-\left(\frac{1}{K+\Lambda}\right)_{xy}, \tag{18}$$

that is,

$$(K+\Lambda)_{xz}D(z,y)=-\delta(x-y). \tag{19}$$

By looking for the stationary solution to the corresponding homogeneous equation, we find that apart from renormalization, the problem has been reduced to that of the scattering of a Klein-Gordon particle in a δ-function potential. [Λ is essentially a δ function (see Eq. (9)).]

In momentum space, (19) is

$$D(p,q)=(2\pi)^4\Delta(p)\delta(p-q) -\frac{ig^2}{(2\pi)^3}\Delta(p)\int D(\mathbf{r},p_0;q)d^3r, \tag{20}$$

where

$$D(p,q)=\int D(x,y)e^{i(px-qy)}dxdy, \tag{21}$$

and

$$\Delta(p)=-i/(p^2+\mu^2-i\epsilon). \tag{22}$$

Since the kernel of this equation factorizes, the solution is easily found to be

$$D(p,q)=(2\pi)^4\Delta(p)\delta(p-q) +2\pi i\delta(p_0-q_0)\Delta(p)\Delta(q)t(p_0), \tag{23}$$

where

$$t(p_0)=-g^2\left[1+\frac{ig^2}{(2\pi)^3}d(p_0)\right]^{-1}, \tag{24}$$

and

$$d(p_0)=\int \Delta(p)d^3p. \tag{25}$$

After renormalization, which is defined to give Born approximation at zero total energy (this is analogous to a Kroll-Ruderman[5] renormalization),

$$t(p_0)=-\bar{g}^2\left[1+\frac{i\bar{g}^2}{(2\pi)^3}\bar{d}(p_0)\right]^{-1}, \tag{26}$$

where

$$\begin{aligned}\bar{d}(p_0)&=d(p_0)-d(0)\\&=2\pi^2[(p_0{}^2-\mu^2+i\epsilon)^{\frac{1}{2}}-i\mu]\\&=2\pi^2(|\mathbf{p}|-i\mu).\end{aligned} \tag{27}$$

Since

$$t(p_0)=\frac{4\pi}{|\mathbf{p}|}e^{i\delta}\sin\delta, \tag{28}$$

$$|\mathbf{p}|\cot\delta=-(4\pi/\bar{g}^2+\mu). \tag{29}$$

This is a well-known result,[6] apart from the extra μ

[4] S. F. Edwards, Proc. Roy. Soc. (London) **A232**, 371 (1955); T. H. R. Skyrme, Proc. Roy. Soc. (London) **A231**, 321 (1955).

[5] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).

[6] S. Wentzel, Helv. Phys. Acta **15**, 111 (1942); A. Klein and B. H. McCormick, Phys. Rev. **98**, 1428 (1955) (see especially their footnote 21).

which comes from our method of charge renormalization. Its derivation serves to illustrate the method in a context where it is exact, and also brings out the fact that the interaction represents nothing but potential scattering.

3. CHARGED LINEAR THEORY

Consider the Lagrangian

$$L=\psi^*\left(i\frac{\partial}{\partial t}-m_0-\tau\tau^*\Delta m\right)\psi(t)-\phi^*(-\Box+\mu^2)\phi(x)$$
$$-g\psi^*(t)[\tau^*\phi(t)+\tau\phi^*(t)]\psi(t). \quad (30)$$

Δm is a difference between the bare mass of the proton and neutron, which is put in to facilitate the renormalization (after renormalization both true masses will be set equal), and $\tau=\frac{1}{2}(\tau_1+i\tau_2)$. Then the nucleon propagator in an external field ϕ for an infinite time interval is

$$G(\phi)=e^{-im_0T}F(\phi), \quad (31)$$

where by iteration

$$F(\phi)=\sum_n\int\int_{-\infty}^{\infty}d\xi_1d\xi_1'\int\int^{\xi_{01}}d\xi_2d\xi_2'\cdots$$
$$\times\int\int^{\xi_{0,n-1}}d\xi_nd\xi_n'\phi^*(1)\Theta(1,1')\phi(1')$$
$$\times\phi^*(2)\Theta(2,2')\phi(2')\cdots\phi^*(n)\Theta(n,n')\phi(n'), \quad (32)$$

and

$$\Theta(n,n')\equiv\Theta(\xi_n,\xi_n')=-g^2\delta(\xi_n)\theta(\xi_{0,n}'-\xi_{0,n})$$
$$\times\exp[i\Delta m(\xi_{0,n}'-\xi_{0,n})]\delta(\xi_n'). \quad (33)$$

We approximate $F(\phi)$ by replacing the upper limits on the integration, $\xi_{0,i}$ by $\xi_{0,i}'$. Then

$$F(\phi)\simeq\sum_n\frac{1}{n!}\left[\int\int_{-\infty}^{\infty}\phi^*(\xi)\Theta(\xi,\xi')d\xi d\xi'\right]^n$$
$$=\exp[\phi^*\Theta\phi]. \quad (34)$$

By considering the effect of this approximation from a graphical point of view, it may be seen that (after the integral over ϕ has been completed) we have included all correct graphs plus a large number of incorrect graphs, all of which involve at least two mesons. In this respect, it is somewhat similar to the Tamm-Dancoff approximation and may be expected to be valid at low energies and reasonably weak coupling.

Substituting (34) into (1), we obtain, as in (7),

$$F=N^{-1}\int\exp[\phi^*(K+\Theta)\phi]\delta\phi\delta\phi^*. \quad (35)$$

The calculation then goes exactly as for pair theory, leading to

$$t(p_0)=\frac{-g^2}{(p_0+\Delta m)}\left[1+\frac{ig^2}{(2\pi)^3}\frac{d(p_0)}{p_0+\Delta m}\right]^{-1}, \quad (36)$$

for the scattering of π^- mesons on protons. This result is renormalized to give the Born approximation at zero energy.

$$t(p_0)=-\frac{g^2}{p_0}\left[1+\frac{ig^2}{(2\pi)^3}\frac{d_c(p_0)}{p_0}\right]^{-1}, \quad (37)$$

where

$$d_c(p_0)=d(p_0)-d(0)$$
$$=(2\pi)^2(|\mathbf{p}|-i\mu). \quad (38)$$

(37) is obtained from (36) by a suitable definition of Δm, viz., $\Delta m=[-ig^2/(2\pi)^3]d(0)$. The bound states of the proton-meson system are given by the singularities of $t(p_0)$ and the neutron appears correctly in the renormalized expression (37) as the singularity at $p_0=0$.

Using (28) and (38), we obtain

$$|\mathbf{p}|\cot\delta=-(4\pi/g^2)p_0-\mu,$$

which is the result obtained previously by Serber and Lee[7] using the "one-meson" approximation of Chew and Low. It is also given by the variational principle of Cini and Fubini.[8]

DISCUSSION

If one calculates, by first Born approximation, the scattering in the charged meson pair theory, the result is equivalent to the first Born approximation for scattering in a δ-function potential. The method employed above makes it very clear that the exact solution to this problem is precisely equivalent to the scattering of a Klein-Gordon particle in a δ-function potential.

By comparison of (35) and (7), one sees that our approximation to the linear theory is equivalent to calculating exactly the scattering in the "potential" Θ. After a Fourier transform and renormalization, this reduces to g^2/p_0 which is the factor which replaces g^2 in (26) to reproduce (37). The "potential" is nonlocal in time. However, it has the essential property that it gives rise only to elastic scattering as a real process, and thus behaves like an ordinary local static potential. It is, in fact, the equivalent potential determined by the Born approximation.

It is to be noted that because we are using the Klein-Gordon equation, it is possible for the particle to be scattered (virtually) "backwards in time" by the potential. However, such effects (scattering "backwards in time") would not be included in the Tamm-Dancoff "one-meson" approximation, which is the

[7] T. D. Lee and R. Serber (unpublished).
[8] M. Cini and S. Fubini, Nuovo cimento **11**, 142 (1954).

reason why the Tamm-Dancoff method does not give the exact solution to the pair theory. (This is just another way of saying that our approximation satisfies the "crossing" symmetry[9] whereas the Tamm-Dancoff approximation does not.)

In Sec. 3 we saw that our method led to the same result as the Chew-Low method applied to the linear theory. This is also true of the pair theory. Applying the analysis of Low[10] to charged scalar meson pair theory, the structure of the quantity called $h(z)$ by Chew and Low $[4\pi h(p_0)=t(p_0)$ in our notation$]$ is quite different from that derived for the linear interaction. For pair theory the neutron is not a possible state of the $\pi^- - p$ system, and hence $h(z)$ has no pole at the origin. This is replaced by a constant term, which comes from the "catastrophic" scattering with no intermediate state, which is now possible. One thus has

$$h(z)=\frac{-g^2}{4\pi}+\int_\mu^\infty\left[\frac{F(x)}{x-z-i\epsilon}+\frac{G(x)}{x+z}\right]dx. \tag{1.1}$$

Applying the Chew-Low method to this equation then leads to the exact answer for pair theory.

Thus our method applied to charged pair theory and our approximation to charged linear theory lead to the same result as that of Chew and Low.[2] These authors have shown that the expression $t(p_0)$ is uniquely determined by the following conditions: (1) unitarity (of S matrix); (2) "crossing" symmetry; (3) the solution is expressible as power series in the renormalized coupling constant,[11] which converges for some fixed g, uniformly with p_0; (4) at zero energy the scattering is given by first order Born approximation (this determines the inhomogeneous term in the Low equation); (5) scattering is the only possible real process (the "one meson" approximation—this implies that $t(z)$, suitably defined, has branch points at $\pm\mu$).

One can then look on the one-meson approximation of Low and Chew (at least for these interactions) as being equivalent to calculating exactly scattering in an equivalent potential, determined by Born approximation.

The solution of the Chew-Low equation is beset by the difficulties pointed out by Castillejo, Dalitz, and Dyson.[11] However, our method leads directly to a unique answer. One might hope that the functional approach offers a method of finding unique solutions to more complicated problems.

We should like to thank Dr. D. J. Candlin for some interesting discussions and one of us (G.F.) would like to thank Dr. J. R. Oppenheimer and the Institute for Advanced Study for their hospitality.

[9] M. Gell-Mann and M. L. Goldberger, *Proceedings of the Fourth Annual Rochester Conference* (University of Rochester Press, Rochester, 1954).

[10] F. Low, Phys. Rev. **97**, 1392 (1954).

[11] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).