Properties of the Quantum Statistical Impedance*

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The transfer function of a general linear dissipative system, which had previously been derived, is examined. It is shown that its real and imaginary parts satisfy relationships which must be true for general passive systems. Examination of the transfer function at low frequencies gives expressions for the equivalent resistance and capacitance.

IN a recent paper by one of the writers[1] the direct response of a linear dissipative system to an external generalized driving force was computed. The transfer function deduced from this computation had a real part which agreed with a previous result of Callen and Welton;[2] a new expression for the imaginary part was obtained. Callen and Welton showed that the real part of the transfer function was always positive, as indeed it should be for a dissipative system. It is the purpose of this note to point out some further properties of this transfer function.

It will be shown that the transfer function satisfies the requirement that it have no poles in the lower half of the ω plane. Expressions will be obtained for the expansion coefficients of the transfer function about zero frequency. This expansion is consistent with the physically expected behavior.

A linear system can be characterized by its transfer function, $F(\omega)$, which has the property that a periodic input, $\exp(i\omega t)$, yields the output $F(\omega)\exp(i\omega t)$. The singularities of $F(\omega)$ are the values of ω for which the homogeneous system equation has solutions. These singularities will in general occur for complex values of ω. The existence of such a singularity in the lower half of the ω plane would imply a temporal exponentially increasing solution of the homogeneous system equation. Thus, the transfer function of a passive system can have no singularities in the lower half ω plane.

As a consequence of this fact, the use of Cauchy's integral theorem gives a necessary and sufficient condition (except for an arbitrary constant in the real part) which must exist between the real and imaginary parts of such a transfer function for real frequencies.[3] This condition is

$$B(\omega)=(2\omega/\pi)\int_0^\infty [A(x)/(x^2-\omega^2)]dx, \qquad (1)$$

where

$$F(\omega)\equiv A(\omega)+iB(\omega); \qquad (1.1)$$

the functions $A(\omega)$ and $B(\omega)$ are real for real frequencies.

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[1] J. L. Jackson, Phys. Rev. **87**, 471 (1952).

[2] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[3] See, e.g., H. W. Bode, *Network Analysis and Feedback Amplifier Design* (D. Van Nostrand Company, Inc., New York, 1945), Chap. XIV.

In Eq. (1) it is to be understood that the Cauchy principal value of the integral is meant.

In reference 1, it is shown that, for a linear dissipative system the real and imaginary parts of the transfer function are given by

$$A(\omega)=\pi\omega\int f(E)\rho(E)[|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega) - |\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)]dE, \qquad (2)$$

and

$$B(\omega)=-2\omega\int\int [f(E)\rho(E)\rho(E')(E-E')|\langle E|Q|E'\rangle|^2 \times[(E-E')^2-(\hbar\omega)^2]^{-1}dEdE'. \qquad (2.1)$$

The notation used is that of reference 1. In order to verify that this transfer function satisfies Eq. (1) the transformation,

$$|E-E'|\equiv\hbar x, \qquad (3)$$

is used to replace E' in Eq. (2.1); Eq. (1) is identically obtained.

It is of interest to note the behavior of this transfer function at low frequencies. The real and imaginary parts have the zero frequency expansions

$$A(\omega)=\alpha\omega^2+O(\omega^4), \qquad (4)$$

$$B(\omega)=\beta\omega+O(\omega^3), \qquad (4.1)$$

where

$$\alpha=(\pi\hbar/kT)\int f(E)\rho^2(E)|\langle E|Q|E\rangle|^2dE, \qquad (4.2)$$

and

$$\beta=2\int\int [f(E)\rho(E)\rho(E')|\langle E|Q|E'\rangle|^2 \times(E'-E)^{-1}dEdE'. \qquad (4.3)$$

The expression for α can be obtained readily from Eq. (4.6) of reference 2. It should be observed that α and β are both positive, the latter because $f(E)$ is smaller than $f(E')$ when E' is less than E. Equations (4) and (4.1) specify an impedance that has a zero frequency expansion of the form

$$Z(\omega)=(i\omega\beta)^{-1}+(\alpha/\beta^2)+O(\omega). \qquad (5)$$

In the language of circuit theory this means that the dissipative system has capacitance β and resistance (α/β^2). Higher order circuit elements can be found by taking into account the higher order terms in the expansion of $A(\omega)$ and $B(\omega)$.

The zero frequency behavior of the transfer function is that which is to be expected for a dissipative system. The resistance is shown to be finite and different from zero. The current vanishes at zero frequency due to the presence of the capacitive term in the impedance.

One might also consider the transfer function $F_n(\omega)$ which gives the response of a single member of the ensemble with unperturbed energy, E_n. The corresponding real and imaginary parts are then clearly

$$A_n(\omega)=\pi\omega[|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) - |\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)], \quad (6)$$

and

$$B_n(\omega)=-2\omega\int (E_n-E')|\langle E_n|Q|E'\rangle|^2 \times[(E_n-E')^2-(\hbar\omega)^2]^{-1}dE'. \quad (6.1)$$

It is not here apparent that the real part of this transfer function, which gives the response of a single member of the ensemble, must be positive. However, it is clear that the relationship of Eq. (1) is fulfilled, even for a single member of the ensemble.

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The Virial Series of the Ideal Bose-Einstein Gas*

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It is suggested that the ideal Bose-Einstein gas be used as the prototype of an imperfect gas in investigating a possible connection between divergence of the virial series and the condensation phenomenon. An explicit expression is obtained for the virial coefficients in this case. Obtaining the radius of convergence of this series appears to be a formidable mathematical problem, and only a lower bound for this radius is established. It is conjectured, however, that the radius of convergence is infinite.

THE ideal Bose-Einstein gas is imperfect, in the sense that its equation of state is not $p=\rho kT$. The imperfection is a quantum effect, and it gives rise to the well-known Einstein condensation. In many respects the p-v isotherms[1] resemble those of an ordinary condensing gas, though there are, of course, important differences.

The problem of finding a possible relationship between the extreme limit of convergence of the virial series, and the point of condensation, is one of considerable importance in the mathematical theory of ordinary imperfect gases.[2] In this connection it would be interesting to find the radius of convergence of the virial series of the ideal Bose gas and observe whether or not it is equal to the density at which condensation is known to occur. I have succeeded in obtaining explicit expressions for the coefficients in this series, but I have only been able to establish a lower bound for the radius of convergence. It may be conjectured, however, that the radius of convergence is infinite, and that the function defined by the virial series bears the same relation to the pressure of the system that a function of van der Waals type (including the unstable portions) bears to the pressure in an ordinary condensing gas.[2]

The dimensionless density $x_1=(h^2/2\pi mkT)^{\frac{3}{2}}\rho$ and the dimensionless pressure $x_2=(1/kT)(h^2/2\pi mkT)^{\frac{3}{2}}p$ are two members of the set of functions[3] x_s given by

$$x_s=\sum_1^\infty n^{-(s+\frac{1}{2})}y^n, \quad (1)$$

where $y=\exp(\mu/kT)$. The virial series is obtained by eliminating y from the expressions for x_2 and x_1, expressing the former as a power series in the latter.[4] From Bürmann's expansion,[5]

$$x_{s+1}(y)=x_{s+1}(0)+\sum_1^\infty \frac{1}{n!}\left\{\frac{d^{n-1}}{dy^{n-1}}\left[\left(\frac{y}{x_s}\right)^n\frac{dx_{s+1}}{dy}\right]\right\}_{y=0} x_s^n.$$

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[1] O. K. Rice, Phys. Rev. 93, 1161 (1954).
[2] H. N. V. Temperley, Proc. Phys. Soc. (London) A67, 233 (1954).
[3] These functions have been studied by E. W. Barnes, Proc. London Math. Soc. (2) 4, 284 (1906); C. Truesdell, Ann. Math. 46, 144 (1945); and others.
[4] A general problem of this kind was considered by J. C. Glashan, Am. J. Math. 3, 190 (1880), but his results are of too implicit a nature to be useful here.
[5] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1927), fourth edition, pp. 128–129.