In the language of circuit theory this means that the dissipative system has capacitance $\beta$ and resistance $(\alpha/\beta^2)$. Higher order circuit elements can be found by taking into account the higher order terms in the expansion of $A(\omega)$ and $B(\omega)$.

The zero frequency behavior of the transfer function is that which is to be expected for a dissipative system. The resistance is shown to be finite and different from zero. The current vanishes at zero frequency due to the presence of the capacitive term in the impedance.

One might also consider the transfer function $F_n(\omega)$ which gives the response of a single member of the ensemble with unperturbed energy, $E_n$. The corresponding real and imaginary parts are then clearly

$$A_n(\omega)=\pi\omega[|\langle E_n+\hbar\omega|Q|E_n\rangle|^2\rho(E_n+\hbar\omega) - |\langle E_n-\hbar\omega|Q|E_n\rangle|^2\rho(E_n-\hbar\omega)], \quad (6)$$

and

$$B_n(\omega)=-2\omega\int(E_n-E')|\langle E_n|Q|E'\rangle|^2 \times[(E_n-E')^2-(\hbar\omega)^2]^{-1}dE'. \quad (6.1)$$

It is not here apparent that the real part of this transfer function, which gives the response of a single member of the ensemble, must be positive. However, it is clear that the relationship of Eq. (1) is fulfilled, even for a single member of the ensemble.

PHYSICAL REVIEW VOLUME 96, NUMBER 1 OCTOBER 1, 1954

# The Virial Series of the Ideal Bose-Einstein Gas*

B. WIDOM†
*Department of Chemistry, University of North Carolina, Chapel Hill, North Carolina*
(Received May 17, 1954)

It is suggested that the ideal Bose-Einstein gas be used as the prototype of an imperfect gas in investigating a possible connection between divergence of the virial series and the condensation phenomenon. An explicit expression is obtained for the virial coefficients in this case. Obtaining the radius of convergence of this series appears to be a formidable mathematical problem, and only a lower bound for this radius is established. It is conjectured, however, that the radius of convergence is infinite.

THE ideal Bose-Einstein gas is imperfect, in the sense that its equation of state is not $p=\rho kT$. The imperfection is a quantum effect, and it gives rise to the well-known Einstein condensation. In many respects the $p$-$v$ isotherms[1] resemble those of an ordinary condensing gas, though there are, of course, important differences.

The problem of finding a possible relationship between the extreme limit of convergence of the virial series, and the point of condensation, is one of considerable importance in the mathematical theory of ordinary imperfect gases.[2] In this connection it would be interesting to find the radius of convergence of the virial series of the ideal Bose gas and observe whether or not it is equal to the density at which condensation is known to occur. I have succeeded in obtaining explicit expressions for the coefficients in this series, but I have only been able to establish a lower bound for the radius of convergence. It may be conjectured, however, that the radius of convergence is infinite, and that the function defined by the virial series bears the same relation to the pressure of the system that a function of van der Waals type (including the unstable portions) bears to the pressure in an ordinary condensing gas.[2]

The dimensionless density $x_1=(h^2/2\pi mkT)^{\frac{3}{2}}\rho$ and the dimensionless pressure $x_2=(1/kT)(h^2/2\pi mkT)^{\frac{3}{2}}p$ are two members of the set of functions[3] $x_s$ given by

$$x_s=\sum_1^\infty n^{-(s+\frac{1}{2})}y^n, \quad (1)$$

where $y=\exp(\mu/kT)$. The virial series is obtained by eliminating $y$ from the expressions for $x_2$ and $x_1$, expressing the former as a power series in the latter.[4] From Bürmann's expansion,[5]

$$x_{s+1}(y)=x_{s+1}(0)+\sum_1^\infty\frac{1}{n!}\left\{\frac{d^{n-1}}{dy^{n-1}}\left[\left(\frac{y}{x_s}\right)^n\frac{dx_{s+1}}{dy}\right]\right\}_{y=0}x_s^n.$$

* Work supported by the U. S. Office of Naval Research.
† Present address: Department of Chemistry, Cornell University, Ithaca, New York.
[1] O. K. Rice, Phys. Rev. **93**, 1161 (1954).
[2] H. N. V. Temperley, Proc. Phys. Soc. (London) **A67**, 233 (1954).
[3] These functions have been studied by E. W. Barnes, Proc. London Math. Soc. (2) 4, 284 (1906); C. Truesdell, Ann. Math. **46**, 144 (1945); and others.
[4] A general problem of this kind was considered by J. C. Glashan, Am. J. Math. **3**, 190 (1880), but his results are of too implicit a nature to be useful here.
[5] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1927), fourth edition, pp. 128–129.

But by Eq. (1), $x_{s+1}(0)=0$ and $dx_{s+1}/dy=x_s/y$. Hence,

$$x_{s+1}=x_s\left\{1+\sum_1^\infty \frac{1}{(n+1)!}\left[\frac{d^n}{dy^n}\left(\frac{x_s}{y}\right)^{-n}\right]_{y=0} x_s{}^n\right\}.$$

Segar[6] has shown how to express, in determinantal form, any derivative of any power of a function.[7] Applying Segar's result, one obtains

$$\left[\frac{d^n}{dy^n}\left(\frac{x_s}{y}\right)^{-n}\right]_{y=0}=(-1)^n D_n,$$

where $D_n$ is the following determinant of order $n$:

$$D_n=\begin{vmatrix} \frac{n}{2^{s+\frac{1}{2}}} & 1 & 0 & 0 & \cdots \\ \frac{2n}{3^{s+\frac{1}{2}}} & \frac{n+1}{2^{s+\frac{1}{2}}} & 2 & 0 & \cdots \\ \frac{3n}{4^{s+\frac{1}{2}}} & \frac{2n+1}{3^{s+\frac{1}{2}}} & \frac{n+2}{2^{s+\frac{1}{2}}} & 3 & \cdots \\ \frac{4n}{5^{s+\frac{1}{2}}} & \frac{3n+1}{4^{s+\frac{1}{2}}} & \frac{2n+2}{3^{s+\frac{1}{2}}} & \frac{n+3}{2^{s+\frac{1}{2}}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}. \tag{2}$$

Finally, then,

$$x_{s+1}=x_s\left[1+\sum_1^\infty (-1)^n\frac{D_n}{(n+1)!}x_s{}^n\right]. \tag{3}$$

When $s=1$, Eq. (3) is the virial series. Having found an explicit formula for the coefficients amounts to having solved the recurrence relations by means of which these coefficients are usually obtained.[8] Note that condensation is known[8] to occur at $x_1=\zeta(3/2)=2.612\cdots$. ($\zeta$ is the Riemann zeta function.)

By writing $y=r\exp(it)$, it is seen from Eq. (1) that, for $r<1$, $|x_1(y)|\leq x_1(r)$, the equality holding when $t=0$. But for $r<1$, $x_1(r)$ is an increasing function of $r$, and as $r\rightarrow 1$, $x_1(r)\rightarrow\zeta(3/2)$. Thus, in the circle $|y|=1$, $|x_1(y)|<\zeta(3/2)$. Also, $x_1(y)$ is regular in this circle. Therefore, by Landau's theorem,[9] the inverse function $y(x_1)$ is regular, and $|y(x_1)|<1$, when $|x_1|<\zeta(3/2)\times\{\zeta(3/2)-[\zeta(3/2)^2-1]^{\frac{1}{2}}\}^2\equiv Q$. But $x_2(y)$ is regular for $|y|<1$. Then, since a regular function of a regular function is regular, the radius of convergence $R$ of the virial series must satisfy $R\geq Q=0.103\cdots$.

Better estimates of $R$ can be obtained by considering the magnitude of $D_n$. There are only $2^{n-1}$ nonvanishing terms in the full expansion of $D_n$. When $s=1$, the largest element of $D_n$ is the element in the $(n-1)$th row and the $n$th column, *viz.*, $(n-1)$. Consequently, $|D_n|\leq 2^{n-1}(n-1)^n$, or $R\geq 1/2e=0.184\cdots$.

Widom[10] has pointed out that for $s=1$ the largest nonzero element of the $k$th row of $D_n$ is $\leq k$ when $k\geq 2(n-1)/3$ and is $\leq 2(n-1)/3$ when $k<2(n-1)/3$, and that, as a consequence,

$$|D_n|\leq\frac{2^{n-1}[2(n-1)/3]^{2(n-1)/3}n!}{(\{2(n-1)/3\}-1)!},$$

where $\{A\}$ means "smallest integer $\geq A$." Thus, $R\geq 1/2e^{\frac{2}{3}}=0.257\cdots$.

I am indebted to Professor O. K. Rice, under whose direction this work was done.

[6] H. W. Segar, Messenger of Math. (2) **21**, 177 (1891–2).

[7] A different, though equivalent, result was obtained by S. Mangeot, Ann. Sci. l'École Norm. Sup. (3) **14**, 247 (1897), but his determinant is slightly less convenient for our purposes.

[8] J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (John Wiley and Sons, Inc., New York, 1940), pp. 288–290, 416–426.

[9] L. Bieberbach, *Lehrbuch der Funktionentheorie* (B. G. Teubner, Leipzig, 1934), fourth edition, Vol. I, pp. 189–190.

[10] H. Widom (private communication).