Relativistic Coulomb Scattering of Electrons and Positrons*

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The numerical tables of Feshbach for the relativistic Coulomb scattering of electrons and positrons by a nucleus of charge Z have been fitted with an analytic function. The accuracy of the analytic function is equivalent to that of the tables.

THE relativistic pure Coulomb scattering of electrons and positrons by a nucleus of charge Z was solved exactly by Mott.[1] The result was expressed in the form of a slowly convergent Legendre series. This series was summed numerically by Bartlett and Watson,[2] McKinley and Feshbach,[3] and Feshbach.[4]

In the course of some work on electron scattering, we have fitted the numerical tables of Feshbach, which are valid for positron and electron energies above about 4 Mev, with an analytical function. Such an analytical representation may be of some use, and we would like to give our result in this note. The accuracy with which our analytic function represents the numerical results is very good. The accuracy of the analytic function is as good as the accuracy of Feshbach's tables, which is about 0.1 percent except where the cross section gets small.

The cross section $\sigma(\theta)$ for the scattering of particles obeying the Dirac equation can be expressed in terms of the two scattering amplitude functions[1] $f_3(\theta)$ and $f_4(\theta)$ as

$$\sigma(\theta)=|f_3|^2+|f_4|^2. \tag{1}$$

At high energies, f_3 and f_4 are related and can be expressed in terms of a single function, $F(\theta)$:

$$f_3=\tfrac{1}{2}(1+\cos\theta)F(\theta), \tag{2}$$

$$f_4=\tfrac{1}{2}\sin\theta e^{i\varphi}F(\theta). \tag{3}$$

This relationship is given by the Born approximation[5] and is also exact in the high-energy limit.[6] In the Born approximation $F(\theta)$ would be given by

$$F=(-2E/4\pi)\cdot\int d\tau\,\exp(-i\mathbf{k}\cdot\mathbf{r})V(\mathbf{r})\exp(i\mathbf{k}_0\cdot\mathbf{r}), \tag{4}$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are the incident and final momenta of the particle, $V(\mathbf{r})$ is the potential, and E is the incident energy of the particle. ($\hbar=c=1$ throughout.)

In terms of F, the cross section is given by

$$\sigma=\cos^2\tfrac{1}{2}\theta|F|^2. \tag{5}$$

We have computed the exact values of $F(\theta)$ from Feshbach's tables and have fitted $F(\theta)$ with an analytic function. In attempting to fit $F(\theta)$ with an analytic function, we have been guided by the fact that the form of $F(\theta)$ is known for small Z and for small scattering angles.

For small Z, which means small α, $\alpha=Ze^2/\hbar c$ and $F(\theta)$ is given by[3]

$$F=\left(\frac{-Ze^2}{2k}\right)\frac{1}{\sin^2\frac{1}{2}\theta}\left\{\exp[i(2\alpha\ln\sin\tfrac{1}{2}\theta+2\eta)]+\tfrac{1}{2}\pi\alpha\frac{\sin\frac{1}{2}\theta}{1+\sin\frac{1}{2}\theta}+i\alpha\tan^2\tfrac{1}{2}\theta\ln\sin\tfrac{1}{2}\theta\right\}, \tag{6}$$

where $\exp(2i\eta)=\Gamma(1-i\alpha)/\Gamma(1+i\alpha)$, and k is the incident momentum of the particle.

For small scattering angles, F is given by[2]

$$F=\frac{-Ze^2}{2k}\frac{1}{\sin^2\frac{1}{2}\theta}\exp[i(2\alpha\ln\sin\tfrac{1}{2}\theta+2\eta)]\times(1+\tfrac{1}{2}\pi\alpha\sin\tfrac{1}{2}\theta e^{i\chi}), \tag{7}$$

where $\exp(2i\chi)=\Gamma(\frac{1}{2}-i\alpha)\Gamma(1+i\alpha)/\Gamma(\frac{1}{2}+i\alpha)\Gamma(1-i\alpha)$. Here, as in all our results, Z is positive for electrons, negative for positrons.

Our analytic expression for $F(\theta)$ must reduce to Eq. (6) for small Z and to Eq. (7) for small θ. A function that will reduce to (6) and (7) in the small-Z and small-θ limits is the function defined by,

$$F_S=\frac{-Ze^2}{2k}\frac{1}{\sin^2\frac{1}{2}\theta}\exp\{i[2\alpha(1+\tfrac{1}{2}\tan^2\tfrac{1}{2}\theta)\ln\sin\tfrac{1}{2}\theta+2\eta+\tfrac{1}{2}\pi\alpha(\sin\chi)y]+\tfrac{1}{2}\pi\alpha(\cos\chi)y\}, \tag{8}$$

$$|F_S|^2=\frac{Z^2e^4}{4k^2}\frac{1}{\sin^4\frac{1}{2}\theta}\exp[\pi\alpha(\cos\chi)y], \tag{9}$$

where $y=\sin\frac{1}{2}\theta/(1+\sin\frac{1}{2}\theta)$.

It turns out that F_S does represent $F(\theta)$ fairly well in the cases of positron scattering and small-Z ($Z\leq 30$) electron scattering. However, for large-Z electron scattering, $F(\theta)$ departs considerably from F_S at the larger angles. For the large-Z large-angle scattering, $F(\theta)$ is well represented by the function F_L defined by

$$F_L=F_S\cdot A\exp(By+i\delta), \tag{10}$$

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[6] Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).

where A, B, and δ depend only on Z and are tabulated below.

In the general case, $F(\theta)$ can be accurately represented by a combination of the small-angle function F_S and the large-angle function F_L as

$$F=\exp(-ay^2)F_S+[1-\exp(-by^2)]F_L, \quad (11)$$

where a and b depend only on Z.

In Tables I and II, the values of the various parameters are given as functions of Z. We found that to the accuracy of Feshbach's tables we could put $A=1$, $B=0$ for all Z in positron scattering. $F(\theta)$ was rather insensitive to the choice of the parameters a and b, particularly in the positron and small-Z electron cases. Thus the values of a and b can be varied by as much as 10 percent in some cases and we will still get the same $F(\theta)$ to the accuracy of Feshbach's tables.

TABLE I. Table of χ and η as functions of Z for electron scattering. For positron scattering, the sign is changed. χ and η were computed with help of *Tables of Coulomb Wave Functions*, National Bureau of Standards, Applied Mathematics Series (U. S. Government Printing Office, Washington, D. C. 1952), No. 17, Vol. I.

Z	χ	η
0	0.0000	0.0000
13	0.2590	0.0544
29	0.5459	0.1184
47	0.8011	0.1827
62	0.9585	0.2275
80	1.0938	0.2687

Although Eq. (11) represents $F(\theta)$ with considerable accuracy, it is somewhat complicated mathematically. There are some special cases where we can find mathematically simpler expressions for $F(\theta)$ which are still fairly accurate.

TABLE II. The parameters A, B, δ, a, and b, as functions of Z for electron and positron scattering. Z is positive for electron scattering.

Z	A	B	δ	a	b
80	0.6172	2.352	0.218	21.1	24.0
62	0.7834	1.316	0.076	27.5	33.4
47	0.9154	0.625	0.019	45.6	49.2
29	0.9764	0.205	−0.009	37.6	38.0
13	0.9928	0.026	−0.0048	7.5	7.5
0	1.0000	0	0		
−13	1.0000	0.0000	−0.0062	10.5	10.5
−29	1.0000	0.0000	−0.030	8.3	8.3
−47	1.0000	0.0000	−0.045	13.8	13.8
−62	1.0000	0.0000	−0.058	13.2	13.2
−80	1.0000	0.0000	−0.062	14.9	14.9

POSITRON SCATTERING

In this case, $F(\theta)$ is given to within a few percent by

$$F=F_S, \quad (12a)$$

$$|F|^2=\frac{Z^2e^4}{4k^2}\frac{1}{\sin^4\frac{1}{2}\theta}\exp[\pi\alpha(\cos\chi)y]. \quad (12b)$$

ELECTRON SCATTERING

For small-Z, that is, $Z\leq 30$, $F(\theta)$ is given to within a few percent by

$$F=F_L \quad (13a)$$

$$|F|^2=\frac{Z^2e^4}{4k^2}\frac{1}{\sin^4\frac{1}{2}\theta}\exp\{(\pi\alpha\cos\chi+2B)y\}. \quad (13b)$$

For the large-Z electron scattering Eq. (13) is valid at the larger angles but is considerably in error at the smaller angles, where $F=F_S$. However, one can say that for large Z, at angles larger than $\theta=60°$, F is given by Eq. (13) to within a few percent.