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Families of Spinor Fields

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It is shown that a system of spinor fields can be classified uniquely into several sets, called families, when the interaction is given. This is made possible by noting that an interaction imposes some restriction on the commutation relation between different spinor fields. The fermion system is then classified, by a method of construction, into a set of families characterized by the following conditions. (1) Spinor fields of the same family anticommute with each other. (2) Commutation relations between any pair of spinor fields of different families are completely unspecified. To this end, we use a mathematical device which is derived from the self-evident requirement that equations of motion for fermions are determined unambiguously by the variation principle. We have applied the present method to the classification of observed spinor fields and discussed the structure of families in some detail.

I. INTRODUCTION

IT is known that reactions of various elementary particles which have heretofore been discovered are subject to several selection rules. One of them is the so-called conservation law of heavy particles which has been noticed in connection with the stability of matter.[1] Although the deeper cause of this law is still beyond our comprehension, it seems to indicate at least that nucleons and V particles form a distinct set from other fermions. Thus, these particles are usually said to form a "nucleon family." Of course the notion of a nucleon family is expected to represent implicitly certain features of the interaction of heavy particles with other elementary particles. It has, however, been a rather obscure concept. It will be worth while to ask whether or not it is possible to give a definite physical meaning to the nucleon family. Furthermore, it will be useful if one can extend the concept to establish the notion of a general family which is applicable to any spinor field.

It was first noticed by Nishizima[2] that a restriction is imposed on the commutation relation between different spinor fields by the integrability condition of the Schrödinger equation in the interaction representation. Generalizing his idea, Oneda and Umezawa[3] tried to use the commutation property of various spinor fields to define the family. Their method is incomplete, however, since they did not give a prescription to determine the commutation relation between different spinor fields. The purpose of this paper is to improve their method and give a satisfactory and unambiguous definition of family on the basis of empirical knowledge about the interaction. We do not intend, however, to give any explanation for the existence of the conservation law of heavy particles.

In Sec. II, we shall give a lemma based on a fundamental requirement of the usual quantized field theory which is useful in studying the interacting spinor fields. In Sec. III, we shall see how a given interaction Lagrangian restricts the commutation relations between different spinor fields and how this result is used to classify them into several families. We shall apply our method to the classification of the actual fermion system observed in recent experiments.

II. PRELIMINARY REMARKS

Let us consider an assembly of n fermion fields described by the operators $\psi^1, \psi^2, \cdots, \psi^n$ and their conjugates interacting with each other and with several boson fields. To describe this, one can of course start from a c-number Lagrangian and introduce commutation relations afterwards so that one obtains a consistent picture of field quantization. Instead, however, we shall use here a q-number Lagrangian L from the beginning.[4] This is a Lorentz-invariant polynomial of fermion and boson operators. It is composed of the free part L_0 and the interaction part L_I. We shall assume that the anticommutation relations,

$$\begin{aligned} \{\psi_\alpha{}^i(\mathbf{x},t),\ \psi_\beta{}^{i*}(\mathbf{x}',t)\} &= \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'), \\ \{\psi_\alpha{}^i(\mathbf{x},t),\ \psi_\beta{}^i(\mathbf{x}',t)\} &= 0, \text{ etc., } i=1, 2, \cdots, n, \end{aligned} \tag{2.1}$$

are already proved to hold for the free fermion fields and that the ordinary commutation relations hold for the free boson fields. On the other hand, the commutation relation between different kinds of spinor fields is not determined by L_0.[5]

In order to study how a given L_I imposes restriction

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[1] It was first noticed by Wigner that a sum of the number of protons, neutrons, and V particles, or more exactly, the difference of the number of these particles and their antiparticles, is conserved in any known reactions, see E. P. Wigner, Proc. Am. Phil. Soc. **93**, 521 (1949); Proc. Nat. Acad. Sci. U. S. **38**, 449 (1952).

[2] K. Nishizima, Progr. Theoret. Phys. (Japan) **5**, 187 (1950).

[3] S. Oneda and H. Umezawa, Progr. Theoret. Phys. (Japan) **9**, 685 (1953).

[4] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

[5] For example, a transformation of free-field variables like

$$\psi'^1=\psi^1,\quad \psi'^2=C\psi^2=\psi^2C,\quad C\psi^1=-\psi^1C,$$

with

$$C=\exp\Big[i\pi\int_{\sigma_x} d\sigma_\mu'\bar{\psi}^1(x')\gamma_\mu{}^1\psi^1(x')\Big],$$

transforms a commutation relation of ψ^1 and ψ^2 (for $n=2$) into an anticommutative one and vice versa without changing any other relations, where σ_x is a space-like surface which passes through the point x. See O. Klein, J. phys. et radium (7), **9**, 1 (1938). L. Michel, Proc. Phys. Soc. (London) **A63**, 514 (1950).

on the commutation relation between different spinor fields, it is convenient to make use of the following lemma:

Lemma 1. *Each term of L_I has to involve an even number of operators (including ψ^i) which anticommute with $\psi^i (i=1, 2, \cdots, n)$.*

This is derived in the following way from the requirement that the equations of motion for all field operators must be determined uniquely from the Lagrangian L by the principle of stationary action. Making use of a two-valued function ϵ_{ij} defined by[6]

$$\psi^i\psi^j=\epsilon_{ij}\psi^j\psi^i, \quad \psi^{i*}\psi^j=\epsilon_{ij}\psi^j\psi^{i*}, \text{ etc.}, \tag{2.2}$$

with

$$\epsilon_{ij}=\begin{cases} 1 \text{ if } \psi^i \text{ and } \psi^j \text{ commute,} \\ -1 \text{ if } \psi^i \text{ and } \psi^j \text{ anticommute,} \end{cases} \tag{2.3}$$

one can rearrange L in such a way that $\bar{\psi}^i$ stands on the left end of all terms. Let us call it L_l. In the same way, $\bar{\psi}^i$ can be brought to the right end of all terms which we shall call L_r. By varying L_l and L_r with respect to $\bar{\psi}^i$, one obtains two equations of motion for ψ^i. The uniqueness postulate requires that they should be identical. It is easy to see that this is satisfied if and only if the relation

$$\textstyle\prod_j \epsilon_{ij}=-1 \tag{2.4}$$

holds for each term of L_I which involves $\bar{\psi}^i$. The left-hand side is a product of functions ϵ_{ij} for the term of L_I considered, and -1 on the right-hand side is the value of the function ϵ_{ii} for L_0. Lemma 1 follows immediately from (2.4). Obviously the same result is obtained for any product of terms of L_I when they are space-like relative to each other.

This lemma is equivalent to the one used by Oneda and Umezawa but is derived in a much simpler way. We shall also notice that Nishizima's result is involved in this lemma since an integrability condition like $[H_1(x), H_2(x')]=0$ for space-like $x-x' (x\neq x')$ follows immediately from it.

III. CLASSIFICATION OF SPINOR FIELDS

We are now ready to classify the spinor fields with a given interaction Lagrangian into several families. Because of the Lorentz invariance, L_I can be written as follows:

$$L_I=L^{(2)}+L^{(4)}+\cdots+L^{(2k)}+\cdots, \tag{3.1}$$

where $L^{(2k)}$ is a sum of all terms which are products of $2k$ spinors.[7] In considering the effect of L_I on the commutation relation, we shall take account of the terms of this expansion successively.

Let ψ^a be an operator which appears in at least one term of $L^{(2)}$. Consider all $\psi^{a'}$ that are partners of ψ^a in $L^{(2)}$. Then consider all $\psi^{a''}$ that are partners of at least one of $\psi^{a'}$ in $L^{(2)}$ and so on. Let us call an assembly of all spinors thus obtained a "2-family" and denote it as

$$F_a^{(2)}\equiv\{\psi^a, \psi^{a'}, \cdots, \psi^{a''}, \cdots\}. \tag{3.2}$$

This is uniquely determined irrespective of which spinor in this set one starts from. By Lemma 1, a pair of spinors forming together a term of $L^{(2)}$ must anticommute with each other. It can further be shown that all spinors belonging to $F_a^{(2)}$ anticommute with each other. One can construct another 2-family $F_b^{(2)}$ starting from a spinor which appears in $L^{(2)}$ but does not belong to $F_a^{(2)}$. Obviously $F_a^{(2)}$ and $F_b^{(2)}$ have no common element. In this manner, one can construct successively several 2-families. When there are spinors which do not appear in $L^{(2)}$, however, they do not belong to any of these families. The commutation property of such a spinor with other spinors cannot be determined by $L^{(2)}$. We shall adopt the definition that each of them forms a 2-family by itself. Then the entire set of spinor fields are classified into several 2-families having the following properties. (1) Spinor fields of the same 2-family anticommute with each other. (2) Commutation relations between spinors of different 2-families are unspecified as far as $L^{(2)}$ is concerned.

The effect of $L^{(4)}$ can be taken into account in a similar but more complicated way. We first notice that if three spinors in a term of $L^{(4)}$ belong to the same 2-family, the fourth spinor must anticommute with the three spinors even if it does not belong to the same 2-family. This can be proved by solving Eq. (2.4) for this term. By Lemma 1, it is easily shown that any pairs of operators from these two 2-families are anticommutative with each other. In general, if there is a term of $L^{(4)}$ which involves three spinors (or one spinor) from a 2-family $F_a^{(2)}$ and one spinor (or three spinors) from a set of 2-families in which any pair of spinors are proved to anticommute with each other, it can be shown that all spinors of $F_a^{(2)}$ anticommute with all spinors of the latter set.

In this way one can construct a maximum set of 2-families in which any pair of spinor fields are anticommutative. We shall call it a 4-family. By the same procedure we shall construct as many 4-families as possible. If there still remain several 2-families which do not belong to any of these 4-families, we shall consider that each such 2-family forms a 4-family by itself. Then the whole set of 2-families can be classified into a set of 4-families in an unambiguous way. Obviously, spinor fields of the same 4-family anticommute with each other. Furthermore, it is easy to see that commutation relations between spinor fields belonging to different 4-families cannot be determined by $L^{(4)}$. Thus, as far as $L^{(4)}$ is concerned, a 4-family defined above is actually maximal in the sense that it cannot be extended to a larger set in which any pair of operators are anticommutative.

The same reasoning can be applied to $L^{(6)}$, $L^{(8)}$, and so on, leading to the classification of spinor fields into

[6] In the case $i=j$, an additional c number may appear in (2.2). We can neglect it, however, since it is not necessary for our purpose.

[7] Being unnecessary, terms involving only boson operators are dropped in (3.1).

6-families, 8-families, and so on. If terms beyond $L^{(2k)}$ do not provide us with a new restriction on the commutation relations, the $2k$-families will be the most precise classification of spinor fields which one can expect from the study of a given L_I. We shall then define a $2k$-family as a *family*. Summarizing the above argument, one can conclude that *when the interaction Lagrangian is given, an assembly of spinor fields is classified uniquely into several families having the following properties. (1) Spinor fields of the same family anticommute with each other. (2) Commutation relations between spinors of different families are completely unspecified.*

It is to be noticed that the distinction between spinor operators and their conjugates is not essential in this result. More detailed information about the fermion system will be obtained if one takes their difference into account. For this purpose it is useful to observe the behavior of ψ and ψ^* with respect to the transformation of their phases. We shall not go into the details of this problem, however, but simply refer to one of the results obtained. It is the following: If there is a conservation law of the number of particles at all, it must be applied to a family or an assembly of families and not to a smaller set of spinor fields.

In applying our method to the actual problem, we must note that what we find from the study of various reactions is in general not the Lagrangian itself but the S matrix. Accordingly, the foregoing discussion might have to be somewhat modified. In fact, however, one can develop essentially the same argument as before making use of the following lemma:

Lemma 2. *Each term in the normal product expansion*[8] *of the S matrix involves an even number of operators which anticommute with* ψ^i.

This can be derived at once by applying Lemma 1 to the operators of the interaction representation.

The existence of the following fermions has been more or less established: electron (e), neutrino (ν), muon (μ), κ particle (κ), proton (p), neutron (n), and V particles (V^0 and V^{ch}). Some of them might turn out to be bosons by further investigation. For the sake of definiteness, we shall assume that they are all fermions. Furthermore, we shall adopt only those interactions which are customarily assumed.[9] Then, it is easy to see that the fermions are classified into the following four 2-families:

$$F_e^{(2)}=\{e\},\quad F_\nu^{(2)}=\{\nu,\mu\},\quad F_\kappa^{(2)}=\{\kappa\},\quad F_p^{(2)}=\{p,n,V^0,V^{\mathrm{ch}}\}. \tag{3.3}$$

By making use of the direct interactions of fermions, they are further classified into the *lepton and nucleon families* as follows:

$$F_e^{(4)}=\{\nu,e,\mu,\kappa\},\quad F_p^{(4)}=\{p,n,V^0,V^{\mathrm{ch}}\}. \tag{3.4}$$

Since we have, at present, no evidence for the existence of any more complicated elementary interaction, this is the most precise classification we can expect from our knowledge about the reactions between various elementary particles.

From our general argument, it is clear that if some new fermions are added to $F_p^{(4)}$ in the future, the conservation of heavy particles should be extended to include them. On the other hand, if an interaction is found which makes the lepton and nucleon family anticommutative, it is impossible to apply the conservation law to nucleons and V particles.[10] It is not known yet whether or not the conservation of particle number should hold for the lepton family too. It is interesting to notice, however, that there are two possible cases characterized by the interactions

$$f\psi^{\mu *}\psi^{\nu *}\psi^{e}\psi^{\nu}+\mathrm{cc}, \tag{3.5}$$

and

$$f'\psi^{\mu *}\psi^{\nu}\psi^{e *}\psi^{\nu}+\mathrm{cc}, \tag{3.6}$$

in which the conservation law can be true for the leptons. Their difference lies in the point that two neutrinos are distinguishable in (3.5) while they are identical in (3.6). Since the electric charge is known to be conserved absolutely in any process, (3.6) is forbidden if ψ^μ and ψ^e have the same electric charge while (3.5) is forbidden if ψ^μ and ψ^e have electric charges of opposite signs. Evidently they are mutually exclusive. An advantage in adopting (3.6) was stressed by Konopinski and Mahmoud[11] since it automatically forbids several unobserved processes like

$$\begin{aligned}&\mu^\pm \leftrightarrow e^\pm+\gamma,\\ &\mu^\pm \leftrightarrow e^\pm+e^+ +e^-,\\ &\mu^\pm+p \text{ (or } n) \leftrightarrow e^\pm+p \text{ (or } n),\end{aligned} \tag{3.7}$$

under the assumption of validity of the conservation law for the lepton family. It is to be noticed, however, that it is impossible to explain other unobserved processes such as the $\pi-e$ decay in this way.

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[8] G. C. Wick, Phys. Rev. **80**, 268 (1950).

[9] In this paper, we shall not discuss such problems as the neutral muon, the Majorana neutrino, or the assumption of the existence of heavy bosons in beta decay suggested by Y. Tanikawa and K. Saeki, Progr. Theoret. Phys. (Japan) **10**, 232 (1953).

[10] Tanikawa proposed just such an interaction in his theory of beta decay. By assuming a special interaction, he could avoid the collapsing of matter though the conservation law of heavy *spinor* particles does not hold in his theory. See reference 9.

[11] E. J. Konopinski and H. M. Mahmoud, Phys. Rev. **92**, 1045 (1953).