Evaluation of the Collision Matrix for Dirac Particles in an External Potential*

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The collision matrix for a quantized Dirac field in an external electromagnetic potential is evaluated by a method which utilizes the close relation between the theory of positrons and the formal one-particle Dirac theory. The main mathematical tool employed is a theorem by Baker and Hausdorff on exponentials of noncommuting quantities. One obtains the collision matrix expressed as a product of exponentials, involving creation and destruction operators in a way suitable for the calculation of matrix elements in the occupation number representation. In the Appendix the evaluation of the collision matrix for a real scalar field in interaction with a given source distribution is carried out by means of a similar technique.

1. INTRODUCTION

THE purpose of the present note is to evaluate the collision matrix for a Dirac field subject to an external time-dependent (unquantized) electromagnetic potential. This problem can be solved by means of the techniques developed by Feynman[1] and Dyson,[2] and has been treated repeatedly by various authors.[3] We shall present here an alternative solution, based on the definition of the collision matrix in the Heisenberg representation given by Yang and Feldman.[4] Our presentation emphasizes the close correspondence between the formal one-particle theory, in which negative energy states are allowed, and the quantized theory of positrons. In Sec. 3 we obtain an exponential and obviously unitary form for the collision matrix and in Sec. 4 we proceed to order the creation and destruction operators. The developments of these two sections parallel those of Friedrichs,[5] and are indeed the relativistic analog of the treatment he devised for the solution of a similar and related problem. In particular we also achieve the ordering of the creation and destruction operators by the use of a theorem by Baker[6] and Hausdorff[7] on exponentials of noncommuting quantities. Although the formula given by Baker and Hausdorff has been occasionally used in field theory in simple cases when the higher commutators vanish (as it happens for instance in the problem of a given source distribution treated in the Appendix of this paper), it will become clear that our use of the theorem is of completely different nature. One of the aims of this paper is just to illustrate this particular type of application.

Our main result is expressed by the ordered form of the collision matrix given in formulas (41) and (42) or (43) below. It is not an S-ordered expression as defined by Wick[8] (in our formulas not all creation operators act after the destruction operators), although it could be transformed into an S-ordered expression. We shall not carry out the transformation, which involves some algebra, since, as we shall see, our solution is useful as it stands.

The problem treated here is the most significant one which we were able to solve in this way. It does not seem easy to generalize the present method to the more complex case of truly nonlinear interactions of quantized fields.

One of the motivations for the present work was the hope that it may represent a first step to a rigorous mathematical treatment of the problem in consideration. However we shall not investigate here the conditions under which our expressions are meaningful and our equations have solutions. In particular this applies to the reciprocals, exponentials and logarithms of the singular operators used in this paper; for small strengths of the potentials one can define these expressions by series expansions. We shall make use without justification of simple formal properties of these expansions.

2. STATEMENT OF THE PROBLEM

We first consider a c-number Dirac field subject to an external potential

$$(\gamma\partial/\partial x+m+Q)\psi=0, \tag{1}$$

$$\gamma\partial/\partial x=\gamma^\mu\partial/\partial x^\mu,\quad Q(x,x')=-ie\gamma^\mu\Phi_\mu(x)\delta(x-x'),\quad \gamma^\mu\gamma^\lambda+\gamma^\lambda\gamma^\mu=2g^{\lambda\mu}. \tag{2}$$

Our time coordinate x^0 is real, $c=\hbar=1$, $g^{00}=-1$, $g^{11}=g^{22}=g^{33}=1$, $g^{\lambda\mu}=0$ for $\lambda\neq\mu$; the matrix γ^0 is anti-Hermitian; γ^1, γ^2, and γ^3 are Hermitian. It has some formal advantages to write the potential, as we have done, as a singular integral operator. We shall also use symbolic integral operators associated with the well known singular functions S, S^+, S^{ret}, etc.: an expression such as $S^{\text{ret}}Q\psi(x)$, for instance, stands for

$$\int\int S^{\text{ret}}(x-x')dx'Q(x',x'')dx''\psi(x'').$$

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[2] F. J. Dyson, Phys. Rev. 75, 1736 (1949).
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[6] H. F. Baker, Proc. London Math. Soc. (2) 3, 24 (1905).
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[8] G. C. Wick, Phys. Rev. 80, 268 (1950).

(The integrations extend over all space-time and the spinor matrices are multiplied in the usual way.) Besides the Pauli adjoint of a spinor $\bar{\psi}=\bar{\psi}_\rho=-\psi_\sigma{}^*\gamma_{\sigma\rho}{}^0$ (sum over repeated spinor indices) we shall define the adjoint of an integral operator

$$\bar{R}=(\bar{R})_{\rho\sigma}(x,x')=-\gamma_{\rho\tau}{}^0 R_{\lambda\tau}{}^*(x',x)\gamma_{\lambda\sigma}{}^0$$

(notice the interchange of x and x'). Expressions like $\int\bar{\chi}(x')R(x',x)dx'$ and $\int\bar{\chi}(x')R(x',x'')\psi(x'')dx'dx''$ will then be abbreviated respectively as $\bar{\chi}R$ and $\bar{\chi}R\psi$. In the particular case of our potential we have

$$\bar{Q}=Q. \qquad (3)$$

The following relations involving the singular kernels will be used[9]

$$S=S^{\mathrm{adv}}-S^{\mathrm{ret}}=S^{+}+S^{-},$$

$$\bar{S}=-S,\quad \bar{S}^{\mathrm{ret}}=S^{\mathrm{adv}},\quad \overline{S^{+}}=-S^{+},\quad \overline{S^{-}}=-S^{-},$$

$$\left(\gamma\frac{\partial}{\partial x}+m\right)S^{\mathrm{ret}}=\left(\gamma\frac{\partial}{\partial x}+m\right)S^{\mathrm{adv}}=-1.$$

For any solution ψ of (1) we define the incoming and outgoing fields,

$$\psi^{\mathrm{in}}=\psi-S^{\mathrm{ret}}Q\psi, \qquad (4)$$

$$\psi^{\mathrm{out}}=\psi-S^{\mathrm{adv}}Q\psi. \qquad (5)$$

It is clear from this definition that both ψ^{in} and ψ^{out} satisfy the free-field equation,

$$(\gamma\partial/\partial x+m)\psi^{\mathrm{in}}=0, \qquad (6)$$
$$(\gamma\partial/\partial x+m)\psi^{\mathrm{out}}=0.$$

In scattering problems one considers (4) as an integral equation for ψ, to be solved for given ψ^{in} satisfying (6). We shall not investigate here the conditions for the existence and the uniqueness of the solution of (4). It is likely that, with suitable restrictions on the potential, both existence and uniqueness will be ensured, and the homogeneous equations corresponding to (4) and (5) will have no solutions.

Eliminating ψ from (4) and (5), the relation between ψ^{in} and ψ^{out} can be written as

$$\psi^{\mathrm{out}}=\psi^{\mathrm{in}}-SQ(1-S^{\mathrm{ret}}Q)^{-1}\psi^{\mathrm{in}}. \qquad (7)$$

The linear operator transforming ψ^{in} into ψ^{out} can be identified with the collision matrix for the formal one particle theory. As we shall see in Sec. 3, it is unitary with respect to the appropriate inner product.

If the Dirac field is quantized all the above relations are still valid as relations between the operators ψ, ψ^{in} and ψ^{out}. The operators ψ^{in} and ψ^{out} both satisfy the free-field equation and the free-field (anti-) commutation relation and we are led to the problem to find a unitary operator $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\psi_\rho{}^{\mathrm{in}}(x)\mathbf{S}=\psi_\rho{}^{\mathrm{out}}(x). \qquad (8)$$

According to Yang and Feldman, $\mathbf{S}$ is the collision matrix; we should like to express it in terms of the incoming fields only. We also should like to find a form for $\mathbf{S}$, in which the creation and destruction operators are ordered in such a way that it is possible to evaluate the effect of the application of $\mathbf{S}$ to a state vector in the occupation number representation.

3. EXPONENTIAL FORM OF THE COLLISION MATRIX

In this section and in Sec. 4 we want to investigate the relation between equations of the type of (7) and (8). We consider general linear transformations operating in the space D of the solutions of

$$(\gamma\partial/\partial x+m)\phi=0. \qquad (9)$$

We shall first assume that the fields are unquantized.

Every solution of (9) and its adjoint can be written, respectively, as

$$\phi=S\phi_1,\quad \bar{\phi}=\bar{\phi}_1\bar{S}=-\bar{\phi}_1S.$$

Here the spinor ϕ_1 is not uniquely determined by ϕ since it can be changed into $\phi_1+\eta$ where $S\eta=0$. The operator S projects a general spinor function of x onto D. We can define the inner product of two solutions ϕ and ξ of (9) in the following way:

$$(\xi,\phi)=-i\bar{\xi}_1S\phi_1. \qquad (10)$$

This inner product depends only upon the solutions ξ and ϕ themselves; it is a complex number, and it would not be difficult to show that it is equal to the more customary expression $\int\xi^*\phi d^3x$ where the three-dimensional integration is extended over a manifold $x^0=$ constant.

If ϕ is in D, a linear relation such as

$$\chi=(1+SY)\phi=(1+SY)S\phi_1, \qquad (11)$$

$$\bar{\chi}=\bar{\phi}(1-\bar{Y}S)=-\bar{\phi}_1S(1-\bar{Y}S), \qquad (12)$$

insures that χ also is in D. If we are interested only in linear operations in D we can consider Y to be to a certain extent arbitrary; it could be changed and still give the same linear relation in D provided SYS remains the same. This situation will come up in various instances because of the presence of a projection operator, such as S, in the formulas.

We shall now require that the linear operator $1+SY$ in (11) is unitary with respect to the inner product (10). The operator is length-preserving if

$$(\xi,\phi)=((1+SY)\xi,\ (1+SY)\phi)$$
$$=-i\bar{\xi}_1(1-S\bar{Y})S(1+YS)\phi_1,$$

for arbitrary ξ_1 and ϕ_1, which means

$$(1-S\bar{Y})S(1+YS)=S. \qquad (13)$$

[9] The symbol $\bar{S}$ is used here to denote the adjoint of the integral operator S, and not, as it is customary in the literature, the singular kernel $\frac{1}{2}(S^{\mathrm{adv}}+S^{\mathrm{ret}})$ for which we reserve the letter G (see Sec. 5).

The operator is also unitary if, in addition,

$$(1+SY)S(1-\bar{Y}S)=S. \tag{14}$$

The relation (7) between ψ^{in} and ψ^{out} is a particular case of (11) for

$$Y=-Q(1-S^{\mathrm{ret}}Q)^{-1}, \tag{15}$$

and one could check, using (3), that for this particular choice of Y both (13) and (14) are verified.

As a consequence of relations (13) and (14) we can assume the existence of an operator P such that

$$(1+SY)S=e^{SP}S \tag{16}$$

and

$$S\bar{P}S=SPS \tag{17}$$

hold; condition (17) means that P is anti-Hermitian in D in the sense of the inner product (10).

We now go over to the case of quantized fields. The conditions (13) and (14) are necessary and sufficient to insure that χ satisfies the free-field commutation relations if ϕ does, and vice versa. For instance, if

$$\{\phi,\bar{\phi}\}=-iS, \tag{18}$$

we have from (14)

$$\{\chi,\bar{\chi}\}=(1+SY)\{\phi,\bar{\phi}\}(1-\bar{Y}S)$$
$$=-i(1+SY)S(1-\bar{Y}S)=-iS.$$

We can express the relation between χ and ϕ by means of a transformation analogous to (8). With the definition

$$[P]=\tfrac{1}{2}i\int\int P_{\rho\lambda}(x,x')\{\bar{\phi}_\rho(x)\phi_\lambda(x') -\phi_\lambda(x')\bar{\phi}_\rho(x)\}dxdx', \tag{19}$$

the commutator identities

$$[\phi_\rho(x),[P]]=(SP\phi)_\rho(x), \tag{20}$$

$$[\bar{\phi}_\rho(x),[P]]=-(\bar{\phi}PS)_\rho(x), \tag{21}$$

follow from (18). [No use is made here of (17), but we shall assume that (17) is satisfied.] As will be shown below, one has as a consequence

$$\chi=e^{SP}\phi=\mathbf{S}^{-1}\phi\mathbf{S}, \tag{22}$$

$$\bar{\chi}=\bar{\phi}e^{-PS}=\mathbf{S}^{-1}\bar{\phi}\mathbf{S}, \tag{23}$$

with

$$\mathbf{S}=\exp[P]. \tag{24}$$

Owing to (17), $\mathbf{S}$ is unitary; formula (24) solves the problem of finding an exponential form for the collision matrix. We see that the operator in the exponent of (24) is the second quantized operator of the operator P in the exponent of (16) in the sense of the theory of positrons (Heisenberg's rule). It should be noted that for $[P]$ only the value of SPS is relevant.

To verify (22), for instance, consider the derivative of the operator

$$f(\lambda)=e^{\lambda[P]}e^{\lambda SP}\phi(x)e^{-\lambda[P]}$$

with respect to the parameter λ. We have

$$f'(\lambda)=e^{\lambda[P]}e^{\lambda SP}\{SP\phi(x)-[\phi(x),[P]]\}e^{-\lambda[P]}$$

and because of (20), $f'(\lambda)=0$; therefore $f(1)=f(0)=\phi$.

4. ORDERED FORM OF THE COLLISION MATRIX

We now proceed to the separation of the creation and destruction operators by suitably decomposing (24) into a product of exponentials. We define as "Lie polynomial" a homogeneous polynomial in certain variables α, β, $\cdots$, which can be constructed from Lie polynomials of lower order by forming commutators and linear combinations with constant coefficients. Any formal series of Lie polynomials will be called a "Lie function."

Making use of (20) and (21) one can easily verify the second commutator identity,

$$[[P_1],[P_2]]=[P], \tag{25}$$

where P satisfies

$$SP_1SP_2S-SP_2SP_1S=SPS. \tag{26}$$

It is then apparent, by repeated application of (25), that, if $l(\alpha,\beta)$ is a Lie function, the relation $SPS=l(SP_1,SP_2)S=Sl(P_1S,P_2S)$ entails $[P]=l([P_1],[P_2])$, and vice versa. This result can be obviously generalized to Lie functions of more than two variables.

Following Friedrichs, we shall now make use of a theorem discovered independently by Baker and Hausdorff: if α, β, and γ satisfy the formal relation $e^\alpha e^\beta=e^\gamma$, then

$$\gamma=\alpha+\beta+\tfrac{1}{2}[\alpha,\beta]+(1/12)([[\alpha\beta]\beta]+[[\beta\alpha]\alpha])+\cdots$$

is a Lie function of α and β.[10] The generalization to the product of more than two exponentials is obvious. The observation of the preceding paragraph allows us to conclude that if

$$e^{SP_1}e^{SP_2}e^{SP_3}S=e^{SP}S, \tag{27}$$

then

$$e^{[P_1]}e^{[P_2]}e^{[P_3]}=e^{[P]}, \tag{28}$$

and vice versa.

Equation (28) can be used to obtain the form of the operator $\mathbf{S}$ ordered in the desired way with respect to creation and destruction operators by having each of the exponentials in the left-hand side contain only certain creation or destruction operators. More precisely, let us define the two projection operators E^+ and E^- which transform a function of the space-time variable x into its positive- and negative-frequency parts, respectively. They satisfy

$$\begin{gathered} E^+f(x)=f^+(x),\quad E^-f(x)=f^-(x),\\ E^++E^-=1,\quad (E^+)^2=E^+,\quad (E^-)^2=E^-,\\ E^+E^-=E^-E^+=0,\quad E^+=\overline{E^+},\quad \overline{E^-}=E^-. \end{gathered} \tag{29}$$

[10] It is not possible to give an explicit formula for γ, but one can give a recursive construction of it. For details see, for instance, W. Magnus, Report No. BR-3, New York University Mathematics Research Group, 1953 (unpublished).

It follows that

$$\bar{f}(x)E^+=\bar{f}^-(x), \quad \bar{f}(x)E^-=\bar{f}^+(x), \tag{30}$$

and that

$$E^+S=S^+=SE^+, \quad E^-S=S^-=SE^-. \tag{31}$$

We shall require that P_1, P_2, and P_3 not only satisfy (27) with the right-hand side given by (16) but also be of the form

$$\begin{aligned} iP_1&=E^+AE^-,\\ iP_2&=E^+BE^+-E^-CE^-,\\ iP_3&=E^-DE^+.\end{aligned} \tag{32}$$

One can then easily see from (29) and (30) that

$$\begin{aligned}[][P_1]&=\bar{\phi}^-A\phi^-,\\ [P_2]&=\bar{\phi}^-B\phi^++(C\phi^-)\bar{\phi}^++\tfrac{1}{2}i\,\mathrm{Tr}BS^++\tfrac{1}{2}i\mathrm{Tr}CS^-,\\ [P_3]&=\bar{\phi}^+D\phi^+;\end{aligned} \tag{33}$$

here we have made use of

$$\{\phi^-,\bar{\phi}^+\}=-iS^-, \quad \{\phi^+,\bar{\phi}^-\}=-iS^+, \tag{34}$$

and the symbol Tr denotes a trace taken by summing over the spinor indices and integrating over the four-dimensional variable x.

Our present problem is to find the operators A, B, C, and D such that (27) is satisfied. Substituting (32) into (27) we obtain the relation which these operators must satisfy. The first exponential in the left-hand side of (27) reduces to

$$\begin{aligned} e^{SP_1}&=\exp(-iSE^+AE^-)\\ &=\exp(-iE^+SAE^-)=1-iE^+SAE^-,\end{aligned}$$

because the higher-order terms in the development of the exponential vanish as a consequence of the property of the projection operators $E^-E^+=0$. Similarly, for the other two exponentials, one has (using also $E^+E^-=0$)

$$e^{SP_3}=\exp(-iSE^-DE^+)=1-iE^-SDE^+$$

and

$$\begin{aligned} e^{SP_2}&=\exp(-iSE^+BE^++iSE^-CE^-)\\ &=\exp(-iE^+SBE^+)\exp(iE^-SCE^-).\end{aligned}$$

From (16), the relation to be satisfied becomes

$$\begin{aligned}&(1-iE^+SAE^-)\exp(-iE^+SBE^+)\exp(iE^-SCE^-)\\ &\qquad\times(1-iE^-SDE^+)S=(1+SY)S.\end{aligned} \tag{35}$$

The presence of the projection operators S, E^+, and E^- in this formula has again the consequence that A, B, C, and D are not uniquely determined by SYS; only their relevant projections are, and this agrees with the fact that only their projections occur in (33). Separation of the various frequency signs in (35) gives the four relations

$$\begin{aligned}&[\exp(iS^-C)-1]S^-=S^-YS^-,\\ &\exp(iS^-C)iS^-DS^+=S^-YS^+,\\ &iS^+A\exp(iS^-C)S^-=S^+YS^-,\\ &[\exp(-iS^+B)-1-S^+A\exp(iS^-C)S^-D]S^+=S^+YS^+.\end{aligned} \tag{36}$$

Before we can proceed to transform (36) we must derive some consequences of the unitarity conditions (13) and (14). We first have the relation

$$S(1-\bar{Y}S^+)YS=S\bar{Y}(1+S^-Y)S, \tag{37}$$

which is equivalent to (13), since $S=S^++S^-$. From (37) we obtain

$$S^+(1-\bar{Y}S^+)YS^-=S^+\bar{Y}(1+S^-Y)S^-,$$

and, multiplying by $(1+YS^-)^{-1}$ on the right and by $(1-S^+\bar{Y})^{-1}$ on the left, we have

$$S^+Y(1+S^-Y)^{-1}S^-=S^+\bar{Y}(1-S^+\bar{Y})^{-1}S^-. \tag{38}$$

In a similar way, from (14), we obtain

$$S^-Y(1+S^-Y)^{-1}S^+=S^-\bar{Y}(1-S^+\bar{Y})^{-1}S^+. \tag{39}$$

We can now proceed to solve the four equations (36). The first three of them give, without difficulty,

$$\begin{aligned} iS^-CS^-&=\{\log(1+S^-Y)\}S^-,\\ iS^-DS^+&=S^-(1+YS^-)^{-1}YS^+=S^-Y(1+S^-Y)^{-1}S^+\\ &=S^-\bar{Y}(1-S^+\bar{Y})^{-1}S^+,\\ iS^+AS^-&=S^+Y(1+S^-Y)^{-1}S^-=S^+\bar{Y}(1-S^+\bar{Y})^{-1}S^-.\end{aligned}$$

From the fourth, one has

$$\exp(-iS^+B)S^+=[1+S^+Y+S^+A\exp(iS^-C)S^-D]S^+;$$

substituting for S^+AS^- the expression just found, and for $\exp(iS^-C)S^-DS^+$ the value given by the third equation (36), one obtains

$$\begin{aligned}&\exp(-iS^+B)S^+\\ &=\{1+S^+Y-S^+\bar{Y}(1-S^+\bar{Y})^{-1}S^-Y\}S^+\\ &=(1-S^+\bar{Y})^{-1}\{(1-S^+\bar{Y})(1+S^+Y)-S^+\bar{Y}S^-Y\}S^+\\ &=(1-S^+\bar{Y})^{-1}\{1+S^+(Y-\bar{Y}-\bar{Y}SY)\}S^+\\ &=(1-S^+\bar{Y})^{-1}.\end{aligned} \tag{40}$$

Here we have used the fact that

$$S^+(Y-\bar{Y}-\bar{Y}SY)S^+=0,$$

which is a consequence of (13). Equation (40) can be solved in the form

$$iS^+BS^+=\{\log(1-S^+\bar{Y})\}S^+.$$

Summarizing our results, we now make use of (24), (28), and (33), and have for the collision matrix

$$\begin{aligned}\mathbf{S}&=\exp(\bar{\phi}^-A\phi^-)\exp[\bar{\phi}^-B\phi^++(C\phi^-)\bar{\phi}^+]\exp(\bar{\phi}^+A\phi^+)\\ &\qquad\times\exp(\tfrac{1}{2}i\,\mathrm{Tr}S^+B+\tfrac{1}{2}i\,\mathrm{Tr}S^-C),\end{aligned} \tag{41}$$

in which we can take

$$\begin{aligned} iA &= Y(1+S^{-}Y)^{-1}, \\ iS^{-}CS^{-} &= \{\log(1+S^{-}Y)\}S^{-}, \\ iS^{+}BS^{+} &= \{\log(1-S^{+}\bar{Y})\}S^{+}. \end{aligned} \tag{42}$$

The ordered form (41) is not S-ordered, because not all destruction operators are to the right of all creation operators. Nevertheless, it can be directly used as it is to calculate matrix elements of **S** between a final and an initial state in which, let us say, only a finite number of particles are present. Expanding the first and third exponential in a power series it is clear that only a finite number of terms of each has to be retained, because the exponents contain only creation and destruction operators, respectively. As for the second exponential, it obviously commutes with the operator which gives the number of particles, and therefore its effect on any vector representing a state in which only a finite number of particles is present can be evaluated without much difficulty and without series expansions. The fourth exponential is just a numerical factor, the vacuum expectation value of the collision matrix. As compared with an S-ordering, the form (41) has the advantage that some terms of the S-ordered expansion are already summed together in the second exponential operator. It is possible, on the other hand, to S-order this second exponential; one would then obtain a closed expression for **S** equivalent to the Feynman-Dyson expansion. We shall not do it here, since it requires some manipulations, and does not represent an improvement over (41).

5. EVALUATION OF THE COLLISION MATRIX

The results of the preceding section can now be applied to the particular case of interest in which Y is given by (15) and $\phi=\psi^{\mathrm{in}}$.

By use of the Feynman function S^F, characterized by

$$\tfrac{1}{2}iS^F=S^{\mathrm{ret}}+S^{-}=S^{\mathrm{adv}}-S^{+},$$

a straightforward calculation gives

$$\begin{aligned} 1+S^{-}Y &= 1-S^{-}Q(1-S^{\mathrm{ret}}Q)^{-1} \\ &= (1-\tfrac{1}{2}iS^FQ)(1-S^{\mathrm{ret}}Q)^{-1}, \\ 1-S^{+}\bar{Y} &= 1+S^{+}Q(1-S^{\mathrm{adv}}Q)^{-1} \\ &= (1-\tfrac{1}{2}iS^FQ)(1-S^{\mathrm{adv}}Q)^{-1}. \end{aligned}$$

Therefore the expressions given in (42) become

$$\left.\begin{aligned} A &= iQ(1-\tfrac{1}{2}iS^FQ)^{-1} \\ iS^{-}C &= \log\{(1-\tfrac{1}{2}iS^FQ)(1-S^{\mathrm{ret}}Q)^{-1}\} \\ iS^{+}B &= \log\{(1-\tfrac{1}{2}iS^FQ)(1-S^{\mathrm{adv}}Q)^{-1}\}, \end{aligned}\right\} \tag{43}$$

dropping a common factor S^+ or S^-.

We can evaluate the traces occurring in (37) making use of the identity $\mathrm{Tr}\log\alpha\beta=\mathrm{Tr}\log\alpha+\mathrm{Tr}\log\beta$, valid also if α and β do not commute. This identity is a consequence of the fact, stated in the Baker-Hausdorff theorem, that $\log\alpha\beta$ is a Lie function of $\log\alpha$ and $\log\beta$; indeed the trace of the commutator of two operators is zero. One has

$$\begin{aligned} \tfrac{1}{2}i\,\mathrm{Tr}S^{+}B+\tfrac{1}{2}i\,\mathrm{Tr}S^{-}C &= \mathrm{Tr}\log(1-\tfrac{1}{2}iS^FQ) \\ &\quad -\tfrac{1}{2}\,\mathrm{Tr}\log(1-S^{\mathrm{ret}}Q)-\tfrac{1}{2}\,\mathrm{Tr}\log(1-S^{\mathrm{adv}}Q). \end{aligned} \tag{44}$$

It is clear that so far in all our considerations we have not made use of the particular form (2) of the operator Q, i.e., our formulas would be true also if Q would not contain a δ-function factor (nonlocal interaction). If however (2) is valid, as we shall assume from now on, the sum of the second and third term in the right-hand side of (44) vanishes. To show this, consider the series development of the logarithms. The terms of first order give

$$\tfrac{1}{2}\,\mathrm{Tr}S^{\mathrm{ret}}Q+\tfrac{1}{2}\,\mathrm{Tr}S^{\mathrm{adv}}Q = \mathrm{Tr}GQ=-ie\,\mathrm{tr}\{G(0)\gamma^{\mu}\}\int\Phi_{\mu}(x)dx, \tag{45}$$

where we have introduced the singular function[9]

$$G=\tfrac{1}{2}(S^{\mathrm{adv}}+S^{\mathrm{ret}}),$$

used the definition (2) of Q, and denoted by tr the trace taken over the spin indices only. But the right-hand side of (45) is zero, owing to

$$\mathrm{tr}\{G(0)\gamma^{\mu}\}=0.$$

The higher-order terms are also zero. For instance

$$\begin{aligned} \mathrm{Tr}S^{\mathrm{ret}}QS^{\mathrm{ret}}Q &= -e^2\int\int S^{\mathrm{ret}}(x-x')\gamma^{\mu}\Phi_{\mu}(x') \\ &\qquad \times S^{\mathrm{ret}}(x'-x)\gamma^{\lambda}\Phi_{\lambda}(x)dxdx' \\ &= 0, \end{aligned}$$

because $S^{\mathrm{ret}}(x-x')$ is zero whenever the point x is outside the future light cone emanating from x', and therefore at least one of the two S^{ret} functions occurring in (46) vanishes.

The vacuum expectation value of the collision matrix now becomes simply $\exp\mathrm{Tr}\log(1-\tfrac{1}{2}iS^FQ)$. This expression has been previously given by various authors.[11] Its expansion is identical with the evaluation of the same quantity given by Feynman;[12] it is known from his work that $\mathrm{Tr}\log(1-\tfrac{1}{2}iS^FQ)$ has an infinite imaginary part, coming from the second term of the expansion (the first term is zero), and is otherwise finite. One can drop this infinite imaginary part; this amounts to a redefinition of the collision matrix by a phase factor, and does not affect transition probabilities. It is connected with the charge renormalization.

In conclusion, the author wishes to express his thanks to Professor K. O. Friedrichs and to Dr. H. E. Moses for many helpful discussions.

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[11] For instance M. Neuman, Phys. Rev. **85**, 129 (1952).

[12] See reference 1, Sec. 5.

APPENDIX. GIVEN SOURCE DISTRIBUTION

We consider briefly the evaluation of the collision matrix for a real Bose field influenced by a given source distribution. For simplicity we take a scalar field

$$(\Box-m^2)\phi(x)=\rho(x). \tag{A.1}$$

The solution of this problem can be obtained, for instance, by use of Feynman's operator calculus,[13] and has been given also by other authors.[14] We want here to make use of the formula

$$e^{\alpha+\beta}=e^{\alpha}e^{\beta}e^{\frac{1}{2}[\beta,\alpha]}, \tag{A.2}$$

valid whenever α and β commute with their commutator, in particular if this commutator is a number. Equation (A.2) is an immediate consequence of the more general formula given by Baker and Hausdorff.

With a notation perfectly analogous to the one used in the main text, we define the incoming and outgoing fields:

$$\phi^{\mathrm{in}}=\phi+\Delta^{\mathrm{ret}}\rho, \tag{A.3}$$

$$\phi^{\mathrm{out}}=\phi+\Delta^{\mathrm{adv}}\rho. \tag{A.4}$$

Here the elimination of ϕ can be completely achieved

$$\phi^{\mathrm{out}}=\phi^{\mathrm{in}}+\Delta\rho, \tag{A.5}$$

and the collision matrix, defined by

$$\mathbf{S}^{-1}\phi^{\mathrm{in}}\mathbf{S}=\phi^{\mathrm{out}}, \tag{A.6}$$

is given by the exponential

$$\mathbf{S}=\exp[-i\rho\phi^{\mathrm{in}}]. \tag{A.7}$$

In the following let us simply write ϕ for ϕ^{in}. To verify (A.7) one can introduce the operator,

$$f(\lambda)=e^{-i\lambda\rho\phi}(\phi+\lambda\Delta\rho)e^{i\lambda\rho\phi},$$

and notice that, as a consequence of

$$f'(\lambda)=e^{-i\lambda\rho\phi}(\Delta\rho-i[\rho\phi,\phi])e^{i\lambda\rho\phi}=0,$$

one has

$$f(1)=f(0).$$

Application of (A.2) immediately gives an ordered form of the collision matrix, if we take $\alpha=-i\rho\phi^-$, $\beta=-i\rho\phi^+$, $[\beta,\alpha]=i\rho\Delta^-\rho$ which is a number. One has

$$\begin{aligned}\mathbf{S}&=\exp[-i\rho(\phi^++\phi^-)]\\&=\exp(-i\rho\phi^-)\exp(-i\rho\phi^+)\exp(\tfrac{1}{2}i\rho\Delta^-\rho).\end{aligned} \tag{A.8}$$

From this ordered form one can easily derive an expression for the probability P_n that n particles are created from the vacuum. It is given by $\sum|(\Phi_n,S\Phi_0)|^2$, if we denote by Φ_0 the vacuum of incoming particles, by Φ_n a state in which n particles are present and sum over all states Φ_n. It is clear from (A.8) that

$$P_n=\sum\left|\left(\Phi_n,\frac{(-i\rho\phi^-)^n}{n!}\Phi_0\right)\exp(\tfrac{1}{2}i\rho\Delta^-\rho)\right|^2; \tag{A.9}$$

indeed, the destruction operator ϕ^+ in the second exponential applied to the vacuum gives zero, and the only term in the expansion of the first exponential which has a matrix element different from zero is the one which creates n particles. In (A.9) the sum $\sum$ can now be extended to states with any number of particles since only matrix elements which are zero are added. One then has

$$P=\frac{1}{(n!)^2}(\Phi_0,(i\rho\phi^+)^n(-i\rho\phi^-)^n\Phi_0)\exp(-\tfrac{1}{2}\rho\Delta^{(1)}\rho); \tag{A.10}$$

here use has been made of $(\Delta^-)^*=\Delta^+$ and of

$$i(\Delta^+-\Delta^-)=\Delta^{(1)}.$$

One can bring the destruction operators in (A.10) to the right of the creation operators by use of the commutation relation. This gives $n!$ terms equal to $(i\rho\Delta^+\rho)^n$; observing that

$$\rho\Delta^+\rho=-\rho\Delta^-\rho,$$

one has finally for P_n the well-known Poisson distribution:[15]

$$P_n=\frac{1}{n!}W^ne^{-W},\quad W=\tfrac{1}{2}\rho\Delta^{(1)}\rho.$$

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[13] See Eq. (49) of Feynman's paper quoted in reference 3.

[14] W. Thirring and B. Touschek, Phil. Mag. (7) 42, 244 (1951); R. J. Glauber, Phys. Rev. 84, 395 (1951).

[15] See, for instance, reference 14.