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Model of the Causal Interpretation of Quantum Theory in Terms of a Fluid with Irregular Fluctuations

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In this paper, we propose a physical model leading to the causal interpretation of the quantum theory. In this model, a set of fields which are equivalent in many ways to a conserved fluid, with density $|\psi|^2$, and local stream velocity, $d\xi/dt=\nabla S/m$, act on a particle-like inhomogeneity which moves with the local stream velocity of the equivalent fluid. By introducing the hypothesis of a very irregular and effectively random fluctuation in the motions of the fluid, we are able to prove that an arbitrary probability density ultimately decays into $|\psi|^2$. Thus, we answer an important objection to the causal interpretation, made by Pauli and others. This result is extended to the Dirac equation and to the many-particle problem.

1. INTRODUCTION

A CAUSAL interpretation of the quantum theory has been proposed,[1,2] involving the assumption that an electron is a particle following a continuous and causally defined trajectory with a well-defined position, $\xi(t)$, accompanied by a physically real wave field, $\psi(\mathbf{x},t)$. To obtain all of the results of the usual interpretation, the following supplementary assumptions had to be made:

1. $\psi(\mathbf{x},t)$ satisfied Schrödinger's equation.
2. $d\xi/dt=\nabla S/m$, where $\psi=R\exp(iS/\hbar)$.
3. The probability distribution in an ensemble of electrons having the same wave function, is $P=|\psi|^2$.

These assumptions were shown to be consistent.

Assumption (3), however, has been criticized by Pauli[3] and others[4] on the ground that such a hypothesis is not appropriate in a theory aimed at giving a causal explanation of the quantum mechanics. Instead, they argue it should be possible to have an arbitrary probability distribution [a special case of which is the function $P=\delta(\mathbf{x}-\mathbf{x}_0)$, representing a particle in a well-defined location], that is at least in principle independent of the ψ field and dependent only on our degree of information concerning the location of the particle.

In a more recent paper,[5] one of us has proposed a means of dealing with this problem by explaining the relation, $P=|\psi|^2$ in terms of random collision processes. It was shown in a simplified case that a statistical ensemble of quantum-mechanical system with an arbitrary initial probability distribution decays in time to an ensemble with $P=|\psi|^2$. This is equivalent to a proof of Boltzmann's H theorem in classical mechanics. Thus, we can answer the objection of Pauli, for no matter what the initial probability distribution may have been (for example, a delta function), it will eventually be given by $P=|\psi|^2$.

In the work cited above, however, certain mathematical difficulties make a generalization of the results to an arbitrary system very difficult. (The difficulties are rather analogous to these appearing in classical statistical mechanics when one tries rigorously to treat the approach of a distribution to equilibrium, by means of demonstrating a quasi-ergodic character of the motion). In the present paper, we shall avoid these difficulties by taking advantage of the fact that the causal interpretation of the quantum theory permits an unlimited number of new physical models, of types not consistent with the usual interpretation, which lead to the usual theory only as an approximation, and which may lead to appreciably different results at new levels (e.g., 10^{-13} cm). The model that we shall propose here furnishes the basis for a simple deduction of the relation, $P=|\psi|^2$; and in addition, gives a possible physical interpretation of the relation $d\xi/dt=\nabla S/m$ (postulate 2), which follows rather naturally from the model. This model is an extension of the causal interpretation of the quantum theory already proposed, which provides a more concrete physical image of the meaning of our postulates than has been available before, and which suggests new properties of matter that may exist at deeper levels.

2. THE HYDRODYNAMIC MODEL

The model that we shall adopt in this paper is an extension of a hydrodynamic model, originally proposed by Madelung[6] and later developed further by Takabayasi[7] and by Schenberg.[8] To obtain this model, we first write down Schrödinger's equation in terms of the

[1] L. de Broglie, Compt. rend. **183**, 447 (1926); **184**, 273 (1927); **185**, 380 (1927).
[2] D. Bohm, Phys. Rev. **85**, 166, 180 (1952).
[3] *Les Savants et le Monde*, Collection dirigée par André George, *Louis de Broglie, Physicien et Penseur* (Editions Albin Michel, Paris, 1953).
[4] J. B. Keller, Phys. Rev. **89**, 1040 (1953).
[5] D. Bohm, Phys. Rev. **89**, 1458 (1953).

[6] E. Madelung, Z. Physik **40**, 332 (1926).
[7] T. Takabayasi, Progr. Theoret. Phys. (Japan) **8**, 143 (1952); **9**, 187 (1953).
[8] M. Schenberg, Nuovo cimento (to be published).

variables, R and S, where $\psi=R\exp(iS/\hbar)$:

$$\partial R^2/\partial t+\operatorname{div}(R^2\nabla S/m)=0, \tag{1}$$

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}+V=0. \tag{2}$$

Now Madelung originally proposed that R^2 be interpreted as the density $\rho(\mathbf{x})$ of a continuous fluid, which had the stream velocity $\mathbf{v}=\nabla S/m$. Thus, the fluid is assumed to undergo only potential flow. Equation (1) then expresses the conservation of fluid, while Eq. (2) determines the changes of the velocity potential S in terms of the classical potential V, and the "quantum potential":

$$\frac{-\hbar^2}{2m}\frac{\nabla^2 R}{R}=\frac{-\hbar^2}{4m}\left[\frac{\nabla^2\rho}{\rho}-\frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right].$$

As shown by Takabayasi[7] and by Schenberg,[8] the quantum potential may be thought of as arising in the effects of an internal stress in the fluid. This stress depends, however, on derivatives of the fluid density, and therefore is not completely analogous to the usual stresses, such as pressures, which are found in macroscopic fluids.

The above model is, however, not adequate by itself; for it contains nothing to describe the actual location, $\xi(t)$, of the particle, which makes possible, as we have seen in previous papers,[2,5] a consistent causal interpretation of the quantum theory. At this point, we therefore complete the model by postulating a particle, which takes the form of a highly localized inhomogeneity that moves with the local fluid velocity, $\mathbf{v}(\mathbf{x},t)$. The precise nature of this inhomogeneity is irrelevant for our purposes. It could be, for example, a foreign body, of a density close to that of the fluid, which was simply being carried along with the local velocity of the fluid as a small floating body is carried along the surface of the water at the local stream velocity of the water. Or else it could be a stable dynamic structure existing in the fluid; for example, a small stable vortex or some other stable localized structure, such as a small pulse-like inhomogeneity. Such structures might be stabilized by some nonlinearity that would be present in a more accurate approximation to the equations governing the fluid motions than is given by (1) and (2).

3. FLUCTUATIONS OF THE MADELUNG FLUID

Thus far we have been assuming that the Madelung fluid undergoes some regular motion, which can in principle be calculated by solving Schrödinger's equation with appropriate boundary conditions. We know, however, that in all real fluids ever met with thus far (and indeed, in all physically real fields also) the motions never take precisely the forms obtained by solving the appropriate equations with the correct boundary conditions. For there always exist random fluctuations. These fluctuations may have many origins. For example, real fluids may be subject to irregular disturbance originating outside the fluid and transmitted to it at the boundaries. Moreover, because the equations of motion flow of the fluid are, in general, nonlinear, the fluid motion may be unstable, so that irregular turbulent motion may arise within the fluid itself. And finally, because of the underlying constitution of the fluid in terms of molecules in random thermal motion, there may exist a residual Brownian movement in the fluid, even for fluid elements that are large enough to contain a great many molecules. Thus, in a real fluid, there are ample reasons why the usual hydrodynamical equations will, in general, describe only some mean or average aspect of the motion, while the actual motion has an addition some very irregular fluctuating components, which are effectively random.

Since the Madelung fluid is being assumed to be some kind of physically real fluid, it is therefore quite natural to suppose that it too undergoes more or less random fluctuations in its motions. Such random fluctuations are evidently consistent within the framework of the causal interpretation of the quantum theory. Thus, there are always random perturbations of any quantum mechanical system which arise outside that system. (Indeed, as we have already shown in a previous paper,[5] the effects of such perturbations are by themselves capable of explaining the probability distribution, $P=|\psi|^2$, at least for certain simple systems.) We may also assume that the equations governing the ψ field have nonlinearities, unimportant at the level where the theory has thus far been successfully applied, but perhaps important in connection with processes involving very short distances. Such nonlinearities could produce, in addition to many other qualitatively new effects, the possibility of irregular turbulent motion. Moreover, we may conceive of a granular substructure of matter underlying the Madelung fluid, analogous to (but not necessarily of exactly the same kind as) the molecular structure underlying ordinary fluids.

We may therefore assume that for any or all of these reasons, or perhaps for still other reasons not mentioned here, our fluid undergoes a more or less random type of fluctuation about the Madelung motion as a mean. Thus, the velocity will not be exactly equal to $\nabla S/m$, nor will the density, ρ, be exactly equal to $|\psi|^2$. All that we require is that the relations $\rho=|\psi|^2$ and $\mathbf{v}=\nabla S/m$ be valid as averages. Indeed, it is not even necessary that the exact velocity be derivable from a potential. Thus, we would have $d\xi/dt=\nabla S'/m+\nabla\times\mathbf{A}$, more generally,[7–9] where $\langle\nabla\times\mathbf{A}\rangle_{\mathrm{Av}}=0$ and $\langle\nabla S'\rangle_{\mathrm{Av}}=\langle\nabla S\rangle_{\mathrm{Av}}$. Hence Schrödinger's equation will not apply to the fluctua-

[9] Such vortex components of the velocity may also explain the appearance of "spin" provided that they could have a regular component as well as a random component. Indeed, in another paper, the Pauli equation will be treated from this point of view. But here we concern ourselves only with a level of precision in which the spin can be neglected, so that Schrödinger's equation is a good approximation for the mean behavior of the fluid.

tions. However, the conservation equation $\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v})=0$ will be assumed to hold even during a fluctuation. Such an equation is implied almost by the very concept of a fluid; for if there were no conservation, then the model of a fluid would lose practically all of its content.

From the above assumptions, it is clear that if we followed a given fluid element, we would discover that it undergoes an exceedingly irregular motion, which is able in time to carry it from any specified trajectory of the mean Madelung motion to practically any other trajectory. Such a random motion of the fluid elements would, if it were the only factor operating, lead eventually to a uniform mean density of the fluid. For it would on the average carry away more fluid from a region of high density than it carried back. The fact that the mean density remains equal to $|\psi|^2$, despite the effects of the random fluctuations, implies then that a systematic tendency must exist for fluid elements to move toward regions of high mean fluid density, in such a way as to maintain the stability of the mean density, $\bar{\rho}=|\psi|^2$. As for the origin of such a tendency, the question is, of course, not important for the problem that we are treating in this paper. We may, however, suggest by way of a possible explanation that the internal stresses in the fluid are such that whenever ρ deviates from $|\psi|^2$, a kind of pressure arises that tends to correct the deviation automatically. Such a behavior is analogous to what would happen, for example, to a gas in irregular turbulent motion in a gravitational field, in which the pressures automatically adjust themselves in such a way as to maintain a local mean density close to $\rho=\rho_0 e^{-mgz/KT}$ if the temperature T is constant. (In this connection, note that as shown in theoretical treatments of turbulence, the irregular turbulent motions themselves raise the effective "pressure" in the fluid, so that the effective "temperature" T is equal to the sum of the mean kinetic energy of random molecular motion and that of irregular turbulent motion.)

We must now make some assumptions concerning the behavior of the particle-like inhomogeneity. We assume that *even in a fluctuation*, it follows the fluid velocity $\mathbf{v}(\mathbf{x},t)$. Such a behavior would result if the inhomogeneity were a very small dynamic structure in the fluid (e.g., a vortex, or a pulse-like inhomogeneity) or if it were a foreign body of about the same density as the fluid, provided that the wavelengths associated with the fluctuations were appreciably larger than the size of the particle. For in this case, the inhomogeneity would have to do more or less as the fluid did, since it would act, for all practical purposes, like a small element of fluid.

The presence of fluctuations with wavelengths smaller than the size of the body could complicate the problem, especially if we were considering inhomogeneities, such as vortices and pulses, which were dynamically maintained structures in the fluid itself. For, such fluctuations would treat different parts of the inhomogeneity differently, and thus, in general, would tend to lead to a dispersal of the inhomogeneity. Let us recall, however, that we are by hypothesis considering only equations having such nonlinearities in them as to lead to *stable* inhomogeneities. It is true that the equations of ordinary hydrodynamics do not do this. But it is not necessary that the sub-quantum-mechanical Madelung fluid should have exactly the same kinds of properties as are possessed by ordinary fluids. Indeed, we have already seen that instead of the usual classical pressure term, it has a quantum-mechanical internal stress, which depends on the derivatives of the fluid density, rather than on the density itself. Thus, we may reasonably postulate that it also has some characteristically new kind of nonlinear term which leads to stable inhomogeneities. Hence, small fluctuations of wave length much less than the size of the body will merely cause irregular oscillations in the inhomogeneities, the effects of which will, for practical purposes, cancel out. Large fluctuations may destroy the inhomogeneity or transform it into new kinds of inhomogeneity. This could, however, represent certain aspects of the "creation," "destruction," and transformation of "elementary" particles, which is characteristic of phenomena connected with very high energies and very short distances. But in the low-energy domain, which we are treating now, where Schrödinger's equation is a good enough approximation, such processes will not occur.

We see then that if there are fluctuations of wavelength a great deal shorter than the size of the body, they will have a negligible effect on the over-all motions of the body (whether it be a foreign body or a stable dynamic structure in the fluid). In this case, the body will follow the mean velocity of the fluid in a small region surrounding it. To take into account the possibility that such fluctuations may exist, we shall therefore hereafter let $\mathbf{v}(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ represent respectively the mean velocity and mean density in a small neighborhood surrounding the body, while $\nabla S(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ represent the means of these quantities in a region that is much larger than the size of the body, but still small enough so that $\psi(\mathbf{x},t)$ does not change appreciably within this region. The consistency of these assumptions evidently requires that the body be very small; but with a choice, for example, of something of the order of 10^{-13} cm for its size, one obtains ample opportunity to satisfy the above assumptions in a consistent way.

It is clear, of course, that fluctuations having a wavelength close to the size of the body will neither cancel out completely, nor will they necessarily cause the body to move exactly with the mean of the fluid velocity in a small neighborhood surrounding it. We may assume, however, that the magnitude of the longer-wavelength fluctuations is so great that we can neglect the effects of fluctuations of these intermediate wavelengths. Thus, a rather wide range exists of kinds of fluctuations that could lead to the type of motion that we are assuming for the inhomogeneity.

On the basis of the above assumptions, it is evident that the inhomogeneity will undergo an irregular motion, analogous to the Brownian motion.[10] Let us now consider a statistical ensemble of fluids, each having in it an inhomogeneity, and let us denote the probability density of such inhomogeneities in the ensemble by $P(\mathbf{x},t)$. Let us further assume that the fluid motion is so irregular that in time a fluid element initially in an arbitrary region $d\mathbf{x}'$ in the domain in which the mean fluid density $|\psi(\mathbf{x},t)|^2$ is appreciable, has a nonzero probability of reaching any other region $d\mathbf{x}$ in this domain. We can then quite easily see in qualitative terms that the probability density $P(\mathbf{x},t)$ must approach $|\psi(\mathbf{x},t)|^2$ as an equilibrium value.

First of all, it is clear that if, for any reason whatever, the distribution $P=|\psi|^2$, is once established, then it will be maintained for all time, despite the random fluctuations in the fluid motion. For the inhomogeneities simply follow the fluid velocity in a small neighborhood surrounding the body. Now by hypothesis the fluid fluctuations are just such as to preserve the equilibrium mean density of $P=|\psi|^2$. Therefore, they must also preserve the equilibrium probability density of particles in the same way.

Let us now consider what happens when P is not equal to $|\psi|^2$. Suppose, for example, that there were a larger number of particles in a specified element of volume than is given by $P=|\psi|^2$. Now, the random motions carry particles away from such an element at a rate proportional to their density in this element. The systematic tendency for particles to come back to the element, which results from their following the fluid, as it drifts back at a rate sufficient to maintain the mean equilibrium density of $\bar{\rho}=|\psi|^2$, will however be just large enough to cancel the loss that would have taken place if the probability density of particles had been $P=|\psi|^2$. Since the density was actually greater than this, more particles are lost than are compensated by the drift back and the density therefore approaches $P=|\psi|^2$. If the probability density of particles in this element had been less than $P=|\psi|^2$, the element would, of course, have tended to gain particles until it had a density of $|\psi|^2$.

In the next section, we shall give a mathematical demonstration of the above result, the correctness of which should however, already be evident from the qualitative considerations cited above.

Finally, we may mention that the picture of a fluid undergoing random motion about a regular mean is only one out of an infinite number of possible models leading to the same general type of theory. Indeed, all the properties that we have assumed for our fluid could equally well belong to some 4-vector field $(\rho,\mathbf{j})$ which was conserved, and which underwent random fluctuations about a mean given (in the nonrelativistic limit) by $\rho=|\psi|^2$ and $\mathbf{j}=(\hbar/2mi)(\psi^*\nabla\psi-\psi\nabla\psi^*)=R^2\nabla S/m$, where ψ is a solution of Schrödinger's equation. And if ρ and $\mathbf{j}$ were assumed to satisfy sufficiently nonlinear equations, there could also exist pulse-like solutions[11] for ρ and $\mathbf{j}$ that moved with a 4-velocity parallel to $(\rho,\mathbf{j})$.

Although it is important to keep in mind these more general possibilities when one is actually trying to formulate a more detailed theory, we have found it convenient in this paper to express our assumptions and results in terms of a hydrodynamical model, because this model not only provides a very natural and vivid physical image of the behavior of the ψ field, but also a simple explanation of the formula, $d\boldsymbol{\xi}/dt=\nabla S/m$, (postulate 2) expressing the velocity of an inhomogeneity in terms of the local mean stream velocity.

4. PROOF THAT PROBABILITY DENSITY APPROACHES FLUID DENSITY IN RANDOM FLUCTUATIONS OF A FLUID

We shall now prove the following theorem. Suppose that we have a conserved fluid that undergoes random fluctuations of the velocity, $\mathbf{v}(\mathbf{x},t)$, and of the density, $\rho(\mathbf{x},t)$, about respective mean values $\mathbf{v}_0(\mathbf{x},t)$ and $\rho_0(\mathbf{x},t)$ [so that $\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{v})=0$ and $\partial\rho_0/\partial t+\operatorname{div}(\rho_0\mathbf{v}_0)=0$]. Suppose in addition that there is an inhomogeneity that follows the fluid motions, with the local stream velocity, $\mathbf{v}(\mathbf{x},t)$. Then if the fluctuations are such that a fluid element starting in an arbitrary element of volume, $d\mathbf{x}'$, in the region where the fluid density is appreciable has a nonzero probability of reaching any other element of volume $d\mathbf{x}$ in this region, it follows that an arbitrary initial probability density of inhomogeneities will in time approach $P=\rho_0(\mathbf{x},t)$.

This theorem is seen to apply to our problems as a special case, in which we set $\rho_0=|\psi(\mathbf{x},t)|^2$ and $\mathbf{v}_0(\mathbf{x},t)=\nabla S(\mathbf{x},t)/m$, where $\psi(\mathbf{x},t)$ satisfies Schrödinger's equation, provided that we regard $\rho(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$ as the mean fluid density and velocity in a small region surrounding the inhomogeneity. This theorem is a generalization of a well-known theorem concerning the approach to equilibrium in a Markow process.[12] Essentially, we have generalized the theorem to treat the time-dependent probabilities of transition and time-dependent limiting distributions with which we have to deal in our problem.

To prove this theorem, we note that, as shown in the previous section, a given fluid element follows an extremely irregular trajectory, in which its density $\rho(\mathbf{x},t)$ fluctuates near the mean density $\rho_0(\mathbf{x},t)$. Now because the volume of a given fluid element is always

[10] Brownian motion models of the quantum theory have already been proposed elsewhere, but on a very different basis. See, I. Fenyes, Z. Physik **132**, 81 (1952); W. Weizel, Z. Physik **134**, 264 (1953); **135**, 270 (1953).

[11] See L. de Broglie, *La Physique Quantique, Restera-t-elle Indeterministe* (Gauthier-Villars, Paris, 1953), where the idea of L. de Broglie and J. P. Vigier on this subject are discussed.

[12] W. Feller, *Probability Theory and Its Applications* (John Wiley and Sons, Inc., New York, 1950).

changing in accordance with the changing mean fluid density in the new regions that it enters, it is rather difficult in rectangular coordinates to keep track of how much fluid is transferred on the average from one element of volume to another. To facilitate the treatment of the problem, we shall therefore take the preliminary step of introducing a new set of coordinates, $\xi_1(\mathbf{x})$, $\xi_2(\mathbf{x})$, $\xi_3(\mathbf{x})$, which are so defined that an elementary cell in the space of ξ_1, ξ_2, ξ_3 always contains a mean quantity of fluid proportional to its volume.

Such a set of coordinates is easily defined. For the mean quantity of fluid in a given volume element is

$$dQ=\rho_0(\mathbf{x},t)d\mathbf{x}=\rho_0(\mathbf{x},t)J(\partial x_\mu/\partial \xi_\nu)d\xi_1 d\xi_2 d\xi_3$$
$$=\rho_0 d\xi_1 d\xi_2 d\xi_3/J_0(\partial \xi_\nu/\partial x_\mu),$$

where $J(\partial \xi_\nu/\partial x_\mu)$ is the Jacobian of the transformation. Now we want to have $J(\partial \xi_\nu/\partial x_\mu)=c\rho_0(\mathbf{x},t)$ (where we shall choose c to be unity for convenience).

Since there is only one equation, it is clear that only one of the ξ_ν can be defined in this way, so that the other two can be chosen according to what is convenient. Thus, if we fix the forms of ξ_2 and ξ_3, we see that the above equation becomes a linear differential equation defining ξ_1, in terms of ξ_2, ξ_3, and ρ_0. Such an equation always has solutions wherever ξ_2, ξ_3, and ρ_0 are regular. There may exist singular points or curves, but we shall later show how these are to be dealt with.

As an example, consider a cylindrically symmetric density function $\rho(R)=e^{-R}/R$. We first express the volume element in cylindrical polar coordinates (with $R^2=X^2+Y^2$):

$$\rho(R)RdRd\phi dZ=e^{-R}dRd\phi dZ.$$

Now we want $e^{-R}dR=d\xi_1$, or $\xi_1=e^{-R}$. As for ξ_2 and ξ_3, we can in this case leave them equal to ϕ and Z respectively.

Here we see that when R goes from 0 to ∞, ξ_1 goes from unity to zero. This is an example of a characteristic property of the ξ_ν space to be limited in volume when the function $\rho_0(R)$ is appreciable only in a limited domain. Such a property is to be expected, because we are mapping the x_μ on the ξ_ν in just such a way that each region maps into a new volume proportional to the amount of fluid originally in that region. Thus even infinite regions of x_μ space may map onto negligible regions of ξ_ν space, if they contain negligible quantities of fluid.

The solution of the differential equation for ξ_1, will lead in general to multiple-valued functions. This, however, causes no trouble, as we need merely establish a convenient cut somewhere which defines which branch of the function that we are using. Thus the transition to cylindrical polar coordinates, $R^2=X^2+Y^2$; $\phi=\tan^{-1}\times(Y/X)$, leads to a multiple valued function for ϕ, but we deal with this problem by establishing a cut, say at $\phi=0$, and then defining the range of variation of ϕ as being from zero to 2π. In order to cover the entire XY plane only once, a similar definition can be made with any multiple-valued function.

If $\rho_0(\mathbf{x},t)$ vanishes at certain points, then at those points we cannot solve for all the ξ_ν in terms of the x_μ (as, for example, in cylindrical polar coordinates we cannot solve for ϕ at $R=0$). As long as $\rho_0(\mathbf{x},t)$ vanishes only at a set of isolated points, or at most, on a set of one-dimensional curves, where will be no real difficulty. For the vanishing of $\rho_0(\mathbf{x},t)$ means only (as in the case of cylindrical polar coordinates) that some of the ξ_ν are not defined along these curves. To avoid any ambiguities arising from the lack of definition, we may surround each of these curves with a tube, as small in radius as we please, and thus exclude them from the region under consideration without excluding any significant physical effects.

If, however, there are two-dimensional surfaces where $\rho_0(\mathbf{x},t)=0$, this creates more serious mathematical difficulties. Since such surfaces do not, in fact, arise in any real problem of interest to us,[13] we shall assume that $\rho_0(\mathbf{x},t)$ vanishes at most on a set of one-dimensional curves.

Finally, let us note that since ρ_0 changes with time, our ξ_ν will change with time correspondingly. Thus, we are adopting a moving set of coordinates (but not in general one that moves with the mean motion of the fluid elements).

In the space of the ξ_ν, the mean fluid density will be a constant which also does not change with time. As a result, the problem of describing the fluctuations will be greatly simplified. For in the ξ_ν space there is no tendency for the fluctuation to favor any special region since the equilibrium density, which was $\rho_0(\mathbf{x},t)$ in rectangular coordinate, is now a constant. Thus, in the ξ_ν space, the fluctuations have a truly random character, independent of the fluid density at any particular point.

We are now ready to set up the equations governing

[13] In the case of interest to us, $\rho_0=|\psi(\mathbf{x},t)|^2$. At first sight, it may seem that we shall have to be concerned with surfaces on which ρ_0 vanished, because in a perfectly stationary state, $\psi(\mathbf{x},t)$ can be zero on certain nodal surfaces. In the case of a perfectly stationary state, ψ can be real [or more generally, writing $\psi=U(\mathbf{x},t)+iV(\mathbf{x},t)$, we may have a functional relationship between $U(\mathbf{x},t)$ and $V(\mathbf{x},t)$ permitting both to vanish on some two-dimensional surface]. However, for the general complex function ψ, which we obtain in a nonstationary state, it may be shown that there is no such functional relation between U and V, so that ψ can vanish at most on a set of one-dimensional curves.

Now a *perfectly* stationary state is an abstraction that never really exists. For all systems that have ever been dealt with are perturbed to some extent by interactions with other systems. Thus, in a gas, a hydrogen atom suffers 10^{12} collisions per second. In a metal, the electrons suffer a correspondingly large number of collisions with each other and with the cores. In the nucleus, there is a continual process of perturbation due to the fluctuating electronic and ionic fields acting on the spin and quadripole moments of the nuclei. Even in interstellar space, atoms undergo at least one collision with electrons in 10^7 seconds. Thus, all states are slightly nonstationary, and no perfectly nodal planes of the ψ function ever really appear in nature.

A set of perfectly nodal surfaces could interfere with our proof that $P\rightarrow|\psi|^2$; for they would represent surfaces that would never be crossed so that the regions on different sides of these surfaces could be completely isolated from each other.

the changes of the probability density $P(\mathbf{x},t)$ for the inhomogeneities. We first transform to the ξ_ν space, writing

$$P(\mathbf{x},t)d\mathbf{x}=\frac{P(\mathbf{x},t)d\xi}{J(\partial\xi_\nu/\partial x_\mu)}=\frac{P(\mathbf{x},t)}{\rho_0(\mathbf{x},t)}d\xi=F(\xi,t)d\xi, \quad (3)$$

where we have defined the vector $\xi=(\xi_1,\xi_2,\xi_3)$ in the ξ_ν space, with the volume element, $d\xi=d\xi_1 d\xi_2 d\xi_3$. The probability density for the space of the ξ_ν is clearly $F=P/\rho_0$. To prove that $P\rightarrow\rho_0$, we then merely have to show that in ξ_ν space, $F(\xi,t)$ approaches a constant.

We now define the probability that fluid in an element $\delta\xi$, centered at the point ξ at the time t, has in the process of fluctuation come from an element $\delta\xi'$ at an earlier time t' with its center ξ' lying in a region $d\xi'$. (Note that $\delta\xi'$ is the magnitude of the volume element,[14] whereas $d\xi'$ is the size of the cell in which the center of the volume element was located at the time t'). This probability is

$$dP=K(\xi,\xi',t,t')d\xi'. \quad (4)$$

Clearly, by definition,

$$\int K(\xi,\xi',t,t')d\xi'=1. \quad (5)$$

Now the exact form of $K(\xi,\xi',t,t')$ will depend on the precise nature of the fluctuations that are taking place in the fluid. We shall see, however, that in order to prove that $F(\xi,t)\rightarrow 1$, it is sufficient to assume that $K(\xi,\xi',t,t')$ fails to be zero over the part of ξ space corresponding to the region of $\mathbf{x}$ space in which $\rho_0(\mathbf{x},t)$ is appreciable. This is clearly just a mathematical expression of the assumption appearing in the first part of this section that there is a nonzero probability that an element starting at any point $\mathbf{x}$ in this region has a nonzero probability of arriving at any other point x' in the region.

Note, however, that the region of $\mathbf{x}$ space in which ρ_0 is appreciable will include, for practical purposes, the whole of the ξ space (except for a region of negligible dimensions). Thus, we may postulate that $K(\xi,\xi',t,t')$ fails to be zero in the whole of ξ space (except possibly along some one-dimensional curves where $\rho_0(\mathbf{x},t)$ may be zero, which we can exclude by means of tubes of negligible dimensions).

As for other properties of K, they are irrelevant for our purposes here, although we shall discuss some of them in Sec. 6, in another connection.[15]

Let us now discuss the motions of the inhomogeneities. Since these latter follow the fluid in its fluctuations, it is easily seen that the probability density of inhomogeneities, $F(\xi,t)$, is just the average of $F(\xi',t')$ weighted with the probability $K(\xi,\xi',t,t')$. Thus,

$$F(\xi,t)=\int K(\xi,\xi',t,t')F(\xi',t')d\xi'. \quad (6)$$

Now, let $\xi_M(t)$ represent the value of ξ for which $F(\xi,t)$ is a maximum, $\xi_m(t)$ the value for which it is a minimum. (If there is more than one pair of such points, let us consider any single pair.) We also let $F_{\max}(\xi,t)=M(t)$, and $F_{\min}(\xi,t)=m(t)$. Setting $\xi=\xi_M(t)$ in Eq. (6), and using (5), we obtain

$$M(t)=\int K(\xi_M(t),\xi',t,t')F(\xi',t')d\xi' \overset{=}{\lessgtr} \int K(\xi_M(t),\xi',t,t')M(t')d\xi'=M(t'); \quad (7)$$

and with $\xi=\xi_m(t)$ in Eq. (6), we get similarly

$$m(t)=\int K(\xi_m(t),\xi',t,t')F(\xi',t')d\xi' \overset{=}{\gtrless} \int K(\xi_m(t),\xi',t,t')m(t')d\xi'=m(t'). \quad (8)$$

Thus,

$$M(t)\overset{=}{<}M(t'), \quad (9a)$$

$$m(t)\overset{=}{>}m(t'). \quad (9b)$$

In order for the equal sign to hold in Eq. (9a), it is necessary that $F(\xi',t')$ be a constant. For by hypothesis, $K(\xi,\xi',t,t')$ fails to vanish anywhere in the ξ space; and if $F(\xi',t')$ is *not* a constant, then the integral (7) must obtain contributions from regions in which $F(\xi',t')<M$. Similarly, we can show that the equal sign can hold in (9b) only if $F(\xi',t')$ is a constant. But if $F(\xi',t')$ is a constant in ξ space, then by (6) we have

$$F(\xi,t)=F(\xi',t')\int K(\xi,\xi',t,t')d\xi'=F(\xi',t').$$

Thus, $F(\xi',t')=$constant is also an equilibrium solution, since it does not change with the passage of time. The result, of course, is more or less to be expected from the physical argument given at the beginning of this section showing that $P=\rho_0(\mathbf{x},t)$ is an equilibrium solution, so that $F=P/\rho_0=$constant must likewise be one. We conclude then that if $F(\xi',t')$ is not a constant, Eqs. (9a) and (9b) must be written as

$$M(t)<M(t'), \quad (10a)$$

$$m(t)>m(t'). \quad (10b)$$

[14] On the average, $\delta\xi$ will not change as the fluid element moves because the fluid density fluctuates near a constant volume in space.

[15] It may be noted at this point that the kernel $K(\xi,\xi',t,t')$ already contains implicit within it a description of the mean fluid velocity $\nabla S/m$. To show this, consider $t-t'=\delta t$ to be a small interval of time. Then $K(x,x',t,t'-\delta t)$ will be large in only a small region of ξ space corresponding in $\mathbf{x}$ space to a region centered around $(x-x'-\nabla S\delta t/m)=0$. The motion of the center of this region describes the mean fluid velocity. The spread of this region describes the random deviations from the mean. In a typical random diffusion process, this width is given by $(\Delta x)^2\sim\delta t$, for short times. For longer times, the functional form of K is determined in a complicated way, which is however of no concern to us in this paper.

Now we can show that Eqs. (10a) and (10b) imply that $F(\mathbf{x},t)$ must approach a constant, with the passage of time. To do this, let us consider a series of times, $t_1, t_2, t_3 \cdots t_n, t_{n+1} \cdots$. We apply (10a) and (10b) from one element of the series of times to the next. Thus

$$M(t_n) < M(t_{n-1}), \tag{11a}$$

$$m(t_n) > m(t_{n-1}). \tag{11b}$$

It is clear that $M(t_n)$ and $m(t_n)$ must each approach constant limits. For $M(t_n)$ is always decreasing and yet remains greater than some fixed number, $m(t_s)$, where t_s is any element of the series such that $t_n > t_s$. Similarly $m(t_n)$ is always increasing and yet less than $M(t_s)$. Now there are just two possibilities: (a) The two constant limits are different; (b) they are the same. We easily see that alternative (a) is self-contradictory. To do this, we denote the two limits by M and m, respectively. Then $M-m=\lim[M(t_n)-m(t_n)]$. But by (11a) and (11b), we have

$$M-m<\lim_{n\to\infty}[M(t_{n-1})-m(t_{n-1})] = \lim_{n\to\infty}[M(t_n)-m(t_n)] = M-m.$$

Because this is a contradiction, alternative (b) must hold. Then $F(\xi,t)$ must approach a constant limit, and $P(\mathbf{x},t)$ must approach $a\rho_0(\mathbf{x},t)$, where a is a constant. If, as happens in quantum theory, the integral of $\rho_0(\mathbf{x},t)$ is normalized to unity, then since by definition the integral of P is also normalized to unity, we must have $a=1$, and

$$P(\mathbf{x},t)\to\rho_0(\mathbf{x},t). \tag{12}$$

5. APPLICATION TO DIRAC EQUATION AND EXTENSION TO MANY-PARTICLE PROBLEM

We may apply the preceding results to the causal interpretation of the Dirac equation,[16] where, as in the Schrödinger equation, we have a stream velocity, $\mathbf{v}_0=\psi^*\boldsymbol{\alpha}\psi/\psi^*\psi$, and a conserved density, $\rho_0=\psi^*\psi$. If we assume a fluid of the same kind as that treated in Sec. 4, and replace $\nabla S/m$ by $\psi^*\boldsymbol{\alpha}\psi/\psi^*\psi$ and $|\psi|^2$ by $\psi^*\psi$, then according to the results of Sec. 4, the probability density will ultimately approach $\psi^*\psi$.

Our results can also be extended very readily to the case of many particles. We first discuss this extension in a purely formal way. We have a wave function, $\psi(\mathbf{x}_1,\mathbf{x}_2\cdots\mathbf{x}_N,t)$, defined in a $3N$-dimensional configuration space. Writing $\psi=R\exp(iS/\hbar)$, we have a set of $3N$ velocity fields, $\mathbf{v}_n=\nabla_n S(\mathbf{x}_1,\mathbf{x}_2,\cdots\mathbf{x}_N,t)$, where ∇_n refers to differentiation with respect to the coordinates of the nth particle. We have a conservation equation in the configuration space.[17] We may now assume that each particle follows the line of flow given by $\mathbf{v}_n(\mathbf{x}_1,\mathbf{x}_2,\cdots\mathbf{x}_N,t)$. Thus, our model is formally just a $3N$-dimensional extension of the model given previously. Hence, if we assume random fluctuations of the $3N$-dimensional velocity field, we shall obtain the result that the probability density in configuration space, $P(\mathbf{x}_1,\mathbf{x}_2,\cdots\mathbf{x}_N,t)$, approaches $|\psi(\mathbf{x}_1,\mathbf{x}_2,\cdots\mathbf{x}_N,t)|^2$.

To obtain a possible physical picture of the meaning of this model, we may use the causal interpretation of the N-particle problem recently proposed by de Broglie.[18] De Broglie has shown that the usual formulation in terms of a wave function in the $3N$-dimensional configuration space can be replaced by an equivalent formulation, according to which each particle is accompanied by its own 3-dimensional wave field, which depends on the precise locations of the other $(N-1)$ particles. Since each wave field satisfies its own Schrödinger's equation, the preceding demonstration still applies.

The above model would imply that each particle moves in its own fluid, and that the fluids interpenetrate each other. For the case of equivalent particles, however, de Broglie has suggested that all particles can be regarded as moving in a common three-dimensional fluid, the velocity of which, at any point $\mathbf{x}$, is dependent on the locations of all the particles, $\mathbf{x}_n$. Thus, we would merely need as many fluids as there are types of particles.

6. ON THE RELATION BETWEEN THE THEORY OF MEASUREMENTS AND FLUCTUATIONS IN THE ψ FIELD

We have demonstrated that with time, the limiting distribution, $P=|\psi|^2$, will be established for any functional form of $K(\xi,\xi',t,t')$, at least within a region which is such that $K(\xi,\xi',t,t')$, does not vanish for any pair of points ξ' and ξ in the region in question. But without a further specification of the $K(\xi,\xi',t,t')$, the rate of approach to the limiting distribution cannot be estimated.

The very fact that no conclusion drawn from the assumption that $P=|\psi|^2$ has as yet been contradicted experimentally, suggests, however, that at least to a fairly high degree of approximation, P is equal to $|\psi|^2$ in all quantum-mechanical systems which have thus far been investigated. Hence, we are led in our model to assume that the existing fluctuations are at least rapid enough to insure the approximate maintenance of the relation, $P=|\psi|^2$ in the very wide variety of systems which has thus far been studied.

In connection with the theory of measurements, however, there arises an important case in which the rate of approach to the equilibrium distribution must be quite slow, if the theory as a whole is to be consistent. This is the case of two wave packets separated by a classical order of distance, throughout which the mean density $|\psi|^2$ is completely negligible.

To show why this case is important, let us recall

[16] D. Bohm, Progr. Theoret. Phys. (Japan) **9**, 273 (1953).
[17] See reference 2, Paper I, Eq. (16).
[18] See reference 11; also Compt. rend. **235**, 1345, 1372 (1953).

briefly some results of the theory of measurements given in a previous paper.[2]

It was shown that in a measurement process, the interaction between measuring apparatus and observed system breaks the wave function into a series of classically separated packets, corresponding to the various possible results of the measurement. The particle, however, enters one of the packets and thereafter remains in it. It is important that the particle remain in this packet; because if it does, the other packets will never play any physical role, so that they can thereafter be neglected and the complete wave function replaced by a simplified one corresponding to the actual result of the measurement. Thus, we understand how a measurement can come to have a definite result, despite the spread of the wave function over a range of possibilities.

Now, if the introduction of a random fluctuation of the ψ field led to an appreciable diffusion of the particle from one of these classically separated packets to another, the above definiteness of the result of a measurement would be destroyed. It is essential therefore for the over-all consistency of the theory that the probability that the particle diffuse across a large region where $\rho_0(\mathbf{x},t)$ is very small shall be negligible.[19]

It is easy to see, however, that almost any reasonable assumptions concerning the fluctuations will lead to this result. For the mean current of particles is $\langle\rho\mathbf{v}\rangle_{\mathrm{Av}}$. Now ρ is everywhere of the order of magnitude of $\rho_0(\mathbf{x},t)$, which is by hypothesis very small in the region between the wave packets. Thus a large probability of a fluctuation that would carry a particle across this space would mean an enormous fluctuation velocity in this region. The mere assumption that fluctuation velocities do not differ by large orders of magnitude in different parts of the fluid is therefore sufficient to insure that the probability of diffusion across this space be very small.

7. CONCLUSION

The essential result of this paper has been to show that the probability density $P=|\psi|^2$ follows from reasonable assumptions concerning random fluctuations of the ψ field. Now, it has already been demonstrated[2] that once the probability distribution $P=|\psi|^2$ has, for any reason whatever, been set up in a statistical ensemble of quantum-mechanical systems, then the results predicted for all measurement processes will be precisely the same in the causal interpretation as in the usual interpretation. The difference between the two points of view, however, is this: in the usual interpretation, the irregular statistical fluctuations in the observed results[20] obtained in general when we make very precise measurements on *individual* atomic systems are assumed, so to speak, to be fundamental elements of reality, since it is supposed that they cannot be analyzed in more detail, and that they cannot be traced to anything else.[21] In the model that we have proposed here, however, the statistical fluctuation in the results of such measurements are shown to be ascribable consistently to an assumed deeper level of irregular motion in the ψ field.

In this paper we have proposed as a possible picture of this deeper level the more specific model of a fluid, undergoing a random fluctuation of its velocity and density about certain mean values determined from Schrödinger's equation, and having in it an inhomogeneity that follows the local stream velocity of the fluid. Of course, this proposal has not yet reached a definitive stage, since we have given only a very general description of the assumed fluctuations and of the properties of the inhomogeneity. Nevertheless, such a model, incompletely defined in character as it is, already suggests a number of interesting questions.

For example, the fluid may have vortex motion. In another paper[22] it will be shown that such vortex motion provides a very natural model for the nonrelativistic wave equation of a particle with spin (the Pauli equation). Work now in progress indicates that a generalization of such a treatment to relativity may yield a model of the Dirac equation.

Another interesting problem to be studied is the possible effects of the assumption of nonlinear equations for the ψ field, which could, as we have seen in Sec. 2, explain the existence of the irregular fluctuations that lead to $P=|\psi|^2$. Such nonlinear equations can lead to many qualitatively new results. For example, it is known that they have a spectrum of stable solutions having localized pulse-like concentrations of field,[23] which could describe inhomogeneities such as we

[19] Note that the slowness of this particular type of diffusion does not interfere with the validity of the relation $P=|\psi|^2$, for the wave function as a whole (i.e., over a whole set of wave packets). For the relation $P=|\psi|^2$ will already have been established by random fluctuations before the measurement took place; and as we have seen, once established, the relationship persists and is not thereafter altered by the fluctuations no matter what happens. But what we have been discussing is *another* probability; namely, the probability that if a particle has entered a given packet, it will within a given time diffuse to another packet. It is this probability that is negligible.

[20] Let us recall that as discussed in reference 5, Sec. 3, there exist real observable large-scale phenomena obtained in a measurement process, which depend on the properties of *individual* atoms (e.g., clicks of a Geiger counter, tracks in a Wilson chamber, etc.)

[21] For example, they cannot in general be ascribed to the uncontrollable actions of the measuring apparatus, as demonstrated by Einstein, Rosen, and Podolsky, Phys. Rev. **47**, 774 (1933) and also D. Bohm, *Quantum Theory* (Prentice Hall Publications, New York, 1951), p. 614. As Bohr has made clear [Phys. Rev. **48**, 696 (1935)] the measuring apparatus plus observed object must be regarded as a single indivisible system which yields a statistical aggregate of irregularly fluctuating observable phenomena. It would be incorrect, however, to suppose that these fluctuations originate in anything at all. They must simply be accepted as fundamental and not further analyzable elements of reality, which do not come from anything else but just exist in themselves. For a complete discussion of this problem, see, *Albert Einstein, Philosopher-Scientist*, Paul Arthur Schilpp, Editor (Library of Living Philosophers, Evanston, 1949).

[22] Bohm, Tiomno, and Schiller (to be published).

[23] Finkelstein, LeLevier, and Ruderman, Phys. Rev. **83**, 326 (1951).

have been assuming in this paper. Such pulse-like concentrations of field would also tend, for many types of field equations, to follow the local stream velocity.[11] The transitions between different possible forms of the inhomogeneous pulse-like part of the solution, combined with transitions between various modes of vibration in the rest of the fluid, could perhaps describe changes from one type of particle to another. Thus, we see that at least in its qualitative aspects, the model seems to have possibilities for explaining some of the kinds of phenomena that are actually found experimentally at the level of very small distances.

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Expansion of Wave Packets

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The Fourier coefficients of a wave packet are proved to be equal to the coefficients obtained when the wave packet is expanded in terms of a set of functions appropriate to a scattering problem.

In a recent paper discussing the use of ingoing waves in scattering problems, Breit and Bethe[1] made use of the fact that when a wave packet is expanded in terms of a set of functions appropriate to a scattering problem, the expansion coefficients are in most cases identical with the Fourier coefficients of the packet. This note indicates a more precise proof of this theorem.

Let $\psi(\mathbf{r})$ be a wave packet which is well localized in both coordinate and momentum space; $\mathbf{r}^c$ and $\mathbf{k}^c$ denote the center of the packet in the two spaces. We assume that the spread of the packet in coordinate (momentum) space is small compared with $\mathbf{r}^c,(\mathbf{k}^c)$. Let $\psi_\mathbf{k}(\mathbf{r})$ be a complete set of wave functions appropriate to a scattering problem. As Breit and Bethe point out, we get a complete set of functions if we choose $\psi_\mathbf{k}$ to behave asymptotically as a plane wave plus an outgoing spherical wave; thus

$$\psi_\mathbf{k}(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + f_\mathbf{k}(\theta,\varphi)e^{ikr}/r. \tag{1}$$

We will expand $\psi(\mathbf{r})$ in terms of $\psi_\mathbf{k}$ and denote the expansion coefficients by $B(\mathbf{k})$; the Fourier coefficients of $\psi(\mathbf{r})$ are $A(\mathbf{k})$. Thus

$$\psi(\mathbf{r}) = (2\pi)^{-\frac{3}{2}} \int B(\mathbf{k})\psi_\mathbf{k}(\mathbf{r})d\mathbf{k}, \tag{2}$$

$$\psi(\mathbf{r}) = (2\pi)^{-\frac{3}{2}} \int A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}. \tag{2'}$$

It can now be proved that, except when $\mathbf{r}^c$ and $\mathbf{k}^c$ are parallel, $B(\mathbf{k}) \simeq A(\mathbf{k})$. In particular, Breit and Bethe used the fact that $B(\mathbf{k}) \simeq A(\mathbf{k})$ when $\mathbf{r}^c$ and $\mathbf{k}^c$ are antiparallel.

Let $C(\mathbf{k}) = B(\mathbf{k}) - A(\mathbf{k})$, and form

$$J(\mathbf{r}^c,\mathbf{k}^c) = (2\pi)^{-\frac{3}{2}} \int |C(\mathbf{k})|^2 d\mathbf{k}.$$

We assume that $\psi(\mathbf{r})$ is far enough removed from the origin so that the asymptotic form of $\psi_\mathbf{k}$ may be used in computing $B(\mathbf{k})$. This gives

$$C(\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \int \psi(\mathbf{r}) f_\mathbf{k}^*(\theta,\varphi) e^{-ikr}/r\, d\mathbf{r}, \tag{3}$$

and

$$J(\mathbf{r}^c,\mathbf{k}^c) = (2\pi)^{-9/2} \int d\mathbf{r}\psi^*(\mathbf{r}) \int d\mathbf{r}'\psi(\mathbf{r}') \times \int d\mathbf{k} f_\mathbf{k}(\theta,\varphi) f_\mathbf{k}^*(\theta',\varphi') \times \exp[ikr - ikr']/rr'. \tag{4}$$

The completeness relation for $\psi_\mathbf{k}$ combined with Eq. (1) yields the result

$$\int d\mathbf{k} f_\mathbf{k}(\theta,\varphi) f_\mathbf{k}^*(\theta',\varphi')(rr')^{-1} \exp[ikr - ikr'] = -\int d\mathbf{k} f_\mathbf{k}(\theta,\varphi) \exp[ikr - i\mathbf{k}\cdot\mathbf{r}']/r - \int d\mathbf{k} f_\mathbf{k}^*(\theta',\varphi') \exp[-ikr' + i\mathbf{k}\cdot\mathbf{r}]/r'. \tag{5}$$

Inserting (5) into (4) and noting that one integration may now be performed, we find that J reduces to

$$J = -(2\pi)^{-3} \int d\mathbf{k} \int d\mathbf{r} \{[\psi^*(\mathbf{r}) f_\mathbf{k}(\theta,\varphi) \times A(\mathbf{k}) e^{ikr}/r] + (\mathrm{cc})\}. \tag{6}$$

[1] G. Breit and H. A. Bethe, Phys. Rev. 93, 888 (1954).