have been assuming in this paper. Such pulse-like concentrations of field would also tend, for many types of field equations, to follow the local stream velocity.[11] The transitions between different possible forms of the inhomogeneous pulse-like part of the solution, combined with transitions between various modes of vibration in the rest of the fluid, could perhaps describe changes from one type of particle to another. Thus, we see that at least in its qualitative aspects, the model seems to have possibilities for explaining some of the kinds of phenomena that are actually found experimentally at the level of very small distances.

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Expansion of Wave Packets

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The Fourier coefficients of a wave packet are proved to be equal to the coefficients obtained when the wave packet is expanded in terms of a set of functions appropriate to a scattering problem.

In a recent paper discussing the use of ingoing waves in scattering problems, Breit and Bethe[1] made use of the fact that when a wave packet is expanded in terms of a set of functions appropriate to a scattering problem, the expansion coefficients are in most cases identical with the Fourier coefficients of the packet. This note indicates a more precise proof of this theorem.

Let $\psi(\mathbf{r})$ be a wave packet which is well localized in both coordinate and momentum space; $\mathbf{r}^c$ and $\mathbf{k}^c$ denote the center of the packet in the two spaces. We assume that the spread of the packet in coordinate (momentum) space is small compared with $\mathbf{r}^c,(\mathbf{k}^c)$. Let $\psi_{\mathbf{k}}(\mathbf{r})$ be a complete set of wave functions appropriate to a scattering problem. As Breit and Bethe point out, we get a complete set of functions if we choose $\psi_{\mathbf{k}}$ to behave asymptotically as a plane wave plus an outgoing spherical wave; thus

$$\psi_{\mathbf{k}}(\mathbf{r})\sim e^{i\mathbf{k}\cdot\mathbf{r}}+f_{\mathbf{k}}(\theta,\varphi)e^{ikr}/r. \tag{1}$$

We will expand $\psi(\mathbf{r})$ in terms of $\psi_{\mathbf{k}}$ and denote the expansion coefficients by $B(\mathbf{k})$; the Fourier coefficients of $\psi(\mathbf{r})$ are $A(\mathbf{k})$. Thus

$$\psi(\mathbf{r})=(2\pi)^{-\frac{3}{2}}\int B(\mathbf{k})\psi_{\mathbf{k}}(\mathbf{r})d\mathbf{k}, \tag{2}$$

$$\psi(\mathbf{r})=(2\pi)^{-\frac{3}{2}}\int A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}. \tag{2'}$$

It can now be proved that, except when $\mathbf{r}^c$ and $\mathbf{k}^c$ are parallel, $B(\mathbf{k})\simeq A(\mathbf{k})$. In particular, Breit and Bethe used the fact that $B(\mathbf{k})\simeq A(\mathbf{k})$ when $\mathbf{r}^c$ and $\mathbf{k}^c$ are antiparallel.

[1] G. Breit and H. A. Bethe, Phys. Rev. **93**, 888 (1954).

Let $C(\mathbf{k})=B(\mathbf{k})-A(\mathbf{k})$, and form

$$J(\mathbf{r}^c,\mathbf{k}^c)=(2\pi)^{-\frac{3}{2}}\int |C(\mathbf{k})|^2 d\mathbf{k}.$$

We assume that $\psi(\mathbf{r})$ is far enough removed from the origin so that the asymptotic form of $\psi_{\mathbf{k}}$ may be used in computing $B(\mathbf{k})$. This gives

$$C(\mathbf{k})=(2\pi)^{-\frac{3}{2}}\int \psi(\mathbf{r})f_{\mathbf{k}}^*(\theta,\varphi)e^{-ikr}/r d\mathbf{r}, \tag{3}$$

and

$$J(\mathbf{r}^c,\mathbf{k}^c)=(2\pi)^{-9/2}\int d\mathbf{r}\psi^*(\mathbf{r})\int d\mathbf{r}'\psi(\mathbf{r}')\times\int d\mathbf{k}f_{\mathbf{k}}(\theta,\varphi)f_{\mathbf{k}}^*(\theta',\varphi')\times\exp[ikr-ikr']/rr'. \tag{4}$$

The completeness relation for $\psi_{\mathbf{k}}$ combined with Eq. (1) yields the result

$$\int d\mathbf{k}f_{\mathbf{k}}(\theta,\varphi)f_{\mathbf{k}}^*(\theta',\varphi')(rr')^{-1}\exp[ikr-ikr'] = -\int d\mathbf{k}f_{\mathbf{k}}(\theta,\varphi)\exp[ikr-i\mathbf{k}\cdot\mathbf{r}']/r - \int d\mathbf{k}f_{\mathbf{k}}^*(\theta',\varphi')\exp[-ikr'+i\mathbf{k}\cdot\mathbf{r}]/r'. \tag{5}$$

Inserting (5) into (4) and noting that one integration may now be performed, we find that J reduces to

$$J=-(2\pi)^{-3}\int d\mathbf{k}\int d\mathbf{r}\{[\psi^*(\mathbf{r})f_{\mathbf{k}}(\theta,\varphi)\times A(\mathbf{k})e^{ikr}/r]+(\mathrm{cc})\}. \tag{6}$$

Since $\psi(\mathbf{r})$ is well localized and far removed from the origin in both coordinate and momentum space, we may write

$$J=-(2\pi)^{-3}\left[(r^c)^{-1}f_{\mathbf{k}^c}(\theta_{\mathbf{r}^c},\varphi_{\mathbf{r}^c})\times\int d\mathbf{k}d\mathbf{r}\psi^*(\mathbf{r})A(\mathbf{k})e^{ikr}+(\mathrm{cc})\right],$$

and we may also expand kr as

$$kr=-k^cr^c+(\mathbf{k}\cdot\mathbf{k}^c)r^c/k^c+(\mathbf{r}\cdot\mathbf{r}^c)k^c/r^c.$$

Thus finally

$$J(\mathbf{r}^c,\mathbf{k}^c)=-[(r^c)^{-1}f_{\mathbf{k}^c}(\theta_{\mathbf{r}^c},\varphi_{\mathbf{r}^c})\exp(-ik^cr^c)\times A^*(\mathbf{r}^c(k^c/r^c))\psi(\mathbf{k}^c(r^c/k^c))+(\mathrm{cc})]. \quad (7)$$

Now since $\psi(\mathbf{r})$ vanishes unless $\mathbf{r}\simeq\mathbf{r}^c$ and since $(r^c/k^c)\mathbf{k}^c$ is a vector of magnitude r^c in the direction of $\mathbf{k}^c$, unless $\mathbf{r}^c$ and $\mathbf{k}^c$ are nearly parallel $\psi((r^c/k^c)\mathbf{k}^c)$ will be very small. A similar argument applies to $A^*((k^c/r^c)\mathbf{r}^c)$. In particular, if $\mathbf{k}^c$ and $\mathbf{r}^c$ are antiparallel J will be essentially zero.

Breit and Bethe make clear the fact that we would also have obtained a complete set for $\psi_{\mathbf{k}}$ by choosing the asymptotic behavior of $\psi_{\mathbf{k}}$ to be that of a plane wave plus an incoming spherical wave. Had we made this choice it is clear that J would be essentially zero except when $\mathbf{r}^c$ and $\mathbf{k}^c$ are antiparallel.

As pointed out by Breit and Bethe, these results may be argued physically by remarking that the outgoing (ingoing) spherical wave part of $\psi_{\mathbf{k}}$ can contribute to the expansion of $\psi(\mathbf{r})$ only if in the past (future) the wave packet passed close enough to the origin to be scattered. In view of this it would be expected that the size of $J(\mathbf{r}^c,\mathbf{k}^c)$ would be determined by the maximum of the probability of finding the particle at the origin at any time in the past (future). If a Gaussian packet is used for $\psi(\mathbf{r})$, it is indeed possible to demonstrate this. For a Gaussian packet with equal relative spread in coordinate and momentum space, $\Delta r/r^c=\Delta k/k^c$, the maximum of the probability of finding the particle at the origin at any time in the past is proportional to $\exp[-2(r^c/\Delta r)^2(1-\cos\gamma)]$, where γ is the angle between $\mathbf{k}^c$ and $\mathbf{r}^c$. For this same packet J is proportional to $\exp[-4(r^c/\Delta r)^2(1-\cos\gamma)]$. In applying this theorem, Breit and Bethe may choose $\Delta r/r^c$ as small as desired, and thus they may say with complete precision that J vanishes unless $\gamma=0$.