Spherically Symmetric Solutions in Nonsymmetrical Field Theories. II

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The skew symmetric tensors satisfying the general criterion of spherical symmetry, derived in Part I, contain certain arbitrary functions which indicate axial symmetry. These functions are removed. It is found that in addition to the "radial" components found by Papapetrou, the tensor has "transverse" components. In Maxwell's electrodynamics and in general relativity there are solutions which represent spherically symmetric fields of skew tensors with these transverse components. But in the unified field theories of Einstein or Schrödinger it is found that such solutions describing fields of skew tensors with nonvanishing transverse components do not exist. Thus the solutions found by Papapetrou, Wyman, and Bonnor are the only spherically symmetric solutions allowed by the unified field theories. The spherically symmetric solution found in general relativity for a radiating star has no counterpart in the unified field theories.

For comparison, it is noted that the same is the case with the field equations of Dirac's new electrodynamics. Spherically symmetric nonstatic solutions of Maxwell's electrodynamics have no counterpart in Dirac's electrodynamics.

1. INTRODUCTION

IN I,[1] we have shown that a spherically symmetric tensor field g_{ik} must satisfy the criterion

$$\xi^\alpha{}_{,i}g_{\alpha k}+\xi^\alpha{}_{,k}g_{i\alpha}+g_{ik,\alpha}\xi^\alpha=0, \tag{1.1}$$

where the ξ^μ are the contravariant components of an infinitesimal rotation of a sphere about a diameter.

$$\xi^1=0,\quad \xi^2=A\cos(\varphi+B),$$
$$\xi^3=-A\sin(\varphi+B)\cot\theta+C,\quad \xi^4=0, \tag{1.2}$$

where A, B, and C are constants.

We have given one tensor field which is skew symmetric and satisfies the criterion (1.1). It is

$$g_{\underset{\smile}{ik}}=\begin{bmatrix} 0 & Pv & -Pu\sin\theta & Hh \\ -Pv & 0 & Ek\sin\theta & Qv \\ Pu\sin\theta & -Ek\sin\theta & 0 & -Qu\sin\theta \\ -Hh & -Qv & Qu\sin\theta & 0 \end{bmatrix}. \tag{1.3}$$

Here capital letters stand for arbitrary functions of (r,t) while small letters are functions of θ, φ, whose forms are specified in I. In addition to the well-known radial components $g_{\underset{\smile}{14}}$ and $g_{\underset{\smile}{23}}$, this tensor also contains the transverse components $g_{\underset{\smile}{12}}$, $g_{\underset{\smile}{24}}$, $g_{\underset{\smile}{13}}$, and $g_{\underset{\smile}{34}}$. Now the transverse components of a skew spherically symmetric field will lie on the tangent plane to the sphere through the field point. From the familiar case of an electromagnetic field, we know that in this tangent plane the transverse components may take up any two orthogonal directions depending on the polarization of the corresponding wave. Thus a polarized electromagnetic wave can choose a preferential direction through polarization even when it is spherically symmetric. This polarization (or preferential choice of a direction) will be exhibited in the tangent plane to the sphere at any field point. Hence effects of polarization will be found on these new transverse components and are not to be traced in the earlier radial components found by Papapetrou. The functions u and v of θ and φ, which occur only in the transverse components of $g_{\underset{\smile}{ik}}$, can therefore be taken to indicate the corresponding polarization of the field.

But there is another way in which functions of θ and φ may occur in our tensor field. If the infinitesimal rotation (1.2) is given about a diameter QOQ' of the sphere, it is clear that those tensor fields which have only axial symmetry round QOQ', but which are not centrally symmetric round O, will also satisfy the criterion (1.1) and so may be present in a solution of Eq. (1.1). It is easy to see that functions indicating axial symmetry round QOQ' will also be functions of θ, φ. The function $w(\theta,\varphi)$, arbitrary functions of which occur in our solution (1.3), can easily be seen to be such a function. We have noted in I that w is proportional to the cosine of the angle POQ, where P is the field point on the sphere at which the value of the tensor is taken. It is therefore clear that arbitrary functions of w (which occur in the radial as well as the transverse components) are indicative of the axial symmetry round the axis of rotation. In I we had taken w, along with u and v, as indicating polarization of the field. This was a mistake. We now remove these functions of w by putting them equal to unity in our tensor. We then get a spherically symmetric skew tensor field of the following form:

$$g_{\underset{\smile}{ik}}=\begin{bmatrix} 0 & 0 & p\sin\theta & H \\ 0 & 0 & E\sin\theta & 0 \\ -p\sin\theta & -E\sin\theta & 0 & -q\sin\theta \\ -H & 0 & q\sin\theta & 0 \end{bmatrix}, \tag{1.4}$$

where p, q, E, and H are functions of r and t. The form (1.4) is obtained by so choosing the orientations of the axes of our polar coordinates that we get simplified values of the functions $u(\theta,\varphi)$ and $v(\theta,\varphi)$:

$$v=0,\quad u=1. \tag{1.5}$$

[1] P. C. Vaidya, Phys. Rev. **90**, 695 (1953). This paper is referred to hereafter as I.

In order to gain increased confidence in these new transverse components of the tensor, we shall give in the next section the spherically symmetric solution in general relativity which describes skew tensor fields with nonzero transverse components. That solution will also provide the background for the corresponding solution of the unified field theory which we are investigating in this paper.

2. SKEW TENSOR FIELDS IN GENERAL RELATIVITY

A spherically symmetric field, in the scheme of general relativity, will be expressible in terms of a line element

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + e^{\nu}dt^2, \qquad (2.1)$$
$$\lambda = \lambda(r,t), \quad \nu = \nu(r,t).$$

From this line element we can calculate

$$T_\mu{}^\nu = -\frac{1}{8\pi}(R_\mu{}^\nu - \tfrac{1}{2}g_\mu{}^\nu R). \qquad (2.2)$$

It is, however, known from Maxwell's electrodynamics that

$$T_\mu{}^\nu = -F_{\mu\alpha}F^{\nu\alpha} + \tfrac{1}{4}g_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta}, \qquad (2.3)$$

where $F_{\mu\nu}$ is the skew symmetric electromagnetic field tensor obtained from a potential four-vector A_μ:

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}. \qquad (2.4)$$

Equating the two values of $T_\mu{}^\nu$, we shall find that[2]

$$F_{14} = 0, \quad F_{23} = 0,$$
$$F_{12} = -(m'/\dot{m})F_{24} = m'(4\pi f)^{-\frac{1}{2}}u, \qquad (2.5)$$
$$F_{13} = -(m'/\dot{m})F_{34} = m'(4\pi f)^{-\frac{1}{2}}v \sin\theta,$$

where $2m = r(1-e^{-\lambda})$, $m' = \partial m/\partial r$, $\dot{m} = \partial m/\partial t$, and f is a function of m. This spherically symmetric skew field has nonvanishing transverse components of the form (1.4). This field satisfies all the equations of Maxwell and the charge current vector J_μ is not zero, but it is null.

$$J_\mu J^\mu = 0. \qquad (2.6)$$

The gravitational field represented by it is the field outside a radiating star.[3] However, our interest in this solution, at present, is that it gives the same form of the spherically symmetric skew tensor as the one obtained by us geometrically in I.

It is further found that it is not necessary to introduce the curved space-time of general relativity to obtain such a spherically symmetric skew field in the scheme of Maxwell's electrodynamics. It can be verified that

$$F_{14} = 0, \quad F_{23} = 0, \quad F_{13} = 0, \quad F_{34} = 0,$$
$$F_{12} = F_{24} = f(t-r),$$

satisfy all the field equations of Maxwell. Here again the charge-current vector J_μ is not zero, but

$$J_\mu J^\mu = 0. \qquad (2.6)$$

This last relation suggests that solutions of this type cannot exist in the scheme of classical electrodynamics recently proposed by Dirac.[4] For in that scheme the vector J_μ is proportional to a unit vector v_μ and therefore it cannot be a null vector as in (2.6). We have actually verified that spherically symmetric solutions with nonvanishing transverse components of the skew tensor $F_{\mu\nu}$ do not exist in Dirac's electrodynamics. This may be a consequence of the nonlinearity introduced by Dirac in the originally linear Maxwell scheme. We now proceed to show that even though the nonlinearity introduced by general relativity allows such solutions, the unified field theories do not allow spherically symmetric skew fields with nonvanishing transverse components.

3. SOLUTIONS IN THE UNIFIED FIELD THEORY OF EINSTEIN

In this section we shall search for solutions in the unified field theories which will correspond to the solution giving the gravitational field of a radiating star mentioned in the foregoing section. For that purpose we take the complete tensor g_{ik} of the form

$$g_{ik} = \begin{pmatrix} -\alpha & 0 & p\sin\theta & a+H \\ 0 & -\beta & E\sin\theta & 0 \\ -p\sin\theta & -E\sin\theta & -\beta\sin^2\theta & -q\sin\theta \\ a-H & 0 & q\sin\theta & \gamma \end{pmatrix}. \qquad (3.1)$$

The different functions introduced here are functions of r and t. It will be seen that we have taken $g_{\underline{ik}}$ from general relativity and $g_{\underset{\smile}{ik}}$ from (1.4). In general, we can impose four coordinate conditions on this g_{ik}. Two of them have already been imposed when we took the orientations of our axes of coordinates for which $v=0$, $u=1$. Two more coordinate conditions can be imposed by properly choosing r and t. Consider an arbitrary transformation,

$$r = r(r',t'), \quad t = t(r',t'), \qquad (3.2)$$

of the coordinates (r,t) to the coordinates (r',t'). It is easy to verify that if we choose r and t of (3.2) to satisfy the equations

$$p(\partial r/\partial t') + q(\partial t/\partial t') = 0, \qquad (3.3)$$

$$(pa+q\alpha)(\partial r/\partial r') - (qa-p\gamma)(\partial t/\partial r') = 0, \qquad (3.4)$$

then after the transformation we shall find that $g_{\underset{\smile}{34}}' = 0$ and $g_{\underline{14}}' = 0$. But this is possible only if the Jacobian of the transformation does not vanish, that is, if

$$p^2\gamma + 2pqa - q^2\alpha \neq 0. \qquad (3.5)$$

Similarly it is possible to find transformations of (r,t)

[2] V. V. Narlikar and P. C. Vaidya, Proc. Natl. Inst. Sci. (India) **14**, 153 (1948).

[3] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951).

[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A212**, 330 (1952).

which will lead to $g_{\underset{\smile}{13}}'=0$ and $g_{\underline{14}}'=0$, again only if (3.5) holds.

However, there is another simplification possible whether the condition (3.5) is satisfied or not. We choose our r and t of (3.2) to satisfy the equations

$$p(\partial r/\partial t')+q(\partial t/\partial t')=0, \tag{3.6}$$

$$\gamma(\partial t/\partial r')^2+2a(\partial t/\partial r')(\partial r/\partial r')-\alpha(\partial r/\partial r')^2=0. \tag{3.7}$$

Then after the transformations we shall find that $g_{\underset{\smile}{34}}'=0$, $g_{11}'=0$. We can factorize the quadratic left-hand member of (3.7) into distinct factors each of which will be linear in the partial derivatives. Now it is always possible to choose one or the other of these two factors in such a way that the selected factor equated to zero, together with Eq. (3.6), will define a transformation of (r,t) into (r',t') whose Jacobian will not vanish. Hence this type of transformation is always possible. It may be noted that these coordinates (r',t') are similar to what we have called "Newtonian" coordinates in general relativity.[5]

We use these "Newtonian" coordinates and take our g_{ik} of the form

$$g_{ik}=\begin{pmatrix} 0 & 0 & p\sin\theta & a+H \\ 0 & -\beta & E\sin\theta & 0 \\ -p\sin\theta & -E\sin\theta & -\beta\sin^2\theta & 0 \\ a-H & 0 & 0 & \gamma \end{pmatrix}. \tag{3.8}$$

One of the field equations is

$$\overset{\smile}{\mathfrak{g}}{}^{is}{}_{,s}=0. \tag{3.9}$$

For $i=4$, this leads to

$$\frac{\partial}{\partial r}[H(\beta^2+E^2)A]-[pEHA]\cot\theta=0, \tag{3.10}$$

$$A^2[\det|g_{ik}|]=-1.$$

It is therefore clear that, as $A\neq 0$,

$$\frac{\partial}{\partial r}[H(\beta^2+E^2)A]=0, \tag{3.11}$$

$$pEH=0. \tag{3.12}$$

The case $p=0$ gives the known solutions of Bonnor. Therefore, from (3.12) we conclude that $EH=0$. We have worked out the detailed solutions in all the three cases (i) $E=0, H=0$; (ii) $E=0, H\neq 0$; (iii) $E\neq 0, H=0$. We found the quantities $\Gamma_{ij}{}^k$ by writing its defining equation,

$$g_{ik,l}-g_{is}\Gamma_{lk}{}^s-g_{sk}\Gamma_{il}{}^s=0, \tag{3.12}$$

in the form

$$g_{\underline{i}s}\Gamma_{\underline{kl}}{}^s=-\tfrac{1}{2}g_{\underline{kl},i}+\tfrac{1}{2}g_{\underset{\smile}{li},k}+\tfrac{1}{2}g_{\underline{ik},l}+g_{\underset{\smile}{ks}}\Gamma_{\underset{\smile}{il}}{}^s+g_{\underset{\smile}{sl}}\Gamma_{\underset{\smile}{ki}}{}^s, \tag{3.13}$$

and

$$g_{\underline{i}s}\Gamma_{\underset{\smile}{kl}}{}^s=\tfrac{1}{2}g_{\underset{\smile}{kl},i}-\tfrac{1}{2}g_{\underline{li},k}-\tfrac{1}{2}g_{\underset{\smile}{ik},l}-g_{\underset{\smile}{ks}}\Gamma_{\underline{il}}{}^s-g_{\underset{\smile}{sl}}\Gamma_{\underline{ki}}{}^s. \tag{3.14}$$

Having found $\Gamma_{kl}{}^i$ from (3.13) and (3.14) for all the three cases mentioned above, we calculated R_{ik} from

$$R_{ik}=\Gamma_{ik}{}^s{}_{,s}-\Gamma_{is}{}^t\Gamma_{tk}{}^s-\tfrac{1}{2}(\Gamma_{\underline{is}}{}^s{}_{,k}+\Gamma_{\underline{ks}}{}^s{}_{,i})+\Gamma_{ik}{}^s\Gamma_{\underline{st}}{}^t. \tag{3.15}$$

It was found that $R_{\underline{ik}}$ contained terms in $\cot^2\theta$. This means that though g_{ik} is spherically symmetric, while R_{ik} is not. The situation is similar to the one discussed by Takeno and others,[6] who were using Papapetrou's form of spherically symmetric $g_{\underset{\smile}{ik}}$. Now in all the unified field theories, there is one field equation of the form

$$R_{\underline{ik}}=\lambda g_{\underline{ik}}. \tag{3.16}$$

[In Schrödinger's theory $\lambda\neq 0$, while in Einstein's theories $\lambda=0$.] In all the cases mentioned above, we have found[7] that (3.16) eventually leads to equations of the type (3.10):

$$f(r,t)+\varphi(r,t)\cot^2\theta=0,$$

which, in their turn, have eventually led to $p=0$ in all cases. Hence solutions with nonvanishing transverse components do not exist under any scheme of field equations which contain equations of the type (3.16).

4. CONCLUSION

We have shown here that spherically symmetric solutions with nonvanishing transverse components of the skew tenor $g_{\underset{\smile}{ik}}$ are not allowed by the unified field theories of Einstein. Corresponding to a solution of the point charge in general relativity there are the solutions of Papapetrou, Wyman, and Bonnor in the unified field theory; but corresponding to the solution of a radiating star in general relativity there are no solutions in the unified field theory.

Here we have an analog in the classical electrodynamics. Corresponding to the well-known solution for a point charge in Maxwell's electrodynamics, we have a solution in the new electrodynamics proposed by Dirac. (It is only a gauge transform of the Maxwellian solution.[8]) However, corresponding to the nonstatic spherically symmetric solution of Maxwell's electrodynamics given here in Sec. 2, there is no solution in Dirac's scheme.

If, however, we want axially symmetric solutions, we can introduce arbitrary functions of w in our $g_{\underset{\smile}{ik}}$ as in Eq. (1.3). Solutions of this type are now being investigated.

[5] P. C. Vaidya, Nature **171**, 260 (1953).

[6] Takeno, Ikeda, and Abe, Progr. Theoret. Phys. Japan **6**, 842 (1951).

[7] The detailed calculations for the three cases are appended here as appendixes I, II, and III.

[8] K. J. Le Couteur, Nature **169**, 146 (1952).

APPENDIX I

Case (i): $E=0,\ H=0$

The tensor is

$$g_{ik}=\begin{bmatrix} 0 & 0 & p\sin\theta & a \\ 0 & -\beta & 0 & 0 \\ -p\sin\theta & 0 & -\beta\sin^2\theta & 0 \\ a & 0 & 0 & \gamma \end{bmatrix}$$

so that

$$g=\det\|g_{ik}\|=-\beta(\beta a^2+\gamma p^2)\sin^2\theta.$$

The 3-index symbols which will ultimately give terms containing $\cot^2\theta$ in R_{11} are $\Gamma_{1k}{}^2$, $\Gamma_{12}{}^k$, $\Gamma_{21}{}^k$, where k may be 1, 3, or 4. Their values can be easily obtained from (3.13), (3.14). We have found that

$$\Gamma_{11}{}^2=(p^2a^2/\beta A)\cot\theta,$$
$$\Gamma_{13}{}^2=-\Gamma_{31}{}^2=(p/2\beta)\cot\theta\sin\theta,\quad \Gamma_{14}{}^2=\Gamma_{41}{}^2=0,$$
$$\Gamma_{12}{}^1=\Gamma_{21}{}^1=-(p^2\gamma/2A)\cot\theta,$$
$$\Gamma_{12}{}^3=-\Gamma_{21}{}^3=(pa^2/2A)\cot\theta\csc\theta,$$
$$\Gamma_{12}{}^4=\Gamma_{21}{}^4=(p^2a/2A)\cot\theta,$$

where $A=a^2\beta-p^2\gamma$. The value of R_{11} is of the following form:

$$R_{11}=f_{11}(r,t)+\frac{p^2a^2(a^2\beta+p^2\gamma)}{2\beta A^2}\cot^2\theta.$$

$R_{11}=\lambda g_{11}$ will now demand that the coefficient of $\cot^2\theta$ in R_{11} should vanish. Since $a^2\beta+p^2\gamma\neq 0$ (because $g\neq 0$), this forces the conclusion $p=0$.

APPENDIX II

Case (ii): $E=0,\ H\neq 0$

The fundamental tensor is

$$g_{ik}=\begin{bmatrix} 0 & 0 & p\sin\theta & a+H \\ 0 & -\beta & 0 & 0 \\ -p\sin\theta & 0 & -\beta\sin^2\theta & 0 \\ a-H & 0 & 0 & \gamma \end{bmatrix}.$$

Of the 64 symbols $\Gamma_{ik}{}^l$ there are 27 (for which either i, k, or l is 2) which contain $\cot\theta$ as a factor in their values. These symbols ultimately lead up to $\cot^2\theta$ terms in R_{ik}. We give here the main results of calculations of these 27 symbols. First, eighteen symbols are evaluated by solving a closed set of eighteen simultaneous equations obtained from Eqs. (3.13) and (3.14). The method of writing these eighteen equations is a simple one. We begin with any one symbol, say $\Gamma_{\underset{\smile}{23}}{}^3$. We write down its value from (3.14). Then on the right-hand side $\Gamma_{\underline{23}}{}^1$ will appear. Next we write the value of $\Gamma_{\underline{23}}{}^1$ from (3.13). Its value will involve two new symbols $\Gamma_{\underline{23}}{}^4$ and $\Gamma_{\underset{\frown}{24}}{}^1$. We write down their values again, using (3.13) and (3.14). We proceed in this way till we come ultimately to an equation which does not involve any further new symbol. The solution of these eighteen equations is a straightforward process. The following values are obtained for the eighteen symbols coming in this set of simultaneous equations:

$$\Gamma_{\underset{\smile}{23}}{}^3=\Gamma_{\underline{23}}{}^1=\Gamma_{\underset{\smile}{24}}{}^3=\Gamma_{\underline{24}}{}^1=0,$$

$$\Gamma_{\underline{24}}{}^4=(H/\gamma)\Gamma_{\underset{\smile}{24}}{}^1=(H/a)\Gamma_{\underset{\smile}{21}}{}^1=(-H/a)\Gamma_{\underset{\smile}{24}}{}^4$$
$$=(H/p)\csc\theta\,\Gamma_{\underline{23}}{}^4=(H^2/a^2)\Gamma_{\underline{21}}{}^1$$
$$=(-2H\gamma/a^2)\Gamma_{\underset{\smile}{21}}{}^4=(-p^2H^2\gamma/2Aa^2)\cot\theta,$$

$$\Gamma_{\underline{24}}{}^3\sin\theta=(H\gamma/a\beta)\Gamma_{\underset{\smile}{23}}{}^4\csc\theta=(-H/\beta)\Gamma_{\underset{\smile}{23}}{}^1\csc\theta$$
$$=(H/p)(\Gamma_{\underline{23}}{}^3-\cot\theta)=(\gamma/a)\Gamma_{\underline{21}}{}^3\sin\theta$$
$$=(-H\gamma/a^2)\Gamma_{\underset{\smile}{21}}{}^3\sin\theta$$
$$=[pH\gamma(a^2+H^2)/2a^2A]\cot\theta,$$

$$\Gamma_{\underline{21}}{}^4=[p^2(a^2-H^2)/2aA]\cot\theta,$$

where $A=(a^2+H^2)\beta-p^2\gamma$. The following five symbols can then be obtained by solving the single equation obtained for each of them from (3.13) or (3.14).

$$\Gamma_{11}{}^2=[p^2(a^2-H^2)/\beta A]\cot\theta,$$
$$\Gamma_{\underline{13}}{}^2=[Hp\{\beta(a^2+H^2)+p^2\gamma\}/2aA\beta]\cot\theta\sin\theta,$$
$$\Gamma_{\underset{\smile}{13}}{}^2=(p/2\beta)\cot\theta,\quad \Gamma_{\underline{14}}{}^2=(-p^2H^2\gamma/2aA\beta)\cot\theta,$$
$$\Gamma_{\underset{\smile}{14}}{}^2=0.$$

The values of R_{11} and R_{22} are of the following form:

$$R_{11}=f_{11}(r,t)+\frac{p^2(a^2+H^2)^3\beta+a^2p^4\gamma(a^2-H^2)}{2\beta a^2A^2}\cot^2\theta,$$

$$R_{22}=f_{22}(r,t)-\frac{p^2\gamma(a^2+H^2)}{2a^2A}\cot^2\theta.$$

$R_{\underline{ik}}=\lambda g_{\underline{ik}}$ will now demand that the coefficients of $\cot^2\theta$ in the above should vanish. This again leads to $p=0$, unless we take $a^2+H^2=0$ and $\gamma=0$ which will make $A=0$ and so the values of the 3-index symbols will need revaluation. We do this. Beginning with $a^2+H^2=0$, $\gamma=0$, we work out the values of these 27 symbols. Most of them become zero. But when we try to find $\Gamma_{\underline{21}}{}^4$ and $\Gamma_{\underset{\smile}{21}}{}^4$, we are faced with the two equations:

$$a\Gamma_{\underset{\smile}{21}}{}^4-H\Gamma_{\underline{21}}{}^4=(Hp^2/2a\beta)\cot\theta,$$

$$H\Gamma_{\underset{\smile}{21}}{}^4+a\Gamma_{\underline{21}}{}^4=(p^2/2\beta)\cot\theta.$$

If neither of $\Gamma_{\underset{\smile}{21}}{}^4$, $\Gamma_{\underline{21}}{}^4$ is to be infinite in value, Hp^2/β must vanish, which again leads to $p=0$ because with $a^2+H^2=0$, H cannot vanish.

Hence in this case the only conclusion is $p=0$.

APPENDIX III

Case (iii): $E \neq 0$, $H = 0$

In this case, if we use the "Newtonian" form (3.8) for g_{ik}, it becomes very difficult to isolate the symbols which involve $\cot\theta$ terms from symbols not involving them. Hence we do not use the form (3.8) but, beginning with the general form (3.1) we take two cases according as (3.5) is satisfied or not.

Sub-Case (i): When (3.5) Is Satisfied

The tensor g_{ik} can be taken in the form

$$g_{ik}=\begin{bmatrix} -\alpha & 0 & p\sin\theta & 0 \\ 0 & -\beta & E\sin\theta & 0 \\ -p\sin\theta & -E\sin\theta & -\beta\sin^2\theta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix}.$$

It is now found that all the 37 symbols $\Gamma_{ik}{}^l$ for which either i or k or l is 4, do not contain $\cot\theta$. $\Gamma_{11}{}^1$ is $\alpha'/2\alpha$. These 38 symbols do not contain $\cot\theta$. Of the remaining 26 symbols, $\Gamma_{11}{}^3$, $\Gamma_{22}{}^2$, $\Gamma_{22}{}^3$, and $\Gamma_{33}{}^3$ vanish. The following are the values of the rest.

$$(-\beta/2p)\Gamma_{11}{}^2=(\alpha/p)\Gamma_{\underline{12}}{}^1=(-\beta/E)(\Gamma_{\underline{12}}{}^2-\beta'/2\beta)$$

$$=\Gamma_{\underset{\smile}{21}}{}^3\sin\theta=(\alpha/2E)(\Gamma_{22}{}^1+\beta'/2\alpha)$$

$$=(-\alpha/p)(\Gamma_{\underline{23}}{}^3-\cot\theta)=x-y\cot\theta,$$

$$\Gamma_{\underset{\smile}{12}}{}^1=\Gamma_{\underset{\smile}{12}}{}^2=\Gamma_{\underline{21}}{}^3=\Gamma_{\underset{\smile}{23}}{}^3=0,$$

$$\Gamma_{13}{}^1=-\Gamma_{31}{}^1=(p/a)[\gamma'/2\gamma+\beta'/2\beta - (E/\beta)(x-y\cot\theta)]\sin\theta,$$

$$\Gamma_{31}{}^2=-\Gamma_{13}{}^2=(E/\beta)[(-p/2E)\cot\theta+E'/2E -\beta'/2\beta+(E/\beta)(x-y\cot\theta)]\sin\theta,$$

$$\Gamma_{31}{}^3=\Gamma_{13}{}^3=B'/B-\alpha'/2\alpha-\beta'/2\beta+(E/\beta)(x-y\cot\theta),$$

$$\Gamma_{32}{}^1=-\Gamma_{23}{}^1=(E/\alpha)[E'/2E-B'/B+\alpha'/2\alpha +(p/2E)\cot\theta-(p^2/E\alpha)(x-y\cot\theta)]\sin\theta,$$

$$\Gamma_{32}{}^2=-\Gamma_{23}{}^2=(p/\alpha)[\beta'/2\beta-(E/\beta)(x-y\cot\theta)]\sin\theta,$$

$$\Gamma_{33}{}^1=[-\beta'/2\alpha+(2p^2/\alpha^2)(\gamma'/2\gamma+\beta'/2\beta) -(2E^2/\alpha\beta)(E'/2E-\beta'/2\beta)+(Ep/\alpha\beta)\cot\theta -2E(p^2\beta+E^2\alpha)(x-y\cot\theta)/\alpha^2\beta^2]\sin^2\theta,$$

$$\Gamma_{33}{}^2=[-(1+p^2/\alpha\beta)\cot\theta-(2Ep/\alpha\beta) \times(E'/2E-B'/B+\alpha'/2\alpha+\beta'/2\beta) +p(p^2\beta+E^2\alpha)(x-y\cot\theta)/\alpha^2\beta^2]\sin^2\theta,$$

where

$$x=(2Ey/p)(B'/B+E'/2E-\alpha'/2\alpha-\beta'/2\beta),$$

$$2y=\alpha\beta p/[\alpha(\beta^2-E^2)-\beta p^2],$$

$$B^2=\alpha(\beta^2+E^2)-\beta p^2,$$

and a prime indicates differentiation with regard to r (e.g., $\alpha'\equiv\partial\alpha/\partial r$). Now $R_{\underline{ik}}$ will contain terms in $\cot\theta$ and $\cot^2\theta$. Here again we shall require these terms to vanish. Once again we shall be led to $p=0$, or else $\alpha(\beta^2-E^2)+\beta p^2=0$ from R_{11} and $\alpha(\beta^2-3E^2)-\beta p^2=0$ from R_{22}. Then, when we write down the value of R_{33} and use the above two results, we get the coefficient of $\cot^2\theta$ therein as $-\frac{1}{4}$ which does not vanish. Hence the other alternative does not hold. The only conclusion s $p=0$.

Sub-Case (2): When (3.5) Is Not Satisfied

In this case we can remove both g_{11} and g_{44} along with g_{34}.[9] Thus we take our tensor as

$$g_{ik}=\begin{bmatrix} 0 & 0 & p\sin\theta & a \\ 0 & -\beta & E\sin\theta & 0 \\ -p\sin\theta & -E\sin\theta & -\beta\sin^2\theta & 0 \\ a & 0 & 0 & 0 \end{bmatrix}.$$

The following 13 nonzero symbols lead to $\cot^2\theta$ terms in R_{11}.

$$\Gamma_{11}{}^2=(p^2/A)\cot\theta+(r,t),$$

$$\Gamma_{12}{}^2=\Gamma_{21}{}^2=(pE/2A)\cot\theta+(r,t),$$

$$\Gamma_{12}{}^3\sin\theta=-\Gamma_{21}{}^3\sin\theta=(p\beta/2A)\cot\theta+(r,t),$$

$$\Gamma_{13}{}^2\csc\theta=-\Gamma_{31}{}^2\csc\theta=(p\beta/2A)\cot\theta+(r,t),$$

$$\Gamma_{13}{}^3=\Gamma_{31}{}^3=(-pE/2A)\cot\theta+(r,t),$$

$$\Gamma_{12}{}^4=\Gamma_{21}{}^4=(p^2\beta/2aA)\cot\theta+(r,t),$$

$$-\Gamma_{13}{}^4\csc\theta=+\Gamma_{31}{}^4\csc\theta=(p^2E/2aA)\cot\theta+(r,t),$$

$$A=\beta^2-E^2,$$

where the (r,t)'s indicate some additional functions of r and t.

The coefficient of $\cot^2\theta$ in R_{11} turns out to be $p^2/2A$, so that the inevitable conclusion is $p=0$.

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[9] If (3.5) is not satisfied, g_{44} automatically becomes zero when we make the transformation for which $g_{\underset{\smile}{34}}$ is zero. We can use the other coordinate condition to make $g_{11}=\check{0}$.