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Radiative Effects in a Constant Field

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A method of avoiding all infrared divergencies in the radiative effects of a constant electromagnetic field is developed. It is used, without the necessity of resumming a nonintegrable series, to calculate the one photon mass operator up to the third order in the external field. The magnetic susceptibility and field dependence of the rest mass of an electron are thereby exhibited to lowest order. The result is then shown in a simple way to provide charged particles with strong short-range forces against the Coulomb field. Particles of unrealistically small mass and charge would carry a large repulsive core, if Coulomb attractive, or a large attractive shell, if Coulomb repulsive.

I. INTRODUCTION

THE present problem is to calculate the one-photon mass operator[1] for the Dirac equation in the presence of a uniform external magnetic field. It was found previously[2] that in the expansion of the mass operator in powers of the uniform external field strength, terms higher than the first contain infrared divergencies. If one asks for the radiative contribution to the behavior of an electron in such a field, one has no other effects which, if properly taken into account, could be expected to cancel such divergencies (such as bremsstrahlung contributions which cancel infrared divergencies in the elastic scattering by a central potential). These divergencies are therefore either real or else due to faulty mathematical technique. Since infrared divergencies are not expected to be real, the latter is the only alternative. It is one of the purposes of this paper to exhibit the faults of the previously used techniques of expanding the mass operator in powers of the external field and to correct them for the present case. Admittedly, the procedure used for this correction in the simple case of a uniform field is not directly applicable to the general one of an arbitrary electromagnetic field. But then, in all other cases of interest one has been able to eliminate the effects of the incorrect expansion by different means (e.g., the introduction of a finite photon mass, which could be allowed to vanish only after other effects, such as inelastic contributions to scattering, were added).

The fact that the customary technique of expanding all terms in powers of the external field is incorrect manifests itself in the proper result by means of a logarithmic dependence on the electromagnetic field. Such a logarithm was first obtained by Gupta[3] and also by Demeur.[4] Both authors, however, used entirely different and less general methods than the one employed here, and their results differ numerically from the present one. (See discussion in Sec. III.)

The renormalized (one-photon) mass operator may be written[5]

$$\Delta M=-ie^2\int_0^\infty dss\int_0^{\prime} duu^{-2}\int (dK)^4(2\pi)^{-4} \times\exp(-ism^2-isu^{-1}K^2)(\mathfrak{M}_1-\mathfrak{M}_0), \quad (1)$$

$$\mathfrak{M}_0=\{2m+(1-u)\gamma\pi,\ \exp[is(1-u)(\gamma\pi)^2]\},$$

$$\mathfrak{M}_1=\tfrac{1}{2}\exp(isK^2)\gamma_\mu\{m-\gamma(\pi-K),\ \exp[is(\gamma(\pi-K))^2]\}\gamma_\mu, \quad (2)$$

where $u=s(s+t)^{-1}$ and t is the "proper time" variable (corresponding to s) for the photon.[6]

After the photon summations (i.e., the K integration and the summation over μ in $\gamma_\mu\cdots\gamma_\mu$) are carried out, ΔM cannot be expressed in a closed form as a function of the external field $F_{\mu\nu}$ and $\gamma\pi$ in such a way that matrix element between electron states can be taken directly. The customary technique is therefore to expand the exponentials in powers of eA and then exhibit the result in gauge invariant form. But it will be noticed that eA occurs in the exponentials in two ways: multiplied by s alone, and multiplied by us. If the former are expanded in powers of eA, correspondingly higher and powers of s occur. The K integration and subsequent vanishing of $(\gamma\pi+m)$ (or $\gamma p+m$) on the extreme left and right by virtue of the Dirac equation introduce a factor of $\exp(-isum^2)$ everywhere, instead of the previous $\exp(-ism^2)$. The s integration will therefore produce higher and higher powers of u^{-1} and consequently divergencies of arbitrarily high order at $u=0$.

This is the origin of the infrared divergencies. They are clearly due to an unallowed expansion in which it is assumed that $seA\ll sum^2$, an expansion in powers of $eAm^{-2}u^{-1}$ rather than the alleged eAm^{-2}. As soon as a small photon mass is introduced, however, the expansion is correct. Such a mass causes, via the photon Green's function, an additional factor of $\exp(-isu^{-1}\epsilon^2)$. The expansion then becomes one in powers of $eA(m^2u$

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[1] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).
[2] R. G. Newton, Phys. Rev. **94**, 1773 (1954).
[3] S. N. Gupta, Nature **163**, 686 (1949).
[4] M. Demeur, Acad. roy. Belg., Classe sci., Mem. **28** (1953).
[5] See reference 2, for example.
[6] The notation is the usual one, with $\hbar=c=1$, $\pi=p-eA$; the Dirac matrices used are such that the Dirac equation reads $(\gamma\pi+M)\psi=0$.

$+\epsilon^2)^{-1}$, which for fixed ϵ can be made uniformly small by assuming eA small enough. If the expansion is done in the presence of a finite photon mass, therefore, as it has been in the past for the purposes of cancelling the divergencies of separate calculations, it is correct (provided the resulting series of finite terms converges). But it is, at least partly, an expansion in powers of $eA\epsilon^{-2}$.

In order to correct this situation one must, then, avoid any expansion in powers of s alone, while one in powers of us is permissable. Such a technique will be introduced below in the special case of a uniform field.

II. CALCULATION

The starting point of the calculation is the following theorem which holds for any uniform electromagnetic field $F_{\mu\nu}=-ie^{-1}[\pi_\mu,\pi_\nu]$ and any 4-vector K_μ, all of whose components commute with all components of π_μ:

$$\begin{aligned}\exp[-is(\pi-K)^2]&\\ &=\exp(-is\pi^2)\exp(-isK\mathrm{E}K)\exp(2isK\mathrm{E}\pi)\\ &=\exp(2is\pi\mathrm{E}K)\exp(-isK\mathrm{E}K)\exp(-is\pi^2),\end{aligned}\tag{3}$$

where

$$\mathrm{E}'(s)_{\mu\nu}=(e^{2esF})_{\mu\nu},\quad \mathrm{E}(s)=s^{-1}\int_0^s ds\mathrm{E}'=(e^{2esF}-1)/(2esF).\tag{4}$$

To prove (3) we first observe that

$$\begin{aligned}[\pi_\mu,\exp(-is\pi^2)]&=-is\int_0^1 dv\exp(-is(1-v)\pi^2)\times[\pi_\mu,\pi^2]\exp(-isv\pi^2)\\ &=2es\int_0^1 dv\exp(-is(1-v)\pi^2)F_{\mu\nu}\pi_\nu\exp(-isv\pi^2)\\ &=2es\exp(-is\pi^2)F_{\mu\nu}\pi_\nu+(2es)^2\int_0^1 dv_1\int_0^1 dv_2v_1\\ &\quad\times\exp(-is(1-v_1v_2)\pi^2)F_{\mu\lambda}F_{\lambda\nu}\pi_\nu\exp(-isv_1v_2\pi^2)\\ &=\cdots=\exp(-is\pi^2)[\exp(2esF)-1]_{\mu\nu}\pi_\nu.\end{aligned}$$

Therefore,

$$\pi_\mu\exp(-is\pi^2)=\exp(-is\pi^2)\mathrm{E}'_{\mu\nu}\pi_\nu.\tag{5}$$

Similarly,

$$\exp(-is\pi^2)\pi_\mu=\pi_\nu\mathrm{E}_{\nu\mu}'\exp(-is\pi^2).\tag{5'}$$

By means of (5) one then obtains

$$\begin{aligned}\frac{\partial}{\partial s}\exp(-is(\pi-K)^2)&=-i(\pi^2+K^2-2K\pi)\\ &\quad\times\exp(-is(\pi-K)^2)=-i(\pi_l{}^2-2K\mathrm{E}'\pi_r+2K\mathrm{E}'K-K^2)\\ &\quad\times\exp(-is(\pi-K)^2),\end{aligned}\tag{6}$$

where the subscripts l and r indicate that π is to stand to the left and right, respectively, of the exponential. It is well known that

$$\frac{\partial}{\partial s}e^{f(s)}=e^{f(s)}\left\{f'+\frac{1}{2!}[f',f]+\frac{1}{3!}[[f',f],f]+\cdots\right\}.$$

Therefore

$$\begin{aligned}e^{-2isK\mathrm{E}\pi}\frac{\partial}{\partial s}e^{2isK\mathrm{E}\pi}&=2i\{K\mathrm{E}'\pi+is[K\mathrm{E}'\pi,K\mathrm{E}\pi]\}\\ &=2i(K\mathrm{E}'\pi-esK\mathrm{E}'F\mathrm{E}^TK)\\ &=2i(K\mathrm{E}'\pi+\tfrac{1}{2}K(1-\mathrm{E}')K).\end{aligned}\tag{7}$$

Equations (6) and (7) yield

$$\begin{aligned}\frac{\partial}{\partial s}\exp(-is(\pi-K)^2)&=\left[\frac{\partial}{\partial s}(-is\pi^2-isK\mathrm{E}K)\right]\\ &\quad\times\exp(-is(\pi-K)^2)+\exp(-is(\pi-K)^2)\\ &\quad\times\exp(-2isK\mathrm{E}\pi)\frac{\partial}{\partial s}\exp(2isK\mathrm{E}\pi).\end{aligned}\tag{8}$$

The differential equation (8) is clearly solved by the first line of (3), with the correct boundary value at $s=0$. The second line of (3) is proved similarly by means of (5').

Equation (3) allows us to separate out the factor $\exp(-is\pi^2)$ from $\mathfrak{M}_1$ in (2) before the K integration is carried out. For a constant field, then

$$\begin{aligned}\exp[is(\gamma(\pi-K))^2]&=\exp(\tfrac{1}{2}ies\sigma F)\exp(-isK\mathrm{E}K)\\ &\quad\times\exp(-is\pi^2)\exp(2isK\mathrm{E}\pi)\\ &=\exp(\tfrac{1}{2}ies\sigma F)\exp(-isK\mathrm{E}K)\\ &\quad\times\exp(2is\pi\mathrm{E}K)\exp(-is\pi^2).\end{aligned}\tag{9}$$

We shall now assume that the electric field vanishes and there is only a constant magnetic field H in the direction of the x axis:

$$F_{23}=-ie^{-1}[\pi_2,\pi_3]=\mathbf{H}_1=H,$$

while all other components and commutators vanish. This yields

$$\begin{aligned}&\mathrm{E}'_{11}=1,\quad \mathrm{E}'_{22}=\mathrm{E}'_{33}=\cos(2esH),\\ &\qquad\qquad \mathrm{E}'_{23}=-\mathrm{E}'_{32}=\sin(2esH),\\ &\mathrm{E}_{11}=1,\quad \mathrm{E}_{22}=\mathrm{E}_{33}=\sin(2esH)/2esH,\\ &\qquad\qquad \mathrm{E}_{23}=-\mathrm{E}_{32}=\sin^2(esH)/esH,\end{aligned}\tag{10}$$

while all other components vanish. If we set

$$\pi'_\mu=\mathrm{E}_{\mu\nu}\pi_\nu,\tag{11}$$

then it is easily checked that

$$[\pi'_2,\pi'_3]=ieH\lambda^2,\quad \lambda=\sin(esH)/esH,\tag{12}$$

$$\pi'_2{}^2+\pi'_3{}^2=(\pi_2{}^2+\pi_3{}^2)\lambda^2.\tag{13}$$

In the following we shall use *a special summation convention: the subscripts "−" shall take the values* 1 *and* 4, *while* "×" *runs through* 2 *and* 3. One may thus handle π_- as c numbers. Because

$$e^{(A+B)}=e^Ae^Be^{-\frac{1}{2}[A,B]}, \tag{14}$$

if $[A,B]$ commutes with A and B, and because of (12), one easily obtains

$$\exp[-is(\pi-K)^2]=\exp(-is\pi^2)\times\exp\{-is[K_-^2+K_\times^2\sin(2esH)/2esH]\}\times\exp(2is^2\lambda^2K_2K_3eH)\exp(2isK_2\pi'_2)\times\exp(2isK_3\pi'_3). \tag{15}$$

Consider the integral

$$I=\int_{-\infty}^{\infty}dK_2\int_{-\infty}^{\infty}dK_3\exp[is(1-u^{-1})K^2]\times\exp(-is\mu u^{-1}K_\times^2)\exp(2isK_2\pi'_2)\exp(2isK_3\pi'_3)\times\exp[2is^2\lambda^2K_2K_3eH], \tag{16}$$

where

$$\mu=1-u[1-(2esH)^{-1}\sin(2esH)]. \tag{17}$$

We first carry out the K_3 integration by shifting K_3 the amount $-u\mu^{-1}[\pi'_3+(esH)^{-1}\sin^2(esH)K_2]$ which is allowable since no commutation between π'_2 and π'_3 is necessary during this integration. Next all terms containing K_2 are moved onto the same exponential by means of (12) and (14), and then K_2 is shifted by the amount

$$-(\pi'_2+u\mu^{-1}\lambda^2eH\pi'_3)u\mu\mu'^{-2},$$

where

$$\mu'=[\mu^2+\lambda^4(useH)^2]^{\frac{1}{2}}. \tag{18}$$

The two K integrations are now ordinary Gaussian integrals. We obtain

$$I=-i\pi us^{-1}\mu^{-1}(\mu'^2/\mu^2)^{-\frac{1}{2}}\times\exp[isu\mu\mu'^{-2}(\pi'_2+u\mu^{-1}\lambda^2eH\pi'_3)^2]\times\exp(isu\mu^{-1}\pi'^2_3). \tag{19}$$

The equation[7] (for $[q,p]=i$),

$$f(p)g(q)=g(q-i\partial/\partial p)f(p), \tag{20}$$

whose consequence is

$$\exp(-ap^2)\exp(bq^2)\exp(ap^2)=\exp[b(q+2iap)^2], \tag{21}$$

allows us to write [since $\pi'_2s(eH)^{\frac{1}{2}}(\sin esH)^{-1}$ and $\pi'_3s(eH)^{\frac{1}{2}}(\sin esH)^{-1}$ obey the same commutation relations as q and p]

$$I=-i\pi s^{-1}\mu^{-1}(\mu'^2/\mu^2)^{-\frac{1}{2}}\exp(\tfrac{1}{2}isu\mu^{-1}\pi'^2_3)\times\exp(isu\mu\mu'^{-2}\pi'^2_2)\exp(\tfrac{1}{2}isu\mu^{-1}\pi'^2_3). \tag{22}$$

[7] See reference 2, Eq. (2.7).

The order in which the integrations are done is, of course, arbitrary. Had we carried them out in the opposite order, π_2' and π_3' would have been interchanged in (22).

The K_1 and K_4 integrations are trivial, and we obtain

$$\int(dK)^4\exp[is(1-u^{-1})K^2]\exp[-is(\pi-K)^2]=-i\pi^2u^2s^{-2}\mu^{-1}(\mu'^2/\mu^2)^{-\frac{1}{2}}\exp(-is\pi^2)\exp(ius\pi_-^2)\times\exp(\tfrac{1}{2}isu\mu^{-1}\pi'^2_3)\exp(isu\mu\mu'^{-2}\pi'^2_2)\exp(\tfrac{1}{2}isu\mu^{-1}\pi'^2_3)=(\pi'_2\text{ and }\pi'_3\text{ interchanged}). \tag{23}$$

So far no approximations have been used, but from this point on we shall restrict ourselves to terms up to the order $(eH)^3$, inclusively. A strict expansion will not be possible, but we shall drop terms which are $o(eH)^3$ as $eH\to 0$.

Now $(\mu^2\mu'^{-2}-1)=\mu'^{-2}(useH)^2\lambda^4$ and $\lambda\leq 1$; μ' and μ can vanish only for $u=1$ and $esH=\frac{1}{2}n\pi$. These zeros of μ' and μ will never cause divergencies.[8] The factor $(\mu^2\mu'^{-2}-1)$ may therefore be handled as one of the second order in eH, since expansion in $(useH)$ is allowed. Similarly $\{[\pi'_2,\pi'_3]us\}$ is of order (eH). Therefore, to the order $(eH)^3$ inclusively,

$$\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)\exp(ius\mu\mu'^{-2}\pi'^2_2)\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)=\tfrac{1}{2}\{\exp[ius\mu^{-1}(\mu^2\mu'^{-2}-1)\pi'^2_2],\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)\times\exp(ius\mu^{-1}\pi'^2_2)\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)\}. \tag{24}$$

It can easily be proved that near $a=0$,

$$\tfrac{1}{2}[\exp(\tfrac{1}{2}ap^2)\exp(aq^2)\exp(\tfrac{1}{2}ap^2)+\exp(\tfrac{1}{2}aq^2)\exp(ap^2)\exp(\tfrac{1}{2}aq^2)]=\exp[a(1-a^2/3!)(p^2+q^2)]+O(a^4), \tag{25}$$

$$\tfrac{1}{2}[\exp(\tfrac{1}{2}ap^2)\exp(aq^2)\exp(\tfrac{1}{2}ap^2)-\exp(\tfrac{1}{2}aq^2)\exp(ap^2)\exp(\tfrac{1}{2}aq^2)]=O(a^3). \tag{26}$$

Equations (24) to (26) and the fact that $(\mu^2\mu'^{-2}-1)=O(eH)^2$ have as a consequence that

$$\tfrac{1}{2}[\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)\exp(ius\mu\mu'^{-2}\pi'^2_2)\times\exp(\tfrac{1}{2}ius\mu^{-1}\pi'^2_3)+(\pi'_2\leftrightarrow\pi'_3)]=\exp[ius(1+\mathrm{A})\pi_\times^2]+o(eH)^3, \tag{27}$$

where

$$\mathrm{A}=\lambda^2\mu^{-1}-1-\tfrac{1}{3}\mu^{-3}\lambda^6(useH)^2. \tag{28}$$

The integral appearing in (23) is also needed with a factor of K in the integrand. This is obtained in a manner similar to the above. By means of (27) and

[8] A simple way of proving this is to envisage the u integration as extending only up to $\frac{1}{2}$, say, for the purposes of the present procedure. The remaining u integration from $\frac{1}{2}$ to 1 is then handled by straightforward expansion in powers of the field, where no trouble at $u=1$ ever arises.

the relation

$$\gamma_\times\pi'_\times=\lambda\exp(\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})\gamma_\times\pi_\times\exp(-\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H}),\quad(29)$$

which is easily proved, the result may be formulated as follows:

$$\begin{aligned}&i\pi^{-2}u^{-2}s^2\int(dK)^4\exp[is(1-u^{-1})K^2]\\&\times\exp[-is(\pi-K)^2](1,\gamma K)=\mu^{-1}[1+\mu^{-2}\lambda^4(useH)^2]^{-\frac{1}{2}}\\&\times\exp(-is\pi^2)\exp(\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})C\exp(-\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})\\&+o(eH)^3=\mu^{-1}[1+\mu^{-2}\lambda^4(useH)^2]^{-\frac{1}{2}}\exp(-\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})C\\&\qquad\times\exp(\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})\exp(-is\pi^2)+o(eH)^3,\quad(30)\end{aligned}$$

where

$$C=\tfrac{1}{2}\{\exp[ius(\pi^2+\mathrm{A}\pi_\times^2)],(1,u\gamma\pi+u\mathrm{B}\gamma_\times\pi_\times)\},$$
$$\mathrm{B}=\lambda\mu^{-1}-1.\quad(31)$$

If (31) is substituted in (1) and (2), the $\gamma\cdots\gamma$ summation carried out, and one sets $\gamma\pi=-m$ on the outside by virtue of the zero-order Dirac equation, one obtains

$$\begin{aligned}\Delta M_1=&-(\alpha/4\pi)\int_0^1du\int_0^\infty dss^{-1}\exp(-iusm^2)\\&\times\{\mu^{-1}[1+\mu^{-2}\lambda^4(useH)]^{-\frac{1}{2}}D+2m(1+u)\}+o(eH)^3,\\D=&\exp(-\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})\\&\times\tfrac{1}{2}\{\exp[ius(e\boldsymbol{\sigma}\cdot\mathbf{H}+\mathrm{A}\pi_\times^2)],4u(\gamma\pi+\mathrm{B}\gamma_\times\pi_\times)\cos(esH)\\&-\{\exp(ise\boldsymbol{\sigma}\cdot\mathbf{H}),m+u\gamma\pi+u\mathrm{B}\gamma_\times\pi_\times\}\}\exp(-\tfrac{1}{2}ise\boldsymbol{\sigma}\cdot\mathbf{H})\\&+2i\sin(esH)[\sigma\gamma_\times\pi_\times,\\&\qquad\exp[ius\mathrm{A}\pi_\times^2-is(1-u)e\boldsymbol{\sigma}\cdot\mathbf{H}]],\quad(32)\end{aligned}$$

where

$$\sigma=\boldsymbol{\sigma}_1=\sigma_{23}=i\gamma_2\gamma_3.$$

The following three relations, in which the zero-order Dirac equation has been used, are easily proved:

$$\begin{aligned}&\{\sigma\gamma_-\pi_-,\exp(ius\mathrm{A}\pi_\times^2)\}\\&\quad=-\{\sigma m+\tfrac{1}{2}i\gamma_\times\pi_\times\sin(2useH\mathrm{A}),\exp(ius\mathrm{A}\pi_\times^2)\},\quad(33)\end{aligned}$$

$$\begin{aligned}&\{\gamma_\times\pi_\times,\exp(ius\mathrm{A}\pi_\times^2)\}\\&\quad=m^{-1}\{(\pi_\times^2-e\boldsymbol{\sigma}\cdot\mathbf{H})[1-i\sigma\mathrm{A}useH-(\mathrm{A}useH)^2\\&\quad+(5/3)i\sigma(\mathrm{A}useH)^3],\exp(ius\mathrm{A}\pi_\times^2)\}\\&\quad-i(16m)^{-1}\sin^3(2\mathrm{A}useH)\\&\qquad\times\{\gamma_\times\pi_\times,\sigma[\gamma_\times\pi_\times,\exp(ius\mathrm{A}\pi_\times^2)]\},\quad(34)\end{aligned}$$

$$\begin{aligned}&[\sigma\gamma_\times\pi_\times,\exp(ius\mathrm{A}\pi_\times^2-is(1-u)e\boldsymbol{\sigma}\cdot\mathbf{H})]\\&\quad=i\{\sin[s(1-u)eH]-\tfrac{1}{2}\cos[s(1-u)eH]\sin(2\mathrm{A}useH)\}\\&\qquad\times\{\gamma_\times\pi_\times,\exp(ius\mathrm{A}\pi_\times^2)\}.\quad(35)\end{aligned}$$

Equations (33) to (35) are used to simplify D. The result is then expanded in powers of $(useH)$, with coefficients which are polynomials in u, whose coefficients in turn are bounded functions of (eHs). Since the integrals are still too complicated to be feasible, we also expand $\exp(ius\mathrm{A}\pi_\times^2)$ in powers of $(us\pi_\times^2)$. We shall assume that $\pi_\times^2=O(eH)$ and retain terms accordingly. The zero-order Dirac equation yields

$$\pi_\times^2=e\boldsymbol{\sigma}\cdot\mathbf{H}-(m^2-p_0^2+p_1^2).\quad(36)$$

Since p_1 and p_0 commute with everything in the present problem, they may be assigned arbitrary values, for example such that $p_0^2-p_1^2=m^2$. Our assumption, that $\pi_\times^2=O(eH)$, is therefore the assumption that

$$(p_0^2-m^2-p_1^2)\text{ is not }\gg eH.\quad(37)$$

After the expansion the u and s integrals can be carried out. They are all of the following form,

$$I=\int_0^1du\int_0^\infty dss^{-1}\exp(-iusm^2)(useH)^nf(u,eHs),\quad n\geq1$$

where f is a polynomial in u, $(eHs)^{-1}$, $\sin(eHs)$, and $\cos(eHs)$ in such a way that it is finite for $eHs=0$. I is easily evaluated by repeated partial integrations on u until all the inverse powers of s have disappeared. The boundary terms, coming from $u=1$, can be expanded in powers of eH, while the remaining double integral can be carried out in straightforward manner. One example will suffice to illustrate the procedure:

$$\begin{aligned}I=&\int_0^1du\int_0^\infty dss^{-1}(useH)\exp(-isum^2)\\&\times[(eHs)^{-1}-(eHs)^{-2}\sin(eHs)]\\&=\int_0^1du\int_0^\infty dse^{-iusK}(s^{-1}-s^{-2}\sin s);\end{aligned}$$

by change of scale in s, $eHs\rightarrow s$, and $K=m^2(eH)^{-1}$.

I is to be evaluated to order K^{-3}. The s^{-1} term is once, and the s^{-2} term twice, partially integrated with respect to u:

$$\begin{aligned}I=&\frac{1}{2}\int_0^\infty dse^{-isK}(s^{-1}-s^{-2}\sin s-\tfrac{1}{3}iKs^{-1}\sin s)\\&+\tfrac{1}{2}iK\int_0^1duu^2\int_0^\infty dse^{-iusK}(1-\tfrac{1}{3}iuK\sin s)\\=&\frac{1}{2}\int_0^\infty dse^{-isK}\left(\frac{1}{6}s-\frac{1}{3}iK+\frac{1}{18}iKs^2+\cdots\right)\\&+\frac{1}{2}\int_0^1du\{\tfrac{2}{3}u-\tfrac{1}{6}K^{-1}[(uK+1)^{-1}+(uK-1)^{-1}]\}\\&\qquad=\tfrac{1}{6}(eH/m^2)^2(\tfrac{5}{6}-\log|eH/m^2|)+O(eH)^4.\end{aligned}$$

After the evaluation of a number of integrals of the foregoing type the result obtained is given in the next section.

III. RESULT

We shall write the mass separator in the following form:

$$\Delta M=\Delta m-\Delta\boldsymbol{\mu}\,\boldsymbol{\sigma}\cdot\mathbf{H}. \tag{38}$$

The suggestiveness of (38) is clear if the Dirac equation, including one-photon radiation processes, is multiplied by $(m-\gamma\pi)$ on the left to read

$$[\pi^2+(m+\Delta m)^2-2m\boldsymbol{\sigma}\cdot\mathbf{H}(\boldsymbol{\mu}+\Delta\boldsymbol{\mu})]\psi=0,$$
$$\boldsymbol{\mu}=e\hbar/2mc.$$

Δm is a field-dependent change in the rest mass of the electron, while $\Delta\boldsymbol{\mu}$ is a field-dependent anomalous magnetic moment. Their values are found to be

$$\frac{\Delta m}{m}=-\frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\right)^2\left\{\frac{13}{9}+\frac{8}{3}\log\left|\frac{2eH}{m^2}\right|\right.$$
$$+\frac{p_0{}^2-m^2-(\mathbf{p}\cdot\mathbf{H}|H|^{-1})^2}{m^2}\left[-\frac{293}{45}+\frac{272}{15}\log2\right.$$
$$\left.\left.+\frac{16}{3}\log\left|\frac{eH}{m^2}\right|\right]+O\left(\left(\frac{eH}{m^2}\right)^2\log\left|\frac{eH}{m^2}\right|\right)\right\}, \tag{39}$$

$$\frac{\Delta\boldsymbol{\mu}}{\boldsymbol{\mu}}=\frac{\alpha}{2\pi}\left\{1+\left(\frac{eH}{m^2}\right)^2\left[-\frac{83}{45}+\frac{332}{15}\log2+\frac{28}{3}\log\left|\frac{eH}{m^2}\right|\right]\right.$$
$$\left.+O\left(\left(\frac{eH}{m^2}\right)^4\log\left|\frac{eH}{m^2}\right|\right)\right\}. \tag{40}$$

The following part of $\Delta m/m$ comes from the expansion of $\exp[-ius(1-\lambda^2)(\pi_-{}^2+m^2)]$ in powers of $(\pi_-{}^2+m^2)/m^2$:

$$-\frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\right)\frac{\pi_-{}^2+m^2}{m^2}\left(-\frac{119}{45}+\frac{116}{15}\log2+\frac{4}{3}\log\left|\frac{eH}{m^2}\right|\right).$$

Notice that $0\leqslant(1-\lambda^2)\leqslant1$, and therefore one could, instead of expanding in $(\pi_-{}^2+m^2)/m^2$, state that for *any* $\pi_-{}^2$, m^2 ought to be replaced everywhere by $m^2+\epsilon(p_0{}^2-p_1{}^2-m^2)$, with $0\leqslant\epsilon\leqslant1$, which is always larger than m^2. But ϵ, of course, differs from integral to integral. It is, nevertheless, clear that for large energies the expansion is one in powers of $eH(p_0{}^2-m^2-p_1{}^2)^{-1}$ rather than eHm^{-2}.

The first term in (40) is the anomalous magnetic moment (first derived by Schwinger[9]), while the second is an induced magnetic moment, as a factor of H a field-dependent magnetic susceptibility of the electron.

A term proportional to $(eHm^{-2})^2\log(eHm^{-2})$ was first obtained by Gupta,[3] but his result differs from ours. Both terms proportional to $(eHm^{-2})^2\log(eHm^{-2})$ and to $(eHm^{-2})^3\log(eHm^{-2})$ were obtained by Demeur,[10] his second-order terms agree with the ones in (40), while our third orders disagree. Both these authors used methods significantly different from ours in two ways. They used a special state of the electron in which $p_0=m$ and $\mathbf{p}\cdot\mathbf{H}=0$, and a special gauge. Their results are therefore neither clearly gauge nor Lorentz invariant and they do not obtain the term in $(p_0{}^2-m^2-\mathbf{p}\cdot\mathbf{H}|H|^{-1})$. Furthermore, an expansion and later resummation of part of the series is used. This latter procedure is never necessary in the present method.

The process used to obtain (39) and (40) is clearly gauge invariant. Potentials, in fact, are never used. It is also evidently Lorentz invariant. Equations (39) and (40) are valid whenever $\mathbf{E}\cdot\mathbf{H}=0$, i.e., $F_{\mu\nu}F_{\mu\nu}{}^*=0$, while in the general Lorentz frame

$$H^2\rightarrow\tfrac{1}{2}F_{\mu\nu}F_{\mu\nu}$$

and

$$H^2[(\mathbf{p}\cdot\mathbf{H}|H|^{-1})^2-p_0{}^2]\rightarrow\pi_\mu F_{\mu\nu}F_{\nu\lambda}\pi_\lambda+\tfrac{1}{2}\pi_\mu{}^2F_{\lambda\nu}F_{\lambda\nu}.$$

Although there exists no physical transformation from a frame in which $E=0$, $H\neq0$, to one in which $H=0$, $E\neq0$, (39) and (40) are clearly valid also in the latter frame. With the Dirac equation,

$$[\pi^2+(m+\Delta m)^2+2m(\boldsymbol{\mu}+\Delta\boldsymbol{\mu})\gamma_4\boldsymbol{\gamma}\cdot\mathbf{E}]\psi=0,$$

Δm and $\Delta\boldsymbol{\mu}$ are obtained from (39) and (40) by the substitution $H^2\rightarrow-E^2$.

IV. DISCUSSION

In the presence of artificially produced fields, the corrections given in (39) and (40) are, of course, much too small to be measurable. The expansion parameter is

$$eHm^{-2}=2.36\times10^{-14}\times H\text{ in gauss,}$$
$$eEm^{-2}=2.36\times10^{-14}\times E\text{ in esu.}$$

In the inner shell of heavy atoms or inside nuclei, however, such corrections are not negligible.

Because of the presence of the logarithms each term in the expansion of $\Delta\boldsymbol{\mu}$ and Δm has an extremum. The maximal magnetic moment correction is attained for a magnetic field strength of about 6.3×10^{12} gauss and its value is -9.6 percent of the anomalous magnetic moment. In view of the fact that at this extremum the expansion parameter has the rather small value 2.06×10^{-2} the next terms will presumably not alter its value appreciably. The maximal correction to the mass has a value of

$$\Delta m/m\approx2.4\times10^{-5},$$

[9] J. Schwinger, Phys. Rev. **73**, 416 (1948).

[10] See reference 4. The discrepancy between our results is presumably due to an error on his part. Starting from (22), p. 79, of Demeur's paper one obtains agreement with the present result if a method of expansion in vx alone is used, which obviates the necessity for resumming the series that leads to divergencies.

which is attained at $(eHm^{-2})^2=3\times 10^{-2}$ (approximately 7.6×10^{12} gauss).

In the $2s$ level of hydrogen the addition to the anomalous magnetic moment is obtained from (40) in an approximate fashion by setting

$$eE/m^2=\alpha^3/n^4=2.44\times 10^{-8},$$

and therefore

$$\Delta^2\boldsymbol{\mu}/\boldsymbol{\mu}=(\alpha/2\pi)\times 0.9\times 10^{-13}.$$

This is to be compared with the fourth-order radiative correction to the magnetic moment,[11]

$$-(\alpha/2\pi)\times 1.38\times 10^{-2}.$$

The contribution to the Lamb shift due to the second term in (40) is therefore entirely negligible.

Apart from a constant change in the magnetic moment, then, the lowest order radiative correction to the behavior of an electron in the presence of a uniform electric field manifests itself in the form of a rest mass change:

$$\frac{\Delta m}{m}=\frac{\alpha}{12\pi}\left(\frac{2eE}{m^2}\right)^2\left[\frac{13}{12}+\log\left(\frac{2eE}{m^2}\right)^2\right]. \tag{41}$$

Let us take this field dependence of the mass for the moment at face value and neglect all other radiative effects due to a spacial inhomogeneity of the field. We may then ask *classically* what the resulting forces between stationary charged particles are by simply taking the radial derivative of the energy. Due to the radiative effects now not only the Coulomb potential energy but also the rest mass is a distance dependent part (via its dependence on the Coulomb field) of the energy. Therefore, with the simplest radiative corrections classically incorporated, the radial derivative of the energy between two charged particles at rest,

$$\partial(m_1+m_2+e_1e_2r^{-1})/\partial r,$$

vanishes, provided that

$$R^3=\mathrm{K}_1(\log R+\mathrm{K}_2), \tag{42}$$

where

$$R=(mc\hbar^{-1})(2\alpha)^{-\frac{1}{3}}\exp(-25/48)r\mathfrak{m}\,|e_1e_2|^{-\frac{1}{3}}(1+\epsilon)^{-\frac{1}{3}},$$

$$\mathrm{K}_1=(8/3\pi)(\alpha/2)^{\frac{1}{3}}\exp(-25/16)e_1^2|e_1e_2|^{-\frac{1}{3}}sgn(e_1e_2),$$

$$\mathrm{K}_2=\tfrac{1}{3}\{\log(1+\epsilon)-\epsilon(1+\epsilon)^{-1}\log[\epsilon(e_1/e_2)^2]\},$$

$$\epsilon=(m_1/m_2)^3(e_2/e_1)^2,$$

and $\mathfrak{m}$ is the mass of particle 1 in units of electron masses, e_1 and e_2 being the two charges in units of the electronic charge.

At the distance which solves (42) two charged particles can stay at rest with respect to each other. The following can be shown easily (in case one particle is much heavier than the other, say): By *decreasing* the charges (in order not to increase the multiple photon effects) the expansion parameter eE/m^2 at the rest distance can be made as small as 0.18 in the case of opposite charges. By decreasing the mass of the lighter particle, we can make the rest distance as large as we please. Hence, if one believes in the convergence of the expansion at $eE/m^2\approx 0.2$ one can conclude that oppositely charged particles of arbitrarily small charge and mass would have repulsive cores of arbitrary extension.

In the case of charges of equal sign, (42) has a solution only if $e_2<10^{-6}e_1^3$. Then there will be two solutions, representing the boundaries of an attractive shell. At the inner boundary $0.09<eE/m^2<0.18$, and at the outer, $eE/m^2<0.09$. Therefore, again, if there were particles of arbitrarily small charge and mass, they would, if of equal sign of charge, have attractive shells of arbitrarily large size.

For physically real particles conclusions as to the existence of a repulsive core cannot be drawn with any assurance because the expansion parameter becomes too large at the rest distance. The first term in the series will then no longer suffice, even if the series still converges. (For an electron-proton pair, $eE/m^2\approx 11$ at the rest distance of $\sim 10^{-12}$ cm.) It is, nevertheless, perhaps of some interest that even on the basis of such a very simple approximation the radiative effects provide charged particles with strong forces opposed to the Coulomb field.

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[11] R. Karplus and N. M. Kroll, Phys. Rev. 77, 536 (1950).