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Vacuum Fluctuation Noise and Dissipation*

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An electron beam interacting with the fields in an enclosed region is considered. The induced noise caused by the vacuum fluctuations and the thermal fluctuations is calculated in terms of the dissipation function $R(\omega)$ and the electron transit time τ. The observable mean squared electromotive force is given by

$$\langle V^2\rangle_{\mathrm{Av}}=\frac{2}{\pi}\int_0^{\omega_{\max}}R(\omega)\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin\frac{1}{2}\omega\tau}{\frac{1}{2}\omega\tau}\right]^2d\omega+\frac{1}{\pi}\int_{\omega_{\max}}^{\infty}\left[\frac{R(\omega)\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin\frac{1}{2}\omega\tau}{\frac{1}{2}\omega\tau}\right]^2d\omega.$$

$\omega_{\max}$ is defined by $\hbar\omega_{\max}=U$, where U is the energy of the incident electrons. The effect of the fluctuations of the vacuum is given by the temperature-independent part of the first integral.

The above expression is employed to calculate the fluctuations induced in an electron beam which interacts with a damped electrical oscillator. The results are valid even if the oscillator is heavily damped. As the damping becomes very large the fluctuations are shown to approach zero. For a weakly damped oscillator the fluctuations reduce to those already given in an earlier paper.

The differential form of the fluctuation dissipation theorem is shown to be very useful for the evaluation of matrix elements.

INTRODUCTION

A PREVIOUS paper[1] considered the interaction of an electron beam with a damped electrical oscillator. The fluctuations of the vacuum were shown to contribute to the electron-beam noise. This observable zero point noise contribution is important because it is one effect of field quantization which is finite in the first nonvanishing approximation.

In order to calculate the noise, the assumption had to be made that the oscillator is very weakly damped. The dissipation was assumed to have no effect on the wave functions for the fields. The coupling to the internal degrees of freedom was considered to have only the effect of establishing thermal equilibrium, between individual electron interactions. Also the electron was assumed to be highly localized. In this paper we consider interaction of a system which may be heavily damped with an electron which is not necessarily highly localized. The fluctuations are shown to depend on the degree of damping, unless the damping is very small. For large damping the fluctuations approach zero.

FLUCTUATIONS MEASURED BY AN ELECTRON BEAM INTERACTING WITH FIELDS IN AN ENCLOSED REGION

For the Hamiltonian of the system of Fig. 1 we take

$$\mathcal{H}=H_0+\frac{P^2}{2m}-\frac{e}{mc}\mathbf{A}\cdot\mathbf{P}. \tag{1}$$

(1) is the Hamiltonian of the system, with one electron in the interaction gap. H_0 is the Hamiltonian for the fields of the enclosed region, $\mathbf{P}$ is the operator corresponding to the momentum of an electron in the beam, m is the mass of the electron, c is the velocity of light, e is the charge on the electron, and $\mathbf{A}$ is the magnetic vector potential for the interaction gap. We assume, at first, that the region is in an eigenstate of its unperturbed Hamiltonian H_0. The electron is to some extent localized and its energy is not known precisely while it is in the interaction gap. A linear combination of free-electron wave functions will therefore be used to represent the electron during the entire interaction time. For the wave functions of the electron and interaction region we assume the expression.

$$\Psi=\sum_{K,M}a_K(t)b_M(t)\psi_K\varphi_M\exp\left(-\frac{i}{\hbar}(E_K+E_M')t\right). \tag{2}$$

In (2) ψ_K is an unperturbed wave function for the fields of the enclosed region. E_K is the Kth eigenvalue of H_0 and E_M' has the corresponding meaning for the electron. φ_M is given[2] by

$$\varphi_M=(1/\sqrt{l})\exp(2\pi iMx/l). \tag{3}$$

(3) is the wave function for a free particle moving in the X direction with momentum given by

$$P_M=2\pi M\hbar/l. \tag{4}$$

M is an integer. The wave function (3) satisfies periodic boundary conditions with period equal to the length l of the interaction gap, and is normalized within the region of the gap. At the beginning of the interaction time, we have: $a_N=1$, $a_K=0$, $K\neq N$. At this time (2) is a summation only over M. At any time after interaction has begun, we have, from perturbation theory,

$$\frac{d}{dt}[a_K(t)b_L(t)]=\frac{e}{mc}(i\hbar)^{-1}\sum_M b_M\exp\left[\frac{i(E_K+E_L'-E_N-E_M')t}{\hbar}\right]\times\int_0^l\psi_K^*\varphi_L^*(\mathbf{A}\cdot\mathbf{P})\varphi_M\psi_N d\tau_\varphi d\tau_\psi. \tag{5}$$

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[1] J. Weber, Phys. Rev. 94, 214 (1954).

[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 49.

We let $\mathbf{A}=\mathbf{A}_0 q(t)$, and assume that the fields have no spacial variation over the interaction gap. $\mathbf{A}_0$ is then a constant. Making use of (3) and the operator for $\mathbf{P}$ we can carry out one integration required by (5) to obtain

$$\int_0^l \varphi_L^*(\mathbf{A}_0\cdot\mathbf{P})\varphi_M dx=\frac{A_{0x}}{l}\int_0^l \exp\left(-\frac{2\pi iLx}{l}\right)\left[-i\hbar\frac{d}{dx}\right]\times\exp\left(\frac{2\pi iMx}{l}\right)dx=A_{0x}P_M\delta_{LM}. \quad (6)$$

We assume the interaction gap to be so oriented that $\mathbf{A}_0$ is parallel to $\mathbf{P}$. Making use of (6), we can write (5) in the form

$$\frac{d}{dt}[a_K(t)b_L(t)]=\frac{e}{mc}(i\hbar)^{-1}\langle E_K|q|E_N\rangle b_L(t)|\mathbf{A}_0|P_L\times\exp\left(\frac{i(E_K-E_N)t}{\hbar}\right). \quad (7)$$

Let $\omega_{KN}=(E_K-E_N)/\hbar$. We assume that the momentum of the electron while in the interaction gap is approximately known to be P_E. The value of L corresponding to P_E can be obtained from (4) and is given by $L=lP_E/2\pi\hbar$. b_L for this value of L will be large and can therefore be considered to remain approximately constant during a not too long interaction time. Under these conditions we can integrate (7) to obtain

$$a_K(t)b_L(t)\approx\frac{e(i\hbar)^{-1}\langle E_K|q|E_N\rangle b_L(t)|\mathbf{A}_0|P_E[\exp(i\omega_{KN}t)-1]}{imc\omega_{KN}}. \quad (8)$$

From (8) we can obtain an expression for $|a_K(t)|^2$ and from this we can calculate the induced noise, since energy is conserved in these transitions. $|a_K(t)|^2$ is given by

$$|a_K(t)|^2=\frac{4e^2|\mathbf{A}_0|^2P_E^2|\langle E_K|q|E_N\rangle|^2\sin^2(\frac{1}{2}\omega_{KN}t)}{m^2c^2\hbar^2\omega_{KN}^2}. \quad (9)$$

In (9) $\langle E_K|q|E_N\rangle$ is the matrix element of q over the quantum states of the fields with eigenvalues E_K and E_N. We now proceed to calculate the matrix elements of q.

We assume that the magnetic vector potential and the scalar potential are not functions of the x coordinate, within the interaction gap, and that the electric field intensity is therefore uniform over the interaction gap. We express the electric field intensity in the interaction region in terms of q and a canonically conjugate variable p following the procedure outlined by Schiff.[3] Using Maxwell's equations, we obtain for the electric field intensity:

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-\frac{1}{c}\mathbf{A}_0\dot{q}=-\frac{1}{c}\mathbf{A}_0 p. \quad (10)$$

The electromotive force V is given by $\int_c\mathbf{E}\cdot d\mathbf{l}$, where the contour is over the interaction gap. For a uniform field over an interaction region of length l, we have

$$V=El=-(1/c)A_0pl. \quad (11)$$

Consider the matrix elements of the operator p. We can write

$$p_{KN}=\frac{dq_{KN}}{dt}=\frac{[H_0q-qH_0]_{KN}}{i\hbar}=\frac{(E_K-E_N)}{i\hbar}\langle E_K|q|E_N\rangle. \quad (12)$$

We substitute (12) in (11) to obtain

$$\langle E_K|V|E_N\rangle=(i/c)A_0\omega_{KN}\langle E_K|q|E_N\rangle l. \quad (13)$$

Let the total time of interaction be τ. Now τ is approximately given by $\tau=ml/P_E$. Evaluating (9) at time $t=\tau$, and making use of (13), we obtain

$$|a_K(\tau)|^2=\frac{e^2|\langle E_K|V|E_N\rangle|^2}{\hbar^2\omega_{KN}^2}\left[\frac{\sin(\frac{1}{2}\omega_{KN}\tau)}{\frac{1}{2}\omega_{KN}\tau}\right]^2. \quad (14)$$

If an electron undergoes a transition such that the region goes to a state E_K, we say the measured electromotive force is V_K, where $eV_K=\hbar\omega_{KN}$. If no transition occurs, the electromotive force is taken to be zero. The mean squared electromotive force measured by a large number M electrons as a result of transitions to the state K is

$$(|V_K|^2)_{\text{Av}}=\frac{[M|a_K(\tau)|^2]|V_K|^2}{M}=\frac{|a_K(\tau)|^2\hbar^2\omega_{KN}^2}{e^2}. \quad (15)$$

Making use of (14), we can write (15) as

$$(|V_K|^2)_{\text{Av}}=|\langle E_K|V|E_N\rangle|^2\left[\frac{\sin(\frac{1}{2}\omega_{KN}\tau)}{\frac{1}{2}\omega_{KN}\tau}\right]^2. \quad (16)$$

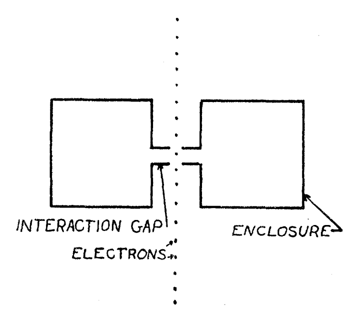

FIG. 1. Electron beam interacting with the fields of an enclosure.

[3] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 362.

The region with which the electrons interact is only partially specified, but its temperature will be known. Also, since dissipation is present ω_{KN} will have a continuous range of values. In order to obtain the total noise contribution (16) has to be summed over all states K and an ensemble average taken. This procedure can be carried out, using the methods of Callen and Welton,[4] in the following way. From the rules for matrix multiplication

$$\langle E_N|V^2|E_N\rangle = \sum_K |\langle E_K|V|E_N\rangle|^2. \tag{17}$$

We can average both sides of (17) over an ensemble, at temperature T. We denote by $[\langle E_N|V^2|E_N\rangle]_{\mathrm{Av},\,d\omega_{KN}}$ the ensemble average of the contribution of the group of matrix elements $\langle E_N|V|E_K\rangle$ in the vicinity of $\pm\omega_{KN}$ within a range $d\omega_{KN}$ to the ensemble average of the operator corresponding to V^2.

A study of Callen and Welton's paper[4] shows that we can write the fluctuation dissipation theorem in differential form as

$$[\langle E_N|V^2|E_N\rangle]_{\mathrm{Av},\,d\omega_{KN}} = [\sum_{d\omega_{KN}} |\langle E_N|V|E_K\rangle|^2]_{\mathrm{Av}} = \frac{R(\omega_{KN})\hbar\omega_{KN}}{\pi}\left[\frac{1}{1-\exp(-\hbar\omega_{KN}/kT)} + \frac{\exp(-\hbar\omega_{KN}/kT)}{1-\exp(-\hbar\omega_{KN}/kT)}\right]d\omega_{KN}. \tag{18}$$

In (18), the first term of the expression within the brackets represents the contribution of matrix elements $\langle E_N|V|E_N+\hbar\omega\rangle$ and the second term represents the contribution of matrix elements $\langle E_N|V|E_N-\hbar\omega\rangle$. $R(\omega_{KN})$ is the real part of the impedance function, which would be seen between the points at which electrons enter and leave the enclosure. We can make use of (16) and (18) to write an expression for the mean squared electromotive force measured by the electron beam. If the energy of all of the electrons before interaction is almost the same and equal to $U=\hbar\omega_{\max}$, then the mean squared electromotive force measured by the electron beam is

$$\langle V^2\rangle_{\mathrm{Av}} = \frac{1}{\pi}\int_0^{\omega_{\max}}\left[\frac{R(\omega)\hbar\omega}{1-\exp(-\hbar\omega/kT)}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega + \frac{1}{\pi}\int_0^{\infty}\left[\frac{R(\omega)\hbar\omega\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega. \tag{19}$$

Equation (19) is a very general expression† for the induced noise on an electron beam which interacts with the fields of an enclosure at temperature T, in terms of the dissipation function $R(\omega)$, and the electron transit time τ.

In (19) the first term represents the effect of transitions in which the electrons give up energy to the fields and the second term represents the effect of transitions in which the electrons gain energy. We can write Eq. (19) in the form

$$\langle V^2\rangle_{\mathrm{Av}} = \frac{2}{\pi}\int_0^{\omega_{\max}} R(\omega)\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega + \frac{1}{\pi}\int_{\omega_{\max}}^{\infty}\left[\frac{R(\omega)\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega. \tag{20}$$

In (20) the term,

$$\frac{2}{\pi}\int_0^{\omega_{\max}} R(\omega)\left[\frac{\hbar\omega}{2}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega,$$

represents the observable effect of the fluctuations of the vacuum fields.[5]

For the damped oscillator of Fig. 2, $R(\omega)$ can be expressed in terms of the conductance G, the capacity C, and the inductance L as

$$R(\omega) = \frac{\omega^2L^2G}{(1-\omega^2LC)^2+(\omega LG)^2}. \tag{21}$$

Employing (21) in (20), we obtain

$$\langle V^2\rangle_{\mathrm{Av}} = \frac{2}{\pi}\int_0^{\omega_{\max}}\left[\frac{\omega^2L^2G}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega + \frac{1}{\pi}\int_{\omega_{\max}}^{\infty}\left[\frac{\omega^2L^2G}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2 d\omega. \tag{22}$$

If G is very small then the integrand will be large only in the vicinity of $\omega=\omega_0$, where ω_0 is the natural frequency of the oscillator, given by $\omega_0^2LC=1$. For $\omega_{\max}\gg\omega_0$, (22) is approximately equal to

$$\langle V^2\rangle_{\mathrm{Av}} \approx \frac{2}{\pi}\left[\frac{\hbar\omega_0}{2}+\frac{\hbar\omega_0}{\exp(\hbar\omega_0/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega_0\tau)}{\frac{1}{2}\omega_0\tau}\right]^2 \times\int_0^{\infty}\frac{\omega^2L^2G\,d\omega}{(1-\omega^2LC)^2+(\omega LG)^2} = \frac{1}{C}\left[\frac{\hbar\omega_0}{2}+\frac{\hbar\omega_0}{\exp(\hbar\omega_0/kT)-1}\right]\left[\frac{\sin(\frac{1}{2}\omega_0\tau)}{\frac{1}{2}\omega_0\tau}\right]^2. \tag{23}$$

[4] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 35 (1951).

† The mean square deviation of the electron velocities after interaction can be obtained by multiplying expressions (19) and (20) by $(e/P)^2$, where e and P are the charge and momentum of the electrons, respectively.

[5] J. Weber, Phys. Rev. **90**, 977 (1953).

Equation (23) is the expression for the mean squared electromotive force measured by an electron beam interacting with a weakly damped oscillator, and agrees with expression (16) of the earlier[1] paper. To discuss the case where the oscillator is very heavily damped we employ (21) in (19) and obtain

$$\langle V^2\rangle_{\mathrm{Av}}=\frac{1}{\pi}\int_0^{\omega_{\max}}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\left[\frac{1}{1-\exp(-\hbar\omega/kT)}\right]d\omega+\frac{1}{\pi}\int_0^{\infty}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\left[\frac{\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}\right]d\omega. \quad (24)$$

The first term of (24) tends to zero as $G\rightarrow\infty$ because the range of integration is finite and the integrand tends to zero. The second term of (24) can be shown to also tend to zero, in the following way:

$$\frac{1}{\pi}\int_0^{\infty}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\left[\frac{\exp(-\hbar\omega/kT)}{1-\exp(-\hbar\omega/kT)}\right]d\omega$$
$$=\frac{1}{\pi}\int_0^{\omega_1}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\left[\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega+\frac{1}{\pi}\int_{\omega_1}^{\infty}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\times\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\left[\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega. \quad (25)$$

As $G\rightarrow\infty$ the first term on the right side of (25) tends to zero because the range of integration is finite and the integrand approaches zero. Consider the second term of (25). In the range $\omega_1<\omega<\infty$, $[\sin(\frac{1}{2}\omega\tau)/\frac{1}{2}\omega\tau]^2$ <1; we can choose two numbers a and b such that

$$(1-\omega^2LC)^2+(\omega LG)^2>a(\omega LG)^2, \quad (26)$$

$$\exp(\hbar\omega/kT)-1>b\exp(\hbar\omega/kT). \quad (27)$$

Using the inequalities (26) and (27), we can write

$$\frac{1}{\pi}\int_{\omega_1}^{\infty}\left[\frac{\omega^2L^2G\hbar\omega}{(1-\omega^2LC)^2+(\omega LG)^2}\right]\left[\frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega\tau}\right]^2\times\left[\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega<\frac{\omega_1kT}{\pi abG}\exp-\left(\frac{\hbar\omega_1}{kT}\right)\times\left[1+\frac{kT}{\hbar\omega_1}\right]. \quad (28)$$

Equation (28) tends to zero as G becomes large. We therefore conclude that the observed electron beam noise caused by both the thermal fluctuations and the vacuum fluctuations tends to zero as the damping becomes very large.

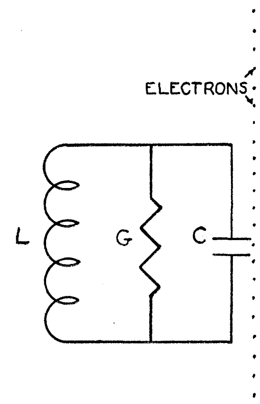

FIG. 2. Electron beam interacting with an oscillator which may be heavily damped.

CONCLUSION

We have studied the interaction of an electron beam with the fields in an enclosed region. It is believed that this model is a good representation for low-temperature noise measurement experiments in which the random changes in velocity of the electrons are measured after interaction. The first term of (20) represents the effect of the vacuum fluctuations. For a weakly damped oscillator this is given by the term $(1/C)(\hbar\omega_0/2)\times[\sin(\frac{1}{2}\omega_0\tau)/\frac{1}{2}\omega_0\tau]^2$ of expression (23). This term represents an observable effect of field quantization which is finite in a first-order theory.