

Now,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-i\eta' u'}}{1-iu'-\rho^2/(1-it')} du' \\ = 2\pi \exp\left[-\eta' + \eta' \frac{\rho^2}{1-it'}\right], \quad \eta' > 0 \\ = 0, \quad \eta' < 0 \quad (\text{since } \rho^2/(1+t^2) < 1). \quad (\text{A-4}) \end{aligned}$$

Therefore,

$$\begin{aligned} W_2(\xi, \eta) = \frac{e^{-\eta'}}{8\pi\sigma_0^2(1-\rho^2)} \\ \times \int_{-\infty}^{\infty} \frac{dt'}{1-it'} \exp\left[-i\xi't' + \eta' \frac{\rho^2}{1-it'}\right]. \quad (\text{A-5}) \end{aligned}$$

If  $x = 1-it'$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dt'}{1-it'} \exp\left[-i\xi't' + \eta' \frac{\rho^2}{1-it'}\right] \\ = -ie^{-\xi'} \int_{1-i\infty}^{1+i\infty} \frac{dx}{x} \exp\left(\xi'x + \frac{\eta'\rho^2}{x}\right) \\ = 2\pi e^{-\xi'} \sum_{m=0}^{\infty} \frac{1}{(m!)^2} (\xi'\eta'\rho^2)^m, \\ (\xi' > 0, \text{ by the Residue Theorem}) \\ = 2\pi e^{-\xi'} I_0(2\rho(\xi'\eta')^{\frac{1}{2}}), \quad (\text{A-6}) \end{aligned}$$

which when substituted into (A-5) gives the probability density, Eq. (15).

## λ Transition of Liquid Helium

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The partition function of liquid helium proposed by Feynman,  $q = \sum g(L) \exp(-aTL)$ , is calculated for a simple cubic lattice using an approximation corresponding to Bethe's method for the Ising model. It is shown that a second-order transition occurs at  $aT_\lambda = \ln 4$ , or  $T_\lambda = 2.9m/m' \text{ }^\circ\text{K}$  ( $m$  and  $m'$  representing the true and the effective masses of a helium atom). The nature of the approximation is discussed.

### I. INTRODUCTION

THE series of papers on liquid helium which Feynman has recently published<sup>1-3</sup> explained many of the so far unsolved properties of this substance. However, the problem of the nature of the  $\lambda$  transformation still does not seem to have been settled. In his papers<sup>1,2</sup> Feynman proposed an expression for the partition function and solved it approximately to obtain a transition of a third order. As it is commonly accepted that the  $\lambda$  transformation of liquid helium is of a second order, Feynman<sup>2</sup> ascribed the disagreement between his results and experiment to the fact that he neglected the correlation among atoms, both in the same cyclic change and in different cyclic changes. Though Chester<sup>4</sup> agreed with Feynman with regard to this interpretation of the discrepancy, Rice,<sup>5</sup> Matsubara,<sup>6</sup> and ter Haar<sup>7</sup> expressed the view that the above neglect of the correlation is not the origin of the discrepancy.

The purpose of the present paper is to show that a technique developed in the order-disorder theories can be applied to this problem to take into account the geometrical correlation, and, though the conclusion is not completely convincing (due to the approximate nature of the technique), that Feynman's partition function does give a second-order transition.

### II. FREE ENERGY

The original expression for the partition function with which we begin is Eq. (7) of reference 2:

$$\begin{aligned} q = \int \sum_P \exp\left[-\frac{m'kT}{2\hbar^2} \sum_i (\mathbf{z}_i - P\mathbf{z}_i)^2\right] \\ \times \rho(\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N) d^N \mathbf{z}_i \quad (1) \end{aligned}$$

in which  $m'$  is the effective mass of a helium atom. For the derivation of this equation and the notation, readers are referred to the original paper by Feynman. When one assumes that the value of  $\rho(\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N)$  is non-vanishing only when the  $\mathbf{z}$ 's are located on a simple cubic lattice and that  $(\mathbf{z}_i - P\mathbf{z}_i)^2$  is neglected except when its value is equal to  $d^2$ ,  $d$  being the lattice constant of the hypothetical lattice, Eq. (1) reduces to Eq. (4)

<sup>1</sup> R. P. Feynman, Phys. Rev. **90**, 1116 (1953).

<sup>2</sup> R. P. Feynman, Phys. Rev. **91**, 1291 (1953).

<sup>3</sup> R. P. Feynman, Phys. Rev. **91**, 1301 (1953); Phys. Rev. **94**, 262 (1954).

<sup>4</sup> G. V. Chester, Phys. Rev. **93**, 1412 (1954).

<sup>5</sup> O. K. Rice, Phys. Rev. **93**, 1161 (1954).

<sup>6</sup> T. Matsubara, Busseiron Kenkyu **72**, 78 (1954).

<sup>7</sup> D. ter Haar, Phys. Rev. **95**, 895 (1954).

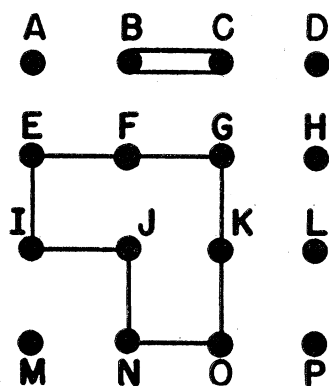


FIG. 1. Examples of allowed polygons for  $g(L)$ .  $BC$  is a two-sided polygon. Each of  $EF, FG, GK, KO, ON$ , etc., is called a "side," and each of  $E, F, G, K$ , etc., is a "vertex" of the polygon.

of reference 1, namely

$$q = \sum_L g(L) \exp(-aTL), \quad (2)$$

where

$$a = m'd^2k/2\hbar^2.$$

$g(L)$  is the total number of ways that polygons can be drawn by connecting nearest neighbor points of a simple cubic lattice so that the total number of "sides" of all the polygons together is  $L$ . Here, by a "side" of a polygon we mean a bond between nearest neighbor lattice points. From the meaning of Eq. (1), the polygons are drawn under the following restrictions: (i) between two adjacent lattice points only one side is drawn, except for two-sided polygons, (ii) polygons must not come into contact, and (iii) polygons except for the two-sided ones are counted twice, corresponding to the two directions of cyclic change.

The terminology is explained in Fig. 1.  $BC$  is an example of a two-sided polygon. The polygon  $EGONJIE$  has eight "sides" by our definition because, for example,  $EG$  contributes two "sides,"  $EF$  and  $FG$ , though this is

contrary to the ordinary usage of the word. In Fig. 2 some of the examples are shown which are not allowed. (a) is a pair of two-sided polygons in contact at the point  $A$ , (b) two polygons in contact at  $B$ , (c) two polygons crossed each other, and (d) two polygons sharing a bond.

The problem we are concerned with in this paper is to evaluate Eq. (2). For that purpose we follow closely the method used by the author in a treatment of the Ising model.<sup>8</sup> We express  $L$  and  $g(L)$  as functions of a certain number of parameters and replace  $q$  of Eq. (2) by its maximum term; in other words, we minimize the free energy,

$$F = -kT \ln[g(L) \exp(-aTL)], \quad (3)^9$$

with respect to these parameters.

Under the restrictions mentioned above concerning polygons, the possible configurations of a lattice point and of a bond are as shown in Tables I and II. The symbols in the tables are to be interpreted as follows: A dot is a lattice point which does not belong to any polygon (like the point  $A$  in Fig. 1). The double line of  $y_4$  indicates a two-sided polygon ( $BC$  in Fig. 1), and the lattice point of  $x_2$  a vertex of this polygon (the point  $B$  or  $C$  in Fig. 1). The angle for  $x_3$  means that this lattice point is a vertex of a multisided polygon (e.g., the point  $G$  in Fig. 1). This symbol also includes a point like  $K$  in Fig. 1.<sup>10</sup> The arrows indicate the directions of cyclic change. The  $x$ 's and  $y$ 's are the probabilities of finding one of the respective configurations and the values of  $\alpha$ 's and  $\beta$ 's are the number of different configurations having the same probability on the basis of symmetry requirement. For instance,  $\alpha_2$  is six because there are six different directions that a two-sided polygon can be drawn with a given point as its vertex.  $\alpha_3$  is equal to the number of ways of selecting two out of six, multiplied by two, corresponding to the two cyclic directions. The  $\beta$ 's are calculated as follows:  $\beta_2 = 2 \times {}_5C_1$ ,  $\beta_3 = ({}_5C_1)^2$ ,  $\beta_5 = 2 \times {}_5C_2 \times 2$ ,  $\beta_6 = 2 \times {}_5C_1 \times {}_5C_2 \times 2$ ,  $\beta_7 = ({}_5C_2 \times 2)^2$ , and  $\beta_8 = ({}_5C_1)^2 \times 2$ .

As is seen in these drawings, every point is either a vertex<sup>10</sup> of a polygon or does not belong to any polygon. In other words, we always have closed polygons. Therefore the figures in Tables I and II satisfy the three requirements mentioned above.

When the number of lattice points in the lattice is  $N$ , the total number  $L$  of "sides" of polygons can be written as

$$L = 3N(2y_4 + 50y_8). \quad (4)$$

<sup>8</sup> R. Kikuchi, Phys. Rev. **81**, 988 (1951). This paper will be referred to as TCP I.

<sup>9</sup> Although  $F$  has the form of  $F = E^* - TS^*$  when one puts  $E^* = akT^2L$  and  $S^* = k \ln g(L)$ ,  $E^*$  and  $S^*$  do not have the meaning of energy and entropy, respectively, because  $E^*$  contains  $T$ . See Eqs. (7) and (18) below.

<sup>10</sup> A point like  $F$  or  $K$  in Fig. 1 is called a vertex, too, because  $EF$  and  $FG$  are each considered separately to be a side.

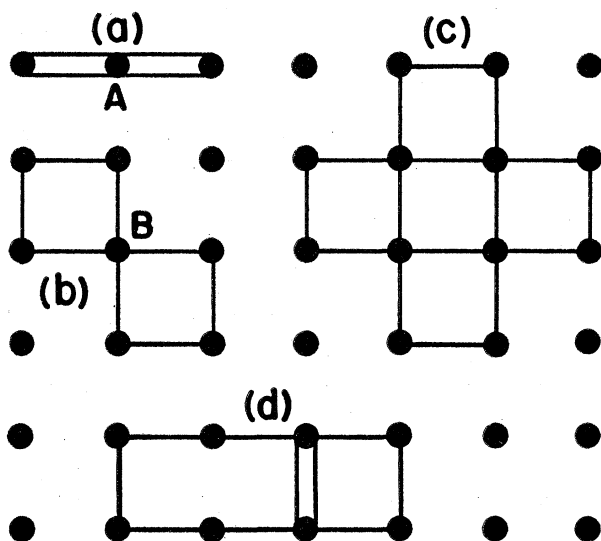


FIG. 2. Examples of polygons not allowed for  $g(L)$ .

This is a rigorous relationship. For  $g(L)$  one has to be satisfied with an approximate functional form. The best approximation using the parameters defined in Tables I and II is the one corresponding to that of Bethe, and the form of  $g(L)$  for that case is given by the last equation of Section B of TCP I, that is

$$g(L) = X_N^{2\omega-1} Y_N^{-\omega} N^{1-\omega+1}, \quad (5)$$

where  $2\omega$  is the coordination number of the lattice, and, as defined in Eq. (A.5) of TCP I,

$$X_N = \prod_{i=1}^3 (x_i N)^{\alpha_i}, \quad (6)$$

$$Y_N = \prod_{i=1}^8 (y_i N)^{\beta_i}.$$

Inserting Eqs. (4), (5), and (6) into (3) and using Stirling's formula, the free energy of the system be-

TABLE I. Configuration of a point.

CONFIGURATION	PROBABILITY	$\alpha_i$
•	$x_1$	1
→	$x_2$	6
↑	$x_3$	30

comes

$$F = 3akT^2N(2y_4 + 50y_8) - kTN(5 \sum_{i=1}^3 \alpha_i x_i \ln x_i - 3 \sum_{i=1}^8 \beta_i y_i \ln y_i) \quad (7)$$

for  $\omega=3$ . The next procedure is to minimize  $F$  with respect to independent parameters, keeping  $T$  constant.

### III. REGION FOR He II

One can derive from Tables I and II the following geometrical relations among  $x$ 's and  $y$ 's:

$$1 = x_1 + 6x_2 + 30x_3, \quad (8)$$

$$x_1 = y_1 + 5y_2 + 20y_5,$$

$$x_2 = y_2 + 5y_3 + 20y_6 = y_4, \quad (9)$$

$$x_3 = y_5 + 5y_6 + 20y_7 = 5y_8.$$

If one chooses  $y_3, y_4, y_6, y_7$ , and  $y_8$  as independent variables, the other variables can be expressed as linear combinations of these, as shown in Table III.

Minimizing  $F$  of Eq. (7) with respect to the independent variables, one obtains the following five funda-

TABLE II. Configuration of a bond.

CONFIGURATION	PROBABILITY	$\beta_i$
• •	$y_1$	1
• →	$y_2$	10
→ •	$y_3$	25
→ →	$y_4$	1
• ↓	$y_5$	40
→ ↓	$y_6$	200
↓ •	$y_7$	400
↓ ↓	$y_8$	50

mental equations:

$$y_1 y_3 = y_2^2, \quad (10)$$

$$y_1 y_6 = y_2 y_5, \quad (11)$$

$$y_1 y_7 = y_5^2, \quad (12)$$

$$e^{2aT} y_2^{10} y_4 / y_1^{11} = (x_2 / x_1)^{10}, \quad (13)$$

$$e^{aT} y_5^4 y_8 / y_1^5 = (x_3 / x_1)^5. \quad (14)$$

These equations together with relations in Table III can be solved without much difficulty when one puts

$$y_2 / y_1 = v, \quad (15)$$

$$y_5 / y_1 = w,$$

yielding

$$v = \frac{1}{5} e^{-aT}, \quad (16)$$

$$w = (4e^{-aT} - 1)/20.$$

The other variable can be written as functions of  $v$  and  $w$ .

As  $y_5$  and  $y_1$  are positive,  $w$  of (15) should be positive. Then Eq. (16) shows that this solution is valid only when the temperature is lower than the critical temperature  $T_\lambda$  defined by

$$aT_\lambda = \ln 4 = 1.386. \quad (17)$$

When Eqs. (10) to (14) are satisfied, or when  $F$  of Eq. (7) is a minimum with respect to the parameters, the free energy  $F$ , the entropy  $S$ , and the energy  $E$  of

TABLE III. Relations between the dependent and the independent variables. The meaning of this table is, for example,  $x_1 = 1 - 6y_4 - 150y_8$ .

	1	$y_3$	$y_4$	$y_6$	$y_7$	$y_8$
$x_1$	1		-6			-150
$x_2$			1			
$x_3$						5
$y_1$	1	25	-11	200	400	-250
$y_2$		-5	1	-20		
$y_3$				-5	-20	5

the system can be written as

$$\begin{aligned} F &= -kTN(5 \ln x_1 - 3 \ln y_1), \\ S &= -3akTN(2y_4 + 50y_8) + kN(5 \ln x_1 - 3 \ln y_1), \\ E &= -3akT^2N(2y_4 + 50y_8). \end{aligned} \quad (18)$$

From these relations one can see that if the  $x$ 's and  $y$ 's are continuous functions of temperature, then so are  $S$  and  $E$ .

The specific heat  $c_{II}$  per atom can be derived from  $E$  of Eq. (18):

$$\frac{c_{II}}{k} = \frac{28.8\theta^2 e^\theta}{(14.4 - e^\theta)^2} - \frac{2\theta(14.4 - 3e^\theta)}{14.4 - e^\theta}, \quad (19)$$

where we put  $\theta = aT$ . At the critical temperature,  $\theta_\lambda = \ln 4$ , Eq. (19) gives

$$\frac{c_{II\lambda}}{k} = \frac{\ln 4}{169} (180 \ln 4 - 78) = 1.407. \quad (20)$$

#### IV. REGION FOR He I

When  $w$  is zero, one can see that  $x_3$  and  $y_5$  to  $y_8$  also vanish. As may be seen from Tables I and II, this is a state in which only two-sided polygons exist. Then, of the five equations, (10) to (14), all but (10) and (13) become trivial identities. We can still use  $v$  as defined in Eq. (15) for this case, but the results are different, namely

$$v = (r-1)/10, \quad (21)$$

where

$$r^2 = 1 + 20e^{-2\theta}.$$

One can use the same expressions (18) for  $S$  and  $E$ , and the specific heat is given by the temperature derivative of

$$\frac{aE}{kN} = \frac{4\theta^2 e^{-2\theta}}{r} \left[ \frac{2(5+6r)}{r+1+12e^{-2\theta}} - \frac{25}{r+1} \right]. \quad (22)$$

At the  $\lambda$  point, the specific heat becomes

$$\frac{c_{I\lambda}}{k} = \frac{\ln 4}{169} (50 \ln 4 - 78) = -0.071. \quad (23)$$

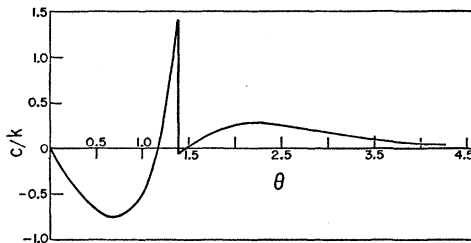


FIG. 3. The specific heat  $c$  per atom versus  $\theta = aT$ . The values at the transition point are:  $\theta_\lambda = 1.386$ ,  $c_{II\lambda} = 1.407k$ , and  $c_{I\lambda} = -0.071k$ .

Comparing this with  $c_{II\lambda}$  of Eq. (20) and considering that  $E$  is continuous at  $T_\lambda$ , one concludes that this temperature corresponds to a second-order transformation.

When one calculates  $a$  of Eq. (2) using  $d = 3.4 \text{ \AA}$ , which is the average distance between atoms in liquid helium,  $T_\lambda$  of Eq. (17) is equal to

$$T_\lambda = 2.9(m/m')^\circ \text{K}, \quad (24)$$

where  $m$  and  $m'$  are the true and the effective masses of a liquid helium atom, respectively. If one equates the expression for  $T_\lambda$  with the observed  $\lambda$  point,  $2.2^\circ \text{K}$ , one obtains  $m'/m = 1.3$  which is not too unreasonable, though slightly smaller than Feynman's estimate.<sup>2</sup>

The curve of specific heat versus  $\theta (= aT)$  is plotted in Fig. 3. Since  $q$  of Eq. (1) is not the total partition function of the system, but is to be multiplied by a function of temperature as is seen in Feynman's paper,<sup>2</sup> it is to be expected that the negative value of the specific heat in Fig. 3 should be increased to a positive value when the remaining factor of the partition function is taken into account; thermodynamics is therefore not contradicted.

From this and the foregoing section and from some other evidence, one can interpret the transition as follows. Above  $T_\lambda$  the main contribution to the partition function comes from polygons of finite size, their sizes increasing as the temperature is lowered. At  $T_\lambda$  or below, polygons of infinite size predominate. In order to verify this picture, a better approximation which uses a square as the basic figure, as was done in TCP I, is desirable. In that approximation we expect that not only two sided polygons but also squares exist in the He I region. The minimum of the specific heat just above  $T_\lambda$ , which we attribute to the nature of the approximation, is expected to disappear or to be reduced by the improved approximation.

As is the case in the approximate treatment of the Ising model, the actual value of  $c_{II\lambda} - c_{I\lambda}$  does not have any physical meaning. Also it should be emphasized again that the present approximation fails near the absolute zero, because we neglected values of  $(\mathbf{z}_i - P\mathbf{z}_i)^2$  larger than  $d^2$ . However, it seems worthwhile to investigate how good the present approximation is at somewhat higher temperatures. This we shall do in the next section by comparison with the Ising model.

#### V. COMPARISON WITH ISING MODEL

One of the specific features of the results in the previous sections is that a second-order transition is obtained without recourse to any long-range order parameters. Although in the ordinary order-disorder theories, the concept of long-range order helps one to understand the physical nature of order-disorder to such an extent that it may be looked upon as the central feature of the phenomenon, a moment's reflection will indicate that there are treatments of order-

disorder problems that do not use the concept of long-range order.

To make the presentation simpler, let us consider in this section a two-dimensional square lattice. The partition function of the Ising model can be written as

$$f = 2^N (\cosh K)^{2N} \sum \Omega_n \tanh^n K, \quad (25)^{11}$$

where  $\Omega_n$  is the number of configurations of polygons when the total number of "sides" of polygons is  $n$ . The difference between  $\Omega_n$  and  $g(L)$  of Eq. (2) is that for  $\Omega_n$  all three restrictions (i), (ii), and (iii) of Sec. II do not hold, and instead the requirements are: (i') between two adjacent lattice points only one side is drawn, and the two-sided polygons are not allowed, (ii') polygons may intersect or touch, but a given bond must be used once only, and (iii') each polygon is counted once. For example, in Fig. 2, (b) and (c) are allowed, but (a) and (d) are not:  $BC$  of Fig. 1 is not allowed.

Although Eq. (25) was used to evaluate  $f$  rigorously,<sup>11</sup> it seems worthwhile, for the present discussion of the nature of the approximations, to show a method of evaluating  $f$  approximately. We shall apply the pair approximation which is analogous to the one used in

TABLE IV. Configuration of a point for the Ising model.

CONFIGURATION	PROBABILITY	$\alpha_i$
•	$x_1$	1
┐	$x_2$	6
+	$x_3$	1

previous sections, and shall show that it gives a result identical to the ordinary Bethe approximation which is the pair approximation applied to the partition function including the long-range order parameter.

The different configurations of a lattice point and of a bond, together with the notation for the probability of finding one of them, are listed in Tables IV and V. One can see from Table V that the number  $n$  of "sides" of polygons is

$$n = 2N(9y_4 + 6y_5 + y_6). \quad (26)$$

For the pair approximation, we use Eqs. (5) and (6) with  $\omega = 2$  for  $\Omega_n$ . Hence from Eq. (25) the free energy contributed from the summation part becomes

$$\begin{aligned} F &= -kT \ln[\Omega_n \tanh^n K] \\ &= 2kTN\Theta(9y_4 + 6y_5 + y_6) \\ &\quad - kTN[3\sum \alpha_i x_i \ln x_i - 2\sum \beta_i y_i \ln y_i], \end{aligned} \quad (27)$$

where

$$\Theta = \ln(\coth K).$$

<sup>11</sup> See, for instance, G. F. Newell and E. W. Montroll, *Revs. Modern Phys.* **25**, 353 (1953); Kurata, Kikuchi, and Watari, *J. Chem. Phys.* **21**, 434 (1953).

TABLE V. Configuration of a bond for the Ising model.

CONFIGURATION	PROBABILITY	$\beta_i$
• •	$y_1$	1
• ┐	$y_2$	6
┐ ┐	$y_3$	9
┐ └	$y_4$	9
┐ ┐	$y_5$	6
┐ ┐	$y_6$	1

Making  $F$  a minimum with respect to the independent variables, one obtains

$$y_2/y_1 = (3 - e^\Theta)/(3e^\Theta - 1). \quad (28)$$

Since this quantity must always be positive, the transition point  $\Theta_c$  is given by

$$\Theta_c = \ln 3. \quad (29)^{12}$$

Above  $T_c$ , all of the  $x$ 's and  $y$ 's vanish except  $x_1$  and  $y_1$ , which means that above  $T_c$ ,  $\Omega_n$  are all zero except  $\Omega_0 = 1$ , giving

$$f = 2^N (\cosh K)^{2N}. \quad (30)$$

This equation is exactly the partition function of the disordered state calculated from the ordinary Bethe approximation. It is just a matter of mathematical manipulation to obtain the solution for the ordered state under Bethe's approximation from Eqs. (27) and (28) when  $y_2 \neq 0$ .

This analysis makes it legitimate to say that the method used in the previous sections corresponds to Bethe's approximation as far as the relation to the rigorous evaluation of the partition function is concerned.

Although we have different restrictions on  $g(L)$  of Sec. II and on  $\Omega_n$  of this section, the difference does not seem to affect appreciably the cooperative nature of the problem. For instance, if we replaced the restrictions on  $g(L)$  by those on  $\Omega_n$  and used the ordinary theories of order-disorder for calculating  $q$ , the transition point for the simple cubic lattice would be

$$aT_\lambda = 1.5, \quad (31)^{13,14}$$

which is very close to the value of Eq. (17).

It is desirable to apply the method developed in TCP I in obtaining an improved approximation to

<sup>12</sup> Using the definition of  $\Theta$  in Eq. (26) and  $K = \epsilon/kT$ , the value of  $\Theta_c$ , Eq. (29), corresponds to  $2\epsilon/kT_c = \ln 2$ , which is the well-known result obtained when Bethe's approximation is applied to the square lattice.

<sup>13</sup> Comparing Eqs. (3) and (27),  $\Theta_c$  is equated to  $aT_\lambda$ . Actually, the value of  $aT_\lambda$  would be 1.57 by Bethe's approximation and 1.52 by Wakefield's estimate of the true transition point. See reference 14.

<sup>14</sup> A. J. Wakefield, *Proc. Cambridge Phil. Soc.* **47**, 799 (1951).

estimate  $q$  of Eq. (2). However, in the next approximation, with a square as the basic figure, the number of independent variables increases to about twenty. But we may take advantage of the close similarity between our  $q$  and the ordinary order-disorder partition function, and may predict from a knowledge of the Ising model what would happen if one improved the approximation for  $q$ .

With improvement in the approximation,  $T_\lambda$  would change slightly, but the real transition point would not be too far from the value of Eq. (17).<sup>15</sup>  $c_{II\lambda}$  and  $c_{I\lambda}$  would both increase, the former moving higher more rapidly than the latter, increasing the discontinuity  $c_{II\lambda} - c_{I\lambda}$ . Although the three-dimensional case of the Ising model has not yet been worked out, it is probable that  $c_{II\lambda}$ , and also  $c_{I\lambda}$ , becomes infinite in the rigorous solution.<sup>14</sup> As  $c_{I\lambda}$  increases, the part of the specific heat curve just above  $T_\lambda$  naturally increases, and could give the shape which Feynman obtained using a different approximation.<sup>2</sup>

In short, the relationship between the various approximate and the rigorous solutions of this problem might be similar to that between the approximate solutions and Onsager's rigorous solution<sup>16</sup> of the problem of the two-dimensional Ising model.

## VI. DISCUSSION

The present calculation is based on the assumption that atoms are localized on lattice points. This may not be too unsatisfactory for discussing the qualitative nature of the transition, in view of the success of the

<sup>15</sup> If the relation between Bethe's approximation and Wakefield's estimate mentioned in reference 13 is transplanted to the present problem, the value of Eq. (17) would have to be lowered less than 0.1 to coincide with the true transition point.

<sup>16</sup> L. Onsager, *Phys. Rev.* **65**, 117 (1944).

lattice model for solid-liquid-gas phase changes.<sup>17</sup> It is probable that, using the lattice model, one can reproduce the qualitative nature of the amorphous phases correctly if one does not take advantage of the long-range regularity of the lattice but only of its local nature.<sup>18</sup> The type of lattice one uses would not affect the result very much. Of course it is desirable to extend the calculation to a continuum model.<sup>19</sup>

As Chester pointed out, Matsubara<sup>20</sup> obtained a second-order transition starting from an approximation quite similar to Feynman's. The difference between the treatment of Matsubara and that of Feynman seems to be that the former assumed a positive constant potential for atoms in cyclic rings. This assumption should be justified before we can accept his theory.

From evidence in the treatment of the ideal Bose-Einstein gas and from Feynman's discussion, it seems likely that the dominance of infinitely large polygons below  $T_\lambda$  corresponds to the condensation to the lowest energy level, though we have not succeeded in showing it clearly.

Summarizing, the present paper indicates, following Feynman's ideas, that the  $\lambda$  transition of liquid helium is essentially a Bose-Einstein condensation of hard-sphere Bose particles.

Thanks are due Professor R. P. Feynman for helpful discussions.

<sup>17</sup> See, for instance, F. Cernushi and H. Eyring, *J. Chem. Phys.* **7**, 547 (1939); J. E. Lennard-Jones and A. F. Devonshire, *Proc. Roy. Soc. (London)* **A163**, 53 (1937); R. Kikuchi, *J. Chem. Phys.* **19**, 1230 (1951).

<sup>18</sup> For instance, the division into sublattices is a kind of long-range regularity and should not be considered in a model for a liquid.

<sup>19</sup> Although a paper by the author (reference 17) shows how to extend this calculation to a continuum, the method is highly impracticable.

<sup>20</sup> T. Matsubara, *Progr. Theoret. Phys.* **6**, 714 (1951).