PHYSICAL REVIEW VOLUME 96, NUMBER 3 NOVEMBER 1, 1954

Nuclear Multipole Transitions in Inelastic Electron Scattering*

L. I. Schiff
Stanford University, Stanford, California
(Received July 26, 1954)

Expressions are obtained for the differential cross sections for inelastic scattering of fast electrons with excitation of various nuclear multipole transitions. The most probable transitions are those that involve collective motion of many nucleons, and in this case the term arising from the transition charge density dominates those that come from the current and magnetization densities. There is then a close relation between the probability for inelastic electron scattering and the probability for the corresponding radiative electric multipole transition, although an assumption must be made as to the shape of the transition charge density. This is illustrated with a detailed discussion of the collective electric quadrupole transitions, using the model of Bohr and Mottelson. When the transition is produced by one or a small number of nucleons, or when it is of magnetic multipole type, there is likely to be little relation between inelastic scattering and radiation probabilities. The electric monopole transition $(0+\rightarrow 0+)$ is also discussed. It is shown how the elastic scattering can be corrected for unresolved inelastic scattering as well as elastic quadrupole scattering before an analysis is made in terms of the spherically symmetric part of the static nuclear charge density, and also how the strength as well as the shape of the transition charge density can be determined experimentally when only relative measurements of inelastic scattering are available.

I. INTRODUCTION

RECENT experimental work of Hofstadter and collaborators[1,2] has demonstrated the possibility of obtaining precise distributions in energy and angle for electrons of very high initial energy scattered by nuclei, and has yielded specific results in a number of cases. Analysis of the angle distributions of such elastically scattered electrons[3] has already provided information on the radial dependence of the static charge density in various nuclei. There are two main reasons why a similar analysis must be made for the inelastically scattered electrons. First, with imperfect energy resolution, some inelastically scattered electrons corresponding to excitation of low-lying nuclear states will be included in the measurement of elastic scattering, and may affect the charge density inferred from these measurements. Second, when inelastically scattered electrons can be resolved in energy, it is anticipated that their distribution in angle can often be used to determine the strength and multipole character of the corresponding nuclear transition.

The information concerning nuclear transitions that can be obtained from inelastic electron scattering is very similar to that which can be obtained from Coulomb excitation by heavy charged particles that are slow enough so that they do not penetrate the nuclear Coulomb barrier appreciably.[4] In both cases, the interaction between incident particle and nucleus is essentially all electromagnetic and is calculable. Thus far, Coulomb excitation experiments have yielded total excitation probabilities as a function of the charge, mass, and velocity of the incident particle, which can be used to determine the strength and multipolarity of nuclear electric multiple transitions. Accurate measurements of the angle dependence of inelastically scattered electrons can determine these quantities independently and can, in addition, give the radial dependence of the transition charge density. In principle, such experiments can also provide similar information concerning magnetic multipole transitions; it will be shown, however, that such transitions are likely to be less probable than electric multipole transitions, and that they are more difficult to interpret.

In the present paper, the nucleus is described by charge, current, and magnetization densities ρ, $\mathbf{j}$, and $\mathbf{M}$, which are treated as classical quantities. Actually, they should be regarded as quantum-mechanical operators of the type discussed by Foldy,[5] in which case appropriate matrix elements between initial and final nuclear states must ultimately be calculated. However, for all except the very simplest nuclei, the nuclear wave functions are not known well enough to warrant such detailed calculations. We shall, therefore, be satisfied with a semiclassical treatment, according to which ρ, $\mathbf{j}$, and $\mathbf{M}$ are c numbers which represent the static densities (expectation values for the nuclear ground state) in the case of elastic scattering, and the transition densities (matrix elements between nuclear ground and excited states) in the case of inelastic scattering. It is then expected that the resulting formulas can be used phenomenologically, perhaps with a hydrodynamical model for the combined contributions of nucleons and mesons to these densities (see Sec. VIII).

It is inherent in the present work that the dynamic interaction between electron and nucleus is treated by

* Supported in part by the Office of Scientific Research, Air Research and Development Command.

[1] Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953); Hofstadter, Hahn, Knudsen, and McIntyre, Phys. Rev. **95**, 512 (1954).

[2] McIntyre, Hahn, and Hofstadter, Phys. Rev. **94**, 1084 (1954).

[3] Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954), and private communication.

[4] T. Huus and Č. Zupančič, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, 1 (1953); C. L. McClelland and C. Goodman, Phys. Rev. **91**, 760 (1953); N. P. Heydenburg and G. M. Temmer, Phys. Rev. **93**, 906 (1954); K. Alder and A. Winther, Phys. Rev. **91**, 1578 (1953).

[5] L. L. Foldy, Phys. Rev. **92**, 178 (1953).

first-order perturbation theory, and this seems to be a reasonable approximation.[6] In addition to this, we shall assume that the static interaction is also small, and use plane wave functions for the incident and scattered electrons; this is justified only for light elements, and is equivalent to the use of the Møller potentials and fields.[7] This latter approximation can be improved by using Coulomb wave functions for the electron like those calculated numerically in connection with elastic scattering,[3] and such calculations are now under way.

The work reported here differs from other recent calculations of radiative transitions[8,9] and of inelastic electron scattering[10] in one or both of two respects: the reduced wavelength $1/q$ associated with the change of momentum $\hbar\mathbf{q}$ of the electron is not necessarily large in comparison with nuclear dimensions, and this wavelength is usually much smaller than $\hbar c$ divided by the energy loss of the electron.[11] The first point means that there may be retardation within the nucleus, so that the radial dependences of the transition densities may be significant. Also, the transition probabilities do not necessarily decrease as the multipole order increases; the expansion in multipoles is nevertheless useful since nuclear selection rules often limit possible transitions to one or two multipole types. The second point means that the probability for inelastic electron scattering is fundamentally different from that for radiation, since in the latter case the reduced wavelength of the photon is equal to $\hbar c$ divided by the energy of the photon. It turns out that there is a close relation between the leading terms in the probabilities for the two processes in the case of electric, but not in the case of magnetic multipole transitions.

II. GENERAL FORMULATION AND MAGNETIC MULTIPOLE CALCULATION

We start from the interaction energy H' between the nuclear densities ρ, $\mathbf{j}$, $\mathbf{M}$, and an arbitrary external electromagnetic field that is described by the potentials φ, $\mathbf{A}$ or the field strengths $\mathfrak{E}$, $\mathfrak{H}$:

$$H' = \int (\rho\varphi - c^{-1}\mathbf{j}\cdot\mathbf{A} - \mathbf{M}\cdot\mathfrak{H})d\tau. \tag{1}$$

[6] An order of magnitude estimate of the second-order dynamic interaction (nuclear dispersion) indicates that it is relatively small.

[7] C. Møller, Z. Physik **70**, 786 (1931).

[8] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Appendix B.

[9] W. Franz, Z. Physik **127**, 363 (1950); P. R. Wallace, Can. J. Phys. **29**, 393 (1951).

[10] Amaldi, Fidecaro, and Mariani, Nuovo cimento **7**, 553, 757 (1950); Thie, Mullin, and Guth, Phys. Rev. **87**, 962 (1952); R. Gatto, Nuovo cimento **10**, 1559 (1953); J. H. Smith, Phys. Rev. **95**, 271 (1954).

[11] The importance of the difference between the short wavelength associated with the electron momentum transfer and the long wavelength associated with the nuclear excitation energy, in the case of high-energy electrons, was pointed out by J. S. Blair in a letter to Richard Wilson, July 1952.

In order that H' result in a transition that conserves the energy, the time variations of the nuclear and field quantities must cancel, so we assume that the nuclear quantities vary like $e^{i\omega t}$ and the field quantities like $e^{-i\omega t}$, where $\hbar\omega$ is the energy of the transition. The electric field strength may be expressed in terms of the potentials in either of the forms

$$\mathfrak{E} = -\partial\mathbf{A}/\partial ct - \mathbf{grad}\ \varphi = (i\omega/c)\mathbf{A} - \mathbf{grad}\ \varphi,$$

and the equation of continuity is

$$\mathrm{div}\mathbf{j} = -\partial\rho/\partial t = -i\omega\rho.$$

Substitution into Eq. (1) and integration by parts leads to

$$H' = \int [(i/\omega)\mathbf{j}\cdot\mathfrak{E} - \mathbf{M}\cdot\mathfrak{H}]d\tau; \tag{2}$$

the partial integration assumes that the nuclear densities have a finite extension in space, so that boundary terms at infinity vanish.

We wish to decompose H' into parts that correspond to electric and magnetic multipole transitions of all possible orders. Since the treatment of the magnetic transitions is the same whether the external field is a free radiation field (emission or absorption of photons) or arises from external sources (inelastic electron excitation), we start with this case. For magnetic multipole radiation of order l, the electric field is transverse and has the parity $(-1)^l$. We therefore substitute for $\mathfrak{H}$ in Eq. (2) from one of Maxwell's equations which is valid whether or not there is an external charge and current density,

$$\mathbf{curl}\,\mathfrak{E} = -\partial\mathfrak{H}/\partial ct = (i\omega/c)\mathfrak{H}, \tag{3}$$

to obtain

$$H' = \int [(i/\omega)\mathbf{j}\cdot\mathfrak{E} + (ic/\omega)\mathbf{M}\cdot\mathbf{curl}\,\mathfrak{E}]d\tau$$
$$= (i/\omega)\int (\mathbf{j} + c\,\mathbf{curl}\,\mathbf{M})\cdot\mathfrak{E}d\tau. \tag{4}$$

Now as discussed by Blatt and Weisskopf,[8] any vector function of the polar angles θ, ϕ can be expanded in terms of three kinds of vector spherical harmonics. The first kind, of total angular momentum l, is

$$\mathbf{X}_{lm}(\theta,\phi) = -i[l(l+1)]^{-\frac{1}{2}}(\mathbf{r}\times\nabla)Y_{lm}(\theta,\phi), \tag{5}$$

where Y_{lm} is the ordinary scalar spherical harmonic. $\mathbf{X}_{lm}$ is transverse and has the parity $(-1)^l$; the other two kinds of the same total angular momentum have the parity $(-1)^{l+1}$. The three kinds together, for all l, constitute a complete orthonormal set of vector functions with respect to integration of θ and ϕ over the sphere. We can therefore write for the part of H' that

gives rise to magnetic l-pole transitions:

$$H'_{lm}{}^{(M)}=(i/\omega)\int(\mathbf{j}+c\,\mathbf{curl}\,\mathbf{M})\cdot\mathbf{X}_{lm}{}^*(\theta,\phi)\times\left[\int\mathbf{X}_{lm}(\theta',\phi')\cdot\mathfrak{E}(r,\theta',\phi')d\Omega'\right]d\tau. \quad (6)$$

Summation of Eq. (6) over l and m, with inclusion of the corresponding terms that come from the other two kinds of vector spherical harmonics will, because of the completeness of these functions, yield Eq. (4). These other two classes of terms give rise to electric multipole transitions, and are more conveniently expressed in terms of an integral that involves $\mathbf{X}_{lm}$ and a different combination of nuclear densities (see Sec. III).

We now evaluate the square-bracketed integral in Eq. (6), assuming that the electric field has the form

$$\mathfrak{E}=\mathbf{e}e^{i\mathbf{q}\cdot\mathbf{r}}=\mathbf{e}\sum_l[4\pi(2l+1)]^{\frac{1}{2}}i^l j_l(qr)Y_{l0}(\theta',\phi'). \quad (7)$$

The vector $\hbar\mathbf{q}$ is the photon momentum in the radiation case, or the momentum change of the electron in the electron scattering case; the way in which Eq. (7) is written assumes that the direction of $\mathbf{q}$ is the polar or z axis with respect to which the spherical harmonics are defined. Substitution of Eqs. (5) and (7) into the square-bracketed integral of Eq. (6) yields, after a partial integration,

$$\int\mathbf{X}_{lm}(\theta',\phi')\cdot\mathfrak{E}(r,\theta',\phi')d\Omega'=[\pi(2l+1)]^{\frac{1}{2}}i^l(e_x\pm ie_y)j_l(qr),$$

if $m=\pm1$, and zero otherwise. Equation (6) now becomes, after use of Eq. (5) and integration by parts,

$$H'_{l,\pm1}{}^{(M)}=(i^l/\omega)[\pi(2l+1)/l(l+1)]^{\frac{1}{2}}(e_x\pm ie_y)\times\int j_l(qr)Y_{l,\pm1}{}^*\mathbf{r}\cdot\mathbf{curl}(\mathbf{j}+c\,\mathbf{curl}\,\mathbf{M})d\tau. \quad (8)$$

III. ELECTRIC MULTIPOLE TRANSITIONS

For electric multipole radiation of order l, the magnetic field is transverse and has the parity $(-1)^l$. We therefore substitute for $\mathfrak{E}$ in Eq. (2) from another of Maxwell's equations,

$$\mathbf{curl}\,\mathfrak{H}=\partial\mathfrak{E}/\partial ct+4\pi\mathbf{j}_e/c=(-i\omega/c)\mathfrak{E}+4\pi\mathbf{j}_e/c,$$

to obtain

$$H'=-\int[(c/\omega^2)\,\mathbf{curl}\,\mathbf{j}+\mathbf{M}]\cdot\mathfrak{H}d\tau+(4\pi/\omega^2)\int\mathbf{j}\cdot\mathbf{j}_e d\tau. \quad (9)$$

Here, $\mathbf{j}_e$ is the transition current density that arises from the change in state of the electron; it is zero for a radiative transition and is given by

$$\mathbf{j}_e=ec\mathbf{a}e^{i\mathbf{q}\cdot\mathbf{r}} \quad (10)$$

in the case of electron scattering, where $\mathbf{a}$ is the matrix element of the Dirac $\boldsymbol{\alpha}$ operator between initial and final electron plane wave states of momenta $\mathbf{p}_0$ and $\mathbf{p}$, and $\hbar\mathbf{q}=\mathbf{p}_0-\mathbf{p}$.

The first integral in Eq. (9) can be reduced in precisely the same way as Eq. (4). We put $\mathfrak{H}=\mathbf{h}e^{i\mathbf{q}\cdot\mathbf{r}}$, and obtain for the electric l-pole part of this integral, which is the entire interaction energy for a radiative transition:[12]

$$H''_{l,\pm1}{}^{(E)}=i^{l+1}[\pi(2l+1)/l(l+1)]^{\frac{1}{2}}(h_x\pm ih_y)\times\int j_l(qr)Y_{l,\pm1}{}^*\,\mathbf{r}\cdot\mathbf{curl}[\mathbf{M}+(c/\omega^2)\,\mathbf{curl}\,\mathbf{j}]d\tau \quad (11a)$$

$$=-i^{l+1}[\pi(2l+1)/l(l+1)]^{\frac{1}{2}}(h_x\pm ih_y)\times\left\{\int j_lY_{l,\pm1}{}^*\operatorname{div}(\mathbf{r}\times\mathbf{M})d\tau-(ic/\omega)\int(j_l+rdj_l/dr)Y_{l,\pm1}{}^*\rho d\tau-(cq^2/\omega^2)\int j_lY_{l,\pm1}{}^*(\mathbf{r}\cdot\mathbf{j})d\tau\right\}. \quad (11b)$$

In the latter form, use has been made of the equation of continuity to replace $\operatorname{div}\mathbf{j}$ by $-i\omega\rho$.

In the electron excitation case, we must also decompose the second integral of Eq. (9) into electric multipole contributions. We cannot now use the symmetry of the magnetic field as a guide, and must rely on the symmetry of the nuclear charge-current density instead. Inspection of the first integral of Eq. (9) and the second integral of Eq. (11b) shows that the $\mathbf{X}_{lm}$ part of $\mathbf{curl}\,\mathbf{j}$ and the Y_{lm} part of ρ give rise to electric l-pole transitions. We therefore transform the second integral of Eq. (9) so that only these combinations of nuclear densities appear. The electron current density can be written as the sum of the gradient of a scalar (irrotational part) and the curl of a divergenceless vector (solenoidal part):

$$\mathbf{j}_e=\mathbf{grad}\,\chi+\mathbf{curl}\,\boldsymbol{\psi},\quad \operatorname{div}\boldsymbol{\psi}=0.$$

From this

$$\operatorname{div}\mathbf{j}_e=\nabla^2\chi,\quad \mathbf{curl}\,\mathbf{j}_e=-\nabla^2\boldsymbol{\psi}.$$

With the form (10) for $\mathbf{j}_e$, we find that

$$\chi=-(i/q^2)(\mathbf{q}\cdot\mathbf{j}_e),\quad \boldsymbol{\psi}=(i/q^2)(\mathbf{q}\times\mathbf{j}_e).$$

[12] Some relations given in reference 8 are useful in deriving Eq. (11b) from (11a).

Thus with the help of some partial integrations and the equation of continuity, we obtain

$$\int \mathbf{j}\cdot\mathbf{j}_e d\tau = (\omega/q^2)\int \rho(\mathbf{q}\cdot\mathbf{j}_e)d\tau + (i/q^2)\int \mathbf{curl}\,\mathbf{j}\cdot(\mathbf{q}\times\mathbf{j}_e)d\tau. \quad (12)$$

Equation (12) can now be broken down into multipole parts in analogy with the decomposition of Eq. (4) into terms of the type (6). The result for the contribution of the second integral of (9) to the electric l-pole transition is

$$H'''_{l,m}{}^{(E)} = (4\pi/\omega q^2)\int \rho Y_{lm}{}^*(\theta,\phi) \times\left[\int Y_{lm}(\theta'\phi')(\mathbf{q}\cdot\mathbf{j}_e)d\Omega'\right]d\tau + (4\pi i/\omega^2 q^2)\int \mathbf{curl}\,\mathbf{j}\cdot\mathbf{X}_{lm}{}^*(\theta,\phi) \times\left[\int \mathbf{X}_{lm}(\theta',\phi')(\mathbf{q}\times\mathbf{j}_e)d\Omega'\right]d\tau.$$

Since the z axis is along $\mathbf{q}$, the first square-bracketed integral vanishes unless $m=0$, and the second square-bracketed integral vanishes unless $m=\pm 1$. Use of the form (10) for $\mathbf{j}_e$ leads to the following results:

$$H'''_{l,0}{}^{(E)} = (8\pi i^l eca_z/\omega q)[\pi(2l+1)]^{\frac{1}{2}} \times\int j_l(qr)Y_{l,0}\rho d\tau, \quad (13)$$

$$H'''_{l,\pm1}{}^{(E)} = -(4\pi i^{l+1}ec/\omega^2 q)(a_y \mp ia_x)[\pi(2l+1)]^{\frac{1}{2}} \times\int j_l\mathbf{X}_{l,\pm1}{}^*\cdot(\mathbf{curl}\,\mathbf{j})d\tau. \quad (14)$$

The external fields $\mathfrak{E}$ and $\mathfrak{H}$ may be found from solution of Maxwell's equations with the electron current (10). The results are the Møller fields,[7] which have the forms used above, with

$$\mathbf{e} = 4\pi ie(k\mathbf{a}+\mathbf{q}a_0)/(q^2-k^2),$$

$$e_x \pm ie_y = 4\pi iek(a_x \pm ia_y)/(q^2-k^2),$$

$$\mathbf{h} = 4\pi ie(\mathbf{q}\times\mathbf{a})/(q^2-k^2),$$

$$h_x \pm ih_y = -4\pi ieq(a_y \mp ia_x)/(q^2-k^2),$$

where $k=\omega/c$ and a_0 is the matrix element of the Dirac unit operator between initial and final electron states. It can now be shown without difficulty that Eq. (14) is just equal to $-(q^2-k^2)/q^2$ times the lm part of the **curl j** term in the first integral of Eq. (9). Thus when Eqs. (11) and (14) are added together, the result is to leave the $\mathbf{M}$ integral in (11a) or (11b) unchanged, and multiply the other integrals by a factor k^2/q^2.

We give here the interaction energies for electric and magnetic l-pole transitions in the electron excitation case. Equation (13) has been changed slightly by making use of the Dirac equation to write $\mathbf{q}\cdot\mathbf{a}+(\omega a_0/c)=0$, or $a_z=-(ka_0/q)$, and a substitution has been made for $e_x \pm ie_y$ in Eq. (8).

$$H'_{l,0}{}^{(E)} = -(4\pi i^l ea_0/q^2)[4\pi(2l+1)]^{\frac{1}{2}} \times\int j_l(qr)Y_{l,0}\rho d\tau; \quad (15)$$

$$H'_{l,\pm1}{}^{(E)} = [4\pi i^l eq(a_y \mp ia_x)/(q^2-k^2)] \times[\pi(2l+1)/l(l+1)]^{\frac{1}{2}}\int j_l(qr)Y_{l,\pm1}{}^* \,\mathbf{r}\cdot\mathbf{curl}\,[\mathbf{M}+(cq^2)^{-1}\,\mathbf{curl}\,\mathbf{j}]d\tau; \quad (16)$$

$$H'_{l,\pm1}{}^{(M)} = [4\pi i^{l+1}e(a_x \pm ia_y)/c(q^2-k^2)] \times[\pi(2l+1)/l(l+1)]^{\frac{1}{2}}\int j_l(qr)Y_{l,\pm1}{}^* \,\mathbf{r}\cdot\mathbf{curl}\,[\mathbf{j}+c\,\mathbf{curl}\,\mathbf{M}]d\tau. \quad (17)$$

Equations (16) and (17) can of course be further transformed in analogy with the change from Eq. (11a) to (11b).

IV. ORDERS OF MAGNITUDE

We now estimate the relative orders of magnitude of the various terms in the above three equations. For this purpose, we note that a_0, a_x and a_y are of order unity and that $k \ll q$ for situations of current interest, and assume that all integrals over nuclear densities can be represented by a common form factor F. Then Eq. (15) is of order eFQ/q^2, where Q is the total charge involved in the transition.

The first (magnetization) term in Eq. (16) involves a derivative operator which, by partial integration, is seen to introduce a factor of order q. The vector $\mathbf{r}$ introduces a factor R, which is an average radial distance from the center of the nucleus for the main contributions to the integral; for large q, R is expected to be somewhat larger than $1/q$ and somewhat smaller than the nuclear radius. Then the order of magnitude of this term is $(eF/q)(qR)(Q'\hbar/Mc)$, where $Q'\hbar/Mc$ is the total magnetization involved in the transition, and M is the nucleonic mass. In the second (current) term of Eq. (16), the two derivative operators introduce a factor q^2, and there is again a factor R. Thus its order of magnitude is $(eF/q)(R/c)Q''v$, where $Q''v$ is the total current involved in the transition, and v is the nuclear convection velocity associated with motion of the charge Q''.

In similar fashion, we find that the current and magnetization terms in Eq. (17) have the orders of magnitude $(eF/cq^2)qRQ''v$ and $(eF/cq^2)(cq^2R)(Q'\hbar/Mc)$, respectively.

If now we call the order of magnitude of Eq. (15) unity, then the relative orders of magnitude of the magnetization and current terms in Eq. (16) are, respectively,

$$(qR)(q\hbar/Mc)(Q'/Q), \quad (qR)(v/c)(Q''/Q),$$

and the orders of magnitude of the current and magnetization terms in Eq. (17) are, respectively,

$$(qR)(v/c)(Q''/Q), \quad (qR)(q\hbar/Mc)(Q'/Q).$$

For large-angle scattering of electrons with about 200-Mev energy, $q\hbar/Mc \sim \frac{1}{3}$ and $qR \sim 3$. For single-particle excitation of a nucleus, Q, Q', and Q'' are all expected to be of order unity, and $v/c \sim \frac{1}{5}$. For a collective mode of excitation, Q' should still be of order unity, Q and Q'' should have the same order of magnitude and be much larger than unity, and v/c should be much less than $\frac{1}{5}$.

We conclude, therefore, that for single-particle excitation in this energy range, all terms of Eqs. (15), (16), and (17) can be of the same order of magnitude. For excitation of collective modes, on the other hand, it is likely that Eq. (15) is not only the leading term for the electric multipole transitions, but is significantly larger than either of the magnetic multipole terms, which are comparable with each other. Further, if we assume that the current density associated with a collective mode is irrotational,[13] then $\mathbf{curl}\,\mathbf{j}=0$ and the current terms in Eqs. (16) and (17) are zero.

It may be noted at this point that summation of Eq. (15) over l yields, with the help of Eq. (7),

$$\sum_l H'_{l,0}{}^{(E)} = -(4\pi e a_0/q^2)\int e^{i\mathbf{q}\cdot\mathbf{r}}\rho d\tau. \tag{18}$$

This is just what would have been obtained if only the first factor in Eq. (1) had been retained, and the static, nonretarded, Coulomb interaction had been used in calculating φ. Thus the present paper provides a justification for the use of this term by itself, at least so far as excitation of collective nuclear oscillations is concerned.[14]

V. DIFFERENTIAL SCATTERING CROSS SECTION

Each of the interaction energies (15) through (18) is of the form AV, where A is one of the quantities a_0, $a_x \pm ia_y$, $a_y \mp ia_x$, which involves only the electron spin functions, and V involves only integrals over nuclear quantities. In order to obtain a differential scattering cross section, it is necessary to sum $|AV|^2$ over final electron spin states, average over initial spin states, multiply by $2\pi/\hbar$ times the energy density of final electron states, and divide by the incident electron flux.

In the extreme relativistic region, the spin sum operation on $|A|^2$ yields $\cos^2\frac{1}{2}\theta$ when $A=a_0$, where θ is the angle between $\mathbf{p}_0$ and $\mathbf{p}$. When $A=a_x \pm ia_y$ or $A=a_y \mp ia_x$, the spin sum leads to

$$[(p_0+p)^2(1-\cos\theta)-p_0p\sin^2\theta]/\hbar^2q^2$$
$$=[1+\sin^2(\tfrac{1}{2}\theta)]\left[1+\frac{\hbar^2k^2}{2p_0p[1+\sin^2(\frac{1}{2}\theta)]}\right]\times\left[1+\frac{\hbar^2k^2}{4p_0p\sin^2(\frac{1}{2}\theta)}\right]^{-1};$$

except for very small scattering angles, this is very nearly equal to $[1+\sin^2(\frac{1}{2}\theta)]$ when the nuclear excitation is moderate. The last group of multiplicative factors involving the incident flux and the density of final states is, in the extreme relativistic region, $(p/2\pi\hbar^2c)^2$. Thus the differential scattering cross section per unit solid angle is equal to $[p\cos(\frac{1}{2}\theta)/2\pi\hbar^2c]^2|V|^2$ when either (15) or (18) is used for V, and is approximately equal to $(p/2\pi\hbar^2c)^2[1+\sin^2(\frac{1}{2}\theta)]|V|^2$ when either (16) or (17) is used for V.

VI. COMPARISON WITH RADIATIVE TRANSITIONS

It may be desirable in some cases to relate the inelastic electron scattering cross section to the radiation probability for the same nuclear transition. We therefore examine the expressions for the electric and magnetic l-pole moments, which are respectively[8]

$$\int r^l Y_{lm}{}^*\rho d\tau - ik(l+1)^{-1}\int r^l Y_{lm}{}^*\,\mathrm{div}(\mathbf{r}\times\mathbf{M})d\tau, \tag{19}$$

$$-[c(l+1)]^{-1}\int r^l Y_{lm}{}^*\,\mathrm{div}(\mathbf{r}\times\mathbf{j})d\tau - \int r^l Y_{lm}{}^*\,\mathrm{div}\mathbf{M}d\tau. \tag{20}$$

Estimates like those of Sec. IV show that the relative orders of magnltude of the two terms of (19) and the two terms of (20) are

$$1, \quad (k\hbar/Mc)(Q'/Q);$$
$$(v/c)(Q''/Q), \quad (\hbar/McR)(Q'/Q).$$

Here, R is the nuclear radius. Since $k\hbar/Mc \ll 1$, the first term of Eq. (19) dominates the electric l-pole moment, whether the nucleus undergoes a single-particle or collective transition. Likewise, the first term of Eq. (20) is the larger in the single-particle case, while the two terms are more nearly comparable for a collective transition.

It follows that there is a close relation between the leading terms for electric multipole transitions in the radiative case [first term of Eq. (19)] and in the electron excitation case [Eq. (15)]. The same quantity ρ appears in both terms, and indeed the former is simply

[13] A. Bohr, *Rotational States of Atomic Nuclei* (Ejnar Munksgaards Forlag, Copenhagen, 1954), Appendix.

[14] Equation (18) has been used in much of the earlier work on inelastic electron scattering (see reference 10).

the first term in the expansion of the latter in powers of q. A similar relation obtains between the first (current) terms of Eqs. (17) and (20) in the magnetic multipole case, since $\mathbf{r}\cdot\mathbf{curl}\,\mathbf{j}=-\mathrm{div}(\mathbf{r}\times\mathbf{j})$. However, the second (magnetization) terms are much less closely related to each other, and this term is more important in the electron excitation than in the radiative case. Thus it is much easier to connect the probabilities for the two types of transitions in the electric than in the magnetic multipole case; an assumption must of course be made concerning the radial dependence of ρ (see for example Sec. VIII).

VII. ELECTRIC MONOPOLE TRANSITIONS

The electric monopole interaction energy $H'_{0,0}{}^{(E)}$ is responsible for the elastic scattering from a static spherically symmetric charge density, which is the expectation value for the nuclear ground state. In this case, either Eq. (15) or Eq. (18) can be used.

The transition between nuclear states with total angular momentum $I=0$ and the same parity (which is even in all known cases), is of considerable interest. It is essential to realize here that the orthogonality of the initial and final nuclear states makes $\int\rho d\tau$ vanish. Thus for small q, the apparently leading terms in Eqs. (15) and (18) are actually zero. As a reminder that this occurs, one should replace the spherical Bessel function in Eq. (15) by $j_0(qr)-1$, and the exponential in Eq. (18) by $\exp(i\mathbf{q}\cdot\mathbf{r})-1$. Thus for small q, the electric monopole and electric quadrupole transitions have the same q dependence.

The difference in behavior between the elastic and inelastic monopole transitions corresponds to the fact that a static spherically symmetric charge density has a field that extends to large distances, whereas a radially oscillating charge density has no time-dependent field external to itself. Therefore interaction with the spherically symmetric part of the electron potential can occur only through its variation over the nucleus, and the leading term, which is independent of r, results in no interaction.

VIII. COLLECTIVE ELECTRIC QUADRUPOLE TRANSITIONS

There is a close relation between the contribution of the static nuclear electric quadrupole moment to the elastic scattering[15] and the contribution of the transition quadrupole moment to the inelastic scattering. This is because the collective model relates both the static and the transition moments to an intrinsic quadrupole moment.[13,16] The relation is worth exploring, since with imperfect energy resolution the two effects may be experimentally indistinguishable.

In both cases we use Eq. (15), and neglect the difference between the two values of q for the same incident electron energy and angle of scattering. We consider first the case of a uniformly charged nucleus of radius R, for which the quadrupole parts of the static and transition charge densities both have the approximate form $\delta(r-R)$.[15,16] Then the integral in Eq. (15) in the static case is equal to that in the transition case for all values of q, except for a multiplicative constant. For small q, the integral is proportional to the static quadrupole moment Q in the elastic case, and to the transition quadrupole moment in the inelastic case, both of which can be expressed in terms of the intrinsic quadrupole moment Q_0. The two integrals can then be expressed in terms of Q_0 for small q, and hence also for all q.

It seems plausible to assume that the situation is similar when the nucleus is not a uniformly charged sphere;[3] this situation is now under study. We assume that the static and transition quadrupole charge densities have the same radial dependence, so that the elastic and inelastic form factors are constant multiples of each other for all q. They can then be related to Q_0 for small q, and hence found for all q if some radial dependence is assumed.

From Eq. (15) and Sec. V, the differential cross section for elastic scattering is given by [compare with Eqs. (2) and (14) of reference 15]

$$\sigma_e(\theta)=[e^2\cos^2(\tfrac{1}{2}\theta)/4E^2\sin^4(\tfrac{1}{2}\theta)]|F_e|^2,$$

$$|F_e|^2=(2I+1)^{-1}\sum_{m_i}\sum_{m_f}\left|(4\pi)^{\frac{1}{2}}\int[j_0(qr)Y_{00}-5^{\frac{1}{2}}j_2(qr)Y_{20}]\rho_e(m_i,m_f)d\tau\right|^2, \tag{21}$$

where $\rho_e(m_i,m_f)$ is the matrix element of the static charge density between initial and final magnetic substates of the nuclear ground state of total angular momentum I.[17] We choose the axis with respect to which the magnetic substates are defined as $\mathbf{q}$, which is the axis of Y_{l0}, in which case the integrals fail to vanish only if $m_f=m_i$. We call this common value m, and note that the Y_{00} part of the integral is independent of m, while the Y_{20} part of the integral depends on m through the factor[18]

$$f_m=[3m^2-I(I+1)]/I(2I-1)$$

[15] L. I. Schiff, Phys. Rev. **92**, 988 (1953).

[16] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, 16 (1953).

[17] Alternatively, $\rho_e(\mathbf{r})$ may be regarded as an operator (see reference 5), and the matrix element taken after the integration is performed over $\mathbf{r}$; all results are the same.

[18] Reference 8, p. 28. The numerator of f_m is the quantum-mechanical transcription of $P_2(\cos\theta)$, and the denominator merely normalizes it so that $f_I=1$. Thus the fact that f_m is indeterminate when $I=0$ or $\frac{1}{2}$ is not significant; in these cases f_m should be regarded as zero since the numerator vanishes. Equation (24) is consistent with this interpretation.

in the following way:

$$|F_e|^2=[4\pi/(2I+1)]\sum_m\left\{\left|\int j_0Y_{00}\rho_e(I,I)d\tau\right|^2 - (20)^{\frac{1}{2}}f_m Re\int j_0Y_{00}\rho_e(I,I)d\tau\overline{\int j_2Y_{20}\rho_e(I,I)d\tau} +5f_m{}^2\left|\int j_2Y_{20}\rho_e(I,I)d\tau\right|^2\right\}.$$

Now $\sum_m f_m=0$, so that the interference term vanishes. Also,

$$\sum_m f_m{}^2=(2I+1)/5A_I{}^2 = (I+1)(2I+1)(2I+3)/5I(2I-1),$$

where A_I is the quantity defined in reference 15. We thus obtain

$$|F_e|^2=4\pi\left\{\left|\int j_0Y_{00}\rho_e(I,I)d\tau\right|^2 +[(I+1)(2I+3)/I(2I-1)] \times\left|\int j_2Y_{20}\rho_e(I,I)d\tau\right|^2\right\}. \quad (22)$$

The first term of Eq. (22) gives the scattering from the spherically symmetric part of the nuclear charge density, and the second term we call the quadrupole part of the elastic scattering.

In order to relate the second term of Eq. (22) to the intrinsic quadrupole moment Q_0, we note that for small q,

$$\int j_2Y_{20}\rho_e(I,I)d\tau\rightarrow(q^2/15) \times\int r^2Y_{20}\rho_e(I,I)d\tau=(q^2/15)(5/16\pi)^{\frac{1}{2}}eQ,$$

where the observed quadrupole moment Q is given in terms of Q_0 by Eq. (V.6) of reference 16:

$$Q=[I(2I-1)/(I+1)(2I+3)]Q_0.$$

Q_0 is calculated in the same way as Q, except that the axis of Y_{20} is the nuclear symmetry axis rather than **I**. Thus if we fix the magnitude of ρ_e by the equation

$$eQ_0=(16\pi/5)^{\frac{1}{2}}\int r^2Y_{20}\rho_e d\tau, \quad (23)$$

and infer its shape in some other way, then the quadrupole part of $|F_e|^2$ is

$$|F_{eq}|^2=4\pi[I(2I-1)/(I+1)(2I+3)] \times\left|\int j_2(qr)Y_{20}\rho_e d\tau\right|^2. \quad (24)$$

It must be remembered that in Eqs. (23) and (24), the axis of Y_{20} is the symmetry axis of the nuclear charge distribution ρ_e.[19] Note that there is no elastic quadrupole scattering if $I=0$ or $\frac{1}{2}$, just as there is no observed quadrupole moment; the charge distribution as viewed by the incident electron is spherically symmetric in this case, even though Q_0 may not be zero.

The differential cross section for inelastic scattering is given by the first of Eqs. (21), with F_e replaced by the quantity F_i:

$$|F_i|^2=(2I_i+1)^{-1}\sum_{m_i}\sum_{m_f}|F_i(m_i,m_f,0)|^2, \quad (25)$$

where

$$F_i(m_i,m_f,m)=-(20\pi)^{\frac{1}{2}}\int j_2(qr)Y_{2m}\rho_i(m_i,m_f)d\tau,$$

and $\rho_i(m_i,m_f)$ is the transition charge density between the initial state I_i, m_i and the final state I_f, m_f. Since the summations in Eq. (25) are carried over all m_i and m_f, the result is independent of the axis chosen for Y_{20}, and hence has the same value if $F_i(m_i,m_f,0)$ is replaced by $F_i(m_i,m_f,m)$ where m is not necessarily equal to zero. Then

$$|F_i|^2=[5(2I_i+1)]^{-1}\sum_m\sum_{m_i}\sum_{m_f}|F_i(m_i,m_f,m)|^2 =(1/5)\sum_m\sum_{m_f}|F_i(m_i,m_f,m)|^2;$$

the second equality holds because the summations over all m and m_f make the result independent of the value of m_i.

For small q,

$$F_i(m_i,m_f,m)\rightarrow-(20\pi)^{\frac{1}{2}}(q^2/15)\int r^2Y_{2m}\rho_i(m_i,m_f)d\tau.$$

The reduced radiative transition probability is defined in Eq. (VII.2) of reference 16 as

$$B(E2)=\sum_m\sum_{m_f}\left|\int r^2Y_{2m}\rho_i(m_i,m_f)d\tau\right|^2,$$

so that for small q,

$$|F_i|^2\rightarrow4\pi(q^2/15)^2B(E2).$$

There are in general two possible quadrupole transitions from the ground state I: to the first excited state $I+1$, and to the second excited state $I+2$.[20] From Eqs. (33) and (36) of reference 13, the reduced transition proba-

[19] The averaging over nuclear orientations was not performed correctly in reference 15; the charge density defined by Eq. (17) of that reference is the same as the quantity ρ_e of the present paper except for a normalization factor Ze, but in Eqs. (20) and (21), ρ_2 should be replaced by $A_I\rho_2$. This does not affect the estimate of the quadrupole scattering below Eq. (22), since only the ratio of (21) to (20) enters there.

[20] If the ground state I is zero, the first excited state has $I=2$, and if the ground state I is $\frac{1}{2}$, the results below may be altered (see references 13 and 16).

bility in the first case is

$$B_1(E2)=(15/16\pi)(eQ_0)^2[I/(I+1)(I+2)],$$

and from Eqs. (34) and (36) of reference 13, this quantity in the second case is

$$B_2(E2)=(15/16\pi)(eQ_0)^2[2/(I+2)(2I+3)].$$

We thus find for the transition $I\rightarrow I+1$ and small q

$$|F_{i1}|^2\rightarrow(5/4)(eQ_0)^2(q^2/15)^2[3I/(I+1)(I+2)], \quad (26)$$

and for the transition $I\rightarrow I+2$

$$|F_{i2}|^2\rightarrow(5/4)(eQ_0)^2(q^2/15)^2[6/(I+2)(2I+3)]. \quad (27)$$

We now assume that the extrapolation from small to large q can be made as in Eq. (24). We then find that the inelastic scattering is also described by Eq. (24), except that the square bracket there must be replaced by the square bracket in Eq. (26) or (27). Some numerical values are given in Table I. It must be remembered that these results may not apply when the ground state I is equal to $\frac{1}{2}$, and that Eq. (27) represents the transition to the first (not the second) excited state when $I=0$.[20]

It is interesting to note that there is a kind of sum rule for the three kinds of quadrupole scattering, which are the only possible kinds

$$[I(2I-1)/(I+1)(2I+3)]+[3I/(I+1)(I+2)]+[6/(I+2)(2I+3)]=1.$$

TABLE I. Values of the square bracket factors in Eqs. (24), (26), and (27), for a few values of I.

I	Elastic $I\rightarrow I$, Eq. (24)	Inelastic $I\rightarrow I+1$, Eq. (26)	Inelastic $I\rightarrow I+2$, Eq. (27)
0	0	0	1
3/2	1/5	18/35	2/7
5/2	5/14	10/21	1/6
7/2	7/15	14/33	6/55

IX. CONCLUSIONS

Excitation of nuclear multipole transitions by inelastic electron scattering occurs along with the elastic scattering of high-energy electrons. If sufficiently accurate measurements of the distribution of the scattered electrons in energy and angle can be made, the elastically scattered electrons give information concerning the static nuclear charge distribution, and the inelastically scattered electrons give information concerning both the strength and shape of the transition charge density. The static quadrupole scattering must be allowed for when interpreting the angle distribution of the elastically scattered electrons in terms of a radial distribution of nuclear charge. If the energy resolution is not good enough to separate the inelastic from the elastic scattering, then it must also be allowed for before the elastic scattering is analyzed.

It is important to note that absolute measurements of the inelastic scattering are not necessary in order to determine the strength of the transition charge density. Analysis of the relative elastic scattering, even if no absolute cross sections are available, yields a shape for the static charge density. The magnitude of the nuclear charge is of course known, so that an absolute elastic cross section can be computed. Then comparison of the relative magnitudes of inelastic and elastic scattering at each angle gives absolute values for the former, from which the strength as well as the shape of the transition charge density can be determined.

The order of magnitude estimates given in Sec. IV are of course not completely reliable, and simply indicate that Eq. (15) or Eq. (18) is likely to be the dominant term, especially for collective transitions, which appear to be much the strongest in any event. For single-particle transitions, where detailed nuclear wave functions are more likely to be available, it would be worth while to make more careful estimates of the relative importance of Eqs. (16) and (17).