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Mach's Principle and Unified Field Theory

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An argument based on the impossibility of satisfying Mach's principle within general relativity is given which leads to the idea of unified field theory. A possible alternative is discussed.

IT is interesting to discuss whether there is an argument, based on classical physics, which would suggest that a unified field theory of some sort is desirable. By unified field theory is meant a theory in which "matter" or the sources of the space-time field are absorbed into the field itself, so that one has a set of homogeneous partial differential equations. I have tried to examine this question from the point of view of Mach's principle, which can be stated: The local inertial frame (the coordinate system in which Newton's laws are obeyed) is determined by the matter in the universe. Mach's principle is a statement concerning the relation between space-time and matter which asserts the subordination of space-time to matter. It is important to determine the status of Mach's principle, because if Mach's principle is either contained in general relativity, or can be satisfied in it; there is no classical argument for the radical approach of unified field theory which usually asserts the identity of space-time and matter. In other words, if Mach's principle holds, we can retain the apparently observed distinction between space-time and matter and at the same time eliminate the absolutes of Newtonian physics. However, if it is impossible to conform to the principle in general relativity, then we can no longer preserve the separation between matter and space-time once the field-theoretic approach of general relativity has been adopted. If matter cannot be consistently regarded as the source of the field, we must treat it as part of the field. In this case, once one has accepted general relativity, one is impelled to go all the way to unified field theory.

It seems apparent that general relativity does not contain Mach's principle in the sense that solutions of the field equations can be found which are inconsistent with the principle. One example of this is the Schwarzschild solution for a centrally symmetric field. In this, space is flat—Minkowski space—at large distances from the gravitating particle, and therefore is a space of definite properties uninfluenced by matter. Also, Taub has found solutions of the field equations which do not represent flat space and yet contain no matter.[1]

The question remains: can one find solutions of the field equations of general relativity which conform to Mach's principle? Another form of this question is: Given arbitrary bodies in motion, can one calculate what will be the local inertial frame and so verify the theory? (The calculation of the local inertial frame can be tested by experiment, for in it the plane of vibration of the Foucault pendulum will be stationary.) That this question should be answered affirmatively is suggested by two results: (1) The calculations of Thirring[2] show that a particle which is placed at the center of a massive, rotating, spherical shell (rotating with respect to the distant matter in the universe) should experience Coriolis and centrifugal forces caused by the rotating shell. (2) Similarly, one can show[3] that the inertia of a test particle is increased by the presence of massive bodies, and that a test body will experience a parallel acceleration when neighboring masses are accelerated.

There is, however, a basic difficulty. The calculations mentioned above are carried out for a situation in which the properties of space-time are largely known in advance, and all we have to do is to determine the effect of a small perturbation. Analysis of these calculations reveals that it is impossible to generalize them so that we can determine the inertial properties of a particle by considering the effect of all other bodies, unless some fundamental assumption is made in advance (which is not contained in general relativity) about the properties of space-time.

The first problem is the question of boundary conditions. The field equations of general relativity are a set of partial differential equations, and in order to specify a solution[4] it is necessary to supply boundary conditions. These conditions are not contained in the theory; they are added from outside. Even the condition of closure of the universe, which is essential to the cosmological satisfaction of Mach's principle, has to be imposed on the theory.[5]

The second and more serious problem is that it is difficult to understand the motion of matter without prior knowledge of the space-time field to be calculated. In order to know the motion of matter, it is not sufficient to specify the coordinates for the particles considered; it is necessary to set up the source function for the gravitational field; the stress energy tensor T_{ik} in the

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[1] A. H. Taub, Ann. Math. **53**, 472 (1951).
[2] H. Thirring, Physik. Z. **19**, 33 (1918); **22**, 29 (1921).
[3] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1953), fourth edition, p. 100.
[4] This can, of course, only be done up to a coordinate transformation.
[5] One may consider adopting a system consisting of differential equations plus boundary conditions. This requires the explicit introduction of an extra hypothesis about space-time.

equation

$$R_{ik}-\tfrac{1}{2}g_{ik}R=kT_{ik}. \quad (1)$$

In order to determine the stress-energy tensor it is necessary to know the laws governing the behavior of matter. Because of the existence of nongravitational forces, these have to be added to general relativity from outside. When these laws are written in a covariant form, they will contain the components of the metric tensor or related quantities. Consequently we will not be able to understand matter apart from space-time. This is in accord with the previously mentioned result that the inertia of a body is increased by the presence of other bodies. For example, consider the case of free particles: The contravariant stress-energy tensor can be written (if pressure is neglected):

$$T^{ik}=\rho\frac{dx^i}{ds}\frac{dx^k}{ds}. \quad (2)$$

Not only do the velocities dx^i/ds depend on the field, but we must in the general case supply a rule by which the particle density ρ may be calculated, and include in this the effect of the gravitational field on the masses of the particles.

The example of the cosmological solutions of general relativity will serve to illustrare some of these problems. In order to begin, it is necessary to impose the cosmological principle or its equivalent: the universe appears homogeneous and isotropic to any (fundamental) observer. This is an extremely powerful assumption as Milne has demonstrated, and is consistent with general relativity but certainly not required by it. From the cosmological principle it follows that the metric of space-time can be put in the form

$$ds^2=dt^2-\frac{R^2(t)}{1+\frac{1}{4}kr^2}[(dx^1)^2+(dx^2)^2+(dx^3)^2]. \quad (3)$$

The equations of general relativity give an equation for $R(t)$ which can be integrated, but it remains necessary to specify the constant k, the curvature of space, as positive, negative, or zero. The case of positive curvature conforms to Mach's principle.

It is apparent that the theory of general relativity is incomplete: it is necessary to supply additional assumptions about the relation between matter and space-time which are not contained in the theory. There seem to be two conceivable solutions to this problem which preserve the field-theoretic approach:

(1) We can abandon Mach's principle and with it attempts to subordinate space-time to matter. Instead, we look for general laws of the space-time field and try to derive from these laws the behavior of matter, which is to be constructed from the space-time field. This is the approach of unified field theory. It is supported by the beautiful result of general relativity, that if we regard gravitating particles as singularities in the field, then the equations of motion of these particles can be obtained from the field equations.

(2) The second approach, suggested by Professor J. A. Wheeler, is less radical. If we knew additional relations between matter and space-time, we could supplement general relativity and hope to obtain a self-consistent scheme. There is at least one situation in which this can be done: the case in which matter consists of an electromagnetic field. The energy tensor is

$$T_{ik}=\frac{1}{4\pi}(F_{i\alpha}F^{\alpha}{}_{k}-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{ik}). \quad (4)$$

The laws governing the electromagnetic field are just Maxwell's equations

$$\frac{\partial F_{ik}}{\partial x^l}+\frac{\partial F_{kl}}{\partial x^i}+\frac{\partial F_{li}}{\partial x^k}=0 \quad (5)$$

and

$$\frac{\partial}{\partial x^\alpha}(\sqrt{(-g)}\,F^{i\alpha})=0. \quad (6)$$

We can hope to be able to construct such a system in any case where the laws of matter are sufficiently well known. Such a theory possesses at least one of the characteristics of a unified field theory; for it is no longer possible to distinguish source from field; matter and space-time enter on an equal basis.[6] Consequently, Mach's principle is not observed. The difficulty with boundary conditions is not surmounted in this kind of a theory. The boundary conditions have to be added explicitly as an extra hypotheses. This is legitimate since Mach's principle has been abandoned. (The situation as regards boundary conditions in unified field theories is still unclear, because nothing is known about allowable solutions.) It may be that a more conservative theory of this sort is preferable at the present stage of science because one is more certain of the component parts.

The conclusions of this argument are the following: It is impossible to satisfy Mach's principle within general relativity because matter cannot be understood apart from knowledge of space-time. If the approach of field theory is accepted, it is necessary to construct a theory in which matter and space-time enter as equals. There are at least two possibilities of doing this.

I wish to thank Professor J. A. Wheeler and Professor A. Einstein for valuable discussions. This note was stimulated by the conflict of their views. I also wish to thank Mr. J. R. H. Dempster for his criticisms and suggestions which have contributed greatly to the clarification of my ideas.

[6] J. A. Wheeler, Phys. Rev. **94**, 773 (1954). The system of Maxwell's equations and the field equations of general relativity is being integrated by Professor Wheeler for a special case.

APPENDIX

Professor Einstein has proposed unified field theory as an alternative to quantum theory because he believes that quantum theory, based essentially on probabilities, cannot give a complete description of nature. Many physicists who are skeptical of the ability of unified field theory to yield all the verified results of quantum theory, do not share this attitude. It is too early to pass judgment on this attempt. However, it may be observed that his theory will either be able to handle quantum phenomena or it will fail completely. I refer in particular to the existence of a sharp value for the elementary electric charge. It has been shown[7] that if solutions of Einstein's unified field theory are admitted which depend continuously on a parameter (i.e., a spread in value for the electronic charge), then the Coulomb law can never be satisfied. It may be concluded from this and other work that the theory is either extremely powerful or useless.

[7] C. P. Johnson, Phys. Rev. **89**, 320 (1953). See also A. Einstein, Phys. Rev. **89**, 321 (1953).

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Direct Proof of the Covariance of Gupta's Indefinite Metric in Quantum Electrodynamics

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Gupta's method of the indefinite metric is at this time for many purposes the most satisfactory method of formulating the principles of quantum electrodynamics. Gupta's indefinite metric, however, depends on the number of scalar photons. This number is no invariant. Yet, the covariance of Gupta's indefinite metric has been proved before. This seems at first surprising. In the present paper we show why the lack of invariance of the number of scalar photons does not matter and how a certain covariance of the occupation numbers together with "repolarization operators" insures the covariance of Gupta's method. In particular, if the norms of the eigenfunctions of occupation numbers are chosen in accordance with Gupta's prescription in one Lorentz frame, they are automatically in accordance with this prescription in a different Lorentz frame.

1. INTRODUCTION

At the time this is written, Gupta's form of quantum electrodynamics is the most satisfactory formulation of quantum electrodynamics for most field-theoretical considerations. The advantages of Gupta's treatment of the problem of longitudinal and "scalar" (or "time-like") photons[1–4] are the following: (1) It avoids state vectors which cannot be normalized.[5] (2) It does not give the photon a small mass.[6] (3) It does not introduce more "redundant" field variables than the longitudinal and "scalar" potentials and their four-dimensional divergence.[7] (4) By its manifest covariance it enables us to perform renormalizations unambiguously in a covariant way.[8] Gupta's theory is at present the only theory combining these four advantages.

It is true that the problem of longitudinal and scalar photons can also be solved by avoiding to introduce it in the first place. For a Maxwell field interacting with a finite number of Dirac particles this was first shown by Pauli.[9] Later, this "gauge-independent" method was further developed and was adapted to positon theory by Belinfante.[10–11] While the covariance of this method can be proved,[9–10] the complicated form of the Lorentz transformation of the field components in this theory has thus far largely obstructed its applicability. In particular, nobody as yet seems to have succeeded in developing a clearly formulated renormalization technique using exclusively gauge-independent methods. True enough, French and Weisskopf based their calculation of the Lamb shift on this type of description of nature.[12] However, a look at their Eq. (13) and at the symmetry in their subsequent treatment of the k occurring in the various terms of this equation shows that they brought in through the back door the Coulomb-interaction-by-way-of-longitudinal-and-scalar-photons which they had thrown out through the front door. Also, their argument for explaining why their treatment is covariant had to be much more complicated and less

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[3] S. N. Gupta, Proc. Phys. Soc. (London) **A66**, 129 (1953).
[4] S. N. Gupta (to be published).
[5] F. J. Belinfante, Phys. Rev. **76**, 226 (1949).
[6] F. J. Belinfante, Progr. Theoret. Phys. (Japan) 4, 165 (1949).
[7] J. C. Valatin, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 13 (1951).
[8] S. N. Gupta, Proc. Phys. Soc. (London) **A64**, 426 (1951). Also reference 3 and references given there.
[9] W. Pauli, in H. Geiger and H. Scheel, *Handbuch der Physik* (Springer, Berlin, 1933), second edition, Vol. 24, Part I, Chap. 2, Sec. B8, p. 269. (Reprinted by Edwards Brothers, Ann Arbor.)
[10] F. J. Belinfante and J. S. Lomont, Phys. Rev. **84**, 541 (1951).
[11] F. J. Belinfante, Phys. Rev. **84**, 644 (1951).
[12] J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949).