APPENDIX

Professor Einstein has proposed unified field theory as an alternative to quantum theory because he believes that quantum theory, based essentially on probabilities, cannot give a complete description of nature. Many physicists who are skeptical of the ability of unified field theory to yield all the verified results of quantum theory, do not share this attitude. It is too early to pass judgment on this attempt. However, it may be observed that his theory will either be able to handle quantum phenomena or it will fail completely. I refer in particular to the existence of a sharp value for the elementary electric charge. It has been shown[7] that if solutions of Einstein's unified field theory are admitted which depend continuously on a parameter (i.e., a spread in value for the electronic charge), then the Coulomb law can never be satisfied. It may be concluded from this and other work that the theory is either extremely powerful or useless.

[7] C. P. Johnson, Phys. Rev. **89**, 320 (1953). See also A. Einstein, Phys. Rev. **89**, 321 (1953).

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Direct Proof of the Covariance of Gupta's Indefinite Metric in Quantum Electrodynamics

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Gupta's method of the indefinite metric is at this time for many purposes the most satisfactory method of formulating the principles of quantum electrodynamics. Gupta's indefinite metric, however, depends on the number of scalar photons. This number is no invariant. Yet, the covariance of Gupta's indefinite metric has been proved before. This seems at first surprising. In the present paper we show why the lack of invariance of the number of scalar photons does not matter and how a certain covariance of the occupation numbers together with "repolarization operators" insures the covariance of Gupta's method. In particular, if the norms of the eigenfunctions of occupation numbers are chosen in accordance with Gupta's prescription in one Lorentz frame, they are automatically in accordance with this prescription in a different Lorentz frame.

1. INTRODUCTION

At the time this is written, Gupta's form of quantum electrodynamics is the most satisfactory formulation of quantum electrodynamics for most field-theoretical considerations. The advantages of Gupta's treatment of the problem of longitudinal and "scalar" (or "time-like") photons[1–4] are the following: (1) It avoids state vectors which cannot be normalized.[5] (2) It does not give the photon a small mass.[6] (3) It does not introduce more "redundant" field variables than the longitudinal and "scalar" potentials and their four-dimensional divergence.[7] (4) By its manifest covariance it enables us to perform renormalizations unambiguously in a covariant way.[8] Gupta's theory is at present the only theory combining these four advantages.

It is true that the problem of longitudinal and scalar photons can also be solved by avoiding to introduce it in the first place. For a Maxwell field interacting with a finite number of Dirac particles this was first shown by Pauli.[9] Later, this "gauge-independent" method was further developed and was adapted to positon theory by Belinfante.[10–11] While the covariance of this method can be proved,[9–10] the complicated form of the Lorentz transformation of the field components in this theory has thus far largely obstructed its applicability. In particular, nobody as yet seems to have succeeded in developing a clearly formulated renormalization technique using exclusively gauge-independent methods. True enough, French and Weisskopf based their calculation of the Lamb shift on this type of description of nature.[12] However, a look at their Eq. (13) and at the symmetry in their subsequent treatment of the k occurring in the various terms of this equation shows that they brought in through the back door the Coulomb-interaction-by-way-of-longitudinal-and-scalar-photons which they had thrown out through the front door. Also, their argument for explaining why their treatment is covariant had to be much more complicated and less

[1] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).
[2] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).
[3] S. N. Gupta, Proc. Phys. Soc. (London) **A66**, 129 (1953).
[4] S. N. Gupta (to be published).
[5] F. J. Belinfante, Phys. Rev. **76**, 226 (1949).
[6] F. J. Belinfante, Progr. Theoret. Phys. (Japan) **4**, 165 (1949).
[7] J. C. Valatin, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 13 (1951).
[8] S. N. Gupta, Proc. Phys. Soc. (London) **A64**, 426 (1951). Also reference 3 and references given there.
[9] W. Pauli, in H. Geiger and H. Scheel, *Handbuch der Physik* (Springer, Berlin, 1933), second edition, Vol. 24, Part I, Chap. 2, Sec. B8, p. 269. (Reprinted by Edwards Brothers, Ann Arbor.)
[10] F. J. Belinfante and J. S. Lomont, Phys. Rev. **84**, 541 (1951).
[11] F. J. Belinfante, Phys. Rev. **84**, 644 (1951).
[12] J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949).

direct than in the manifestly covariant theory, where this covariance is seen by inspection.

No doubt exists that the gauge-independent treatment comes closer to the way many physicists think about phenomena involving electromagnetic radiation besides Coulomb interaction. The advantage of the gauge-independent method lies not only in the fact that it avoids redundant field variables altogether, but also in the fact that in many applications one *wants* to treat Coulomb interaction as a first-order, preferably even zeroth-order effect. Yet, to this date, the manifestly covariant method remains the single mathematical tool by which the theoretician has successfully attacked the more complicated problems of quantum electrodynamics and arrived at results comparable with experiment.[13] Therefore, it is important to find a consistent interpretation of this covariant method.

This has not been possible within the framework of a quantum theory operating in a Hilbert space of positive definite metric. The many published attempts at such interpretation all contain either errors or inconsistencies, or other unsatisfactory features.[14] Therefore, for a justification of modern quantum electrodynamics without eliminating the part of Hilbert space describing the states of longitudinal and scalar photons, one must replace Hilbert space by what I may call *Gupta space*, that is, the more general type of Hilbert space in which the condition of a positive definite metric has been dropped.[4]

2. THE PROBLEM OF COVARIANCE OF GUPTA'S "RULE OF NORMS"

As the gauge-independent theory, thus also at first Gupta's theory gives to the unsuspecting reader the impression of being noncovariant. This is due to the choice of the metric in Gupta space. It is convenient to choose the eigenfunctions of the incident-photon occupation numbers[15–17] as basic vectors in Gupta space. Gupta has shown, then, that the metric in Gupta space must be such that the norm of these basic vectors is -1 $(+1)$, as the number of incident "scalar" photons is odd (even).[1] We shall call this Gupta's "rule of norms." Now, everybody knows that the number of incident "scalar" photons is not invariant under Lorentz transformation. Therefore, the average reader will at first be skeptical about the possibility of covariance of such rule of normals. However, here as for gauge-independent quantum electrodynamics, the covariance of the theory has been proved nevertheless.

By a proof of covariance, of course, we mean that two physicists, moving with respect to each other and each using the same rule of norms in his own different coordinate system, will both ascribe the same norm to the same basic vector in Gupta space. This may be expressed differently as follows. Physicist A may draw his attention for instance to a state,[18] in which from his point of view there is one scalar photon besides, perhaps, several vector photons. He gives it the norm -1. This same basic vector, according to physicist B, describes a mixture of states, in which there are $0, 1, 2, \cdots$, scalar photons present. Anyhow, the norm of this mixed state from A's observation was already known to be -1, and B cannot change that. Similarly, there is an infinite number of other states, which look like mixed states to B, but for which the norms are known, because for A these states happen to have a simpler aspect allowing A the application of the rule of norms. The norms of this complete set of mixed states, then, already will determine completely the metric of Gupta space for B. In particular, it will also fix the norms of states which look simple to B, and to which he would have liked to apply the rule of norms himself. The question then, is whether, for the latter cases, the metric thus by A already fixed for B will automatically guarantee or will automatically contradict the validity of the rule of norms for B.

This is not just a question of "elegance" of the theory. This is a question of necessity for the covariance of the theory. For it is not sufficient that field equations and commutation relations are consistent and covariant. In the first place, it is also necessary that expectation values of observables will transform in exactly the same way as the corresponding q numbers do. Therefore, no difference in expectation value should be caused by differences in methods for obtaining expectation values from q numbers in different Lorentz systems. The calculation of expectation values involves use of the metric. The same metric should, therefore, be used by different observers. This is the reason why we said above that the metric for observer B is completely determined by the metric for observer A, and that B cannot change that.

In the second place, if the special relativity principle is to hold, the laws of nature for B should be the same

[13] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

[14] In my own attempt [F. J. Belinfante, Physica **12**, 17 (1946)], the asymmetry between the treatment of the products of F into Ψ and into $\Psi^\dagger$ in expressions like $(\Psi^\dagger F\Psi)$ was an unsatisfactory feature which can be remedied. In that attempt, a basic idea of the gauge-independent theory was smuggled half-heartedly into the manifestly covariant theory, by defining Ψ in a subspace of Hilbert space only. While such a "half-hearted" theory possibly could be further developed, this has never been done.

[15] By "incident-photon" occupation numbers we understand the occupation numbers which are derived from the Fourier components of the potentials given in an interaction representation coinciding with the Heisenberg representation at the "time of incidence" (usually $t=-\infty$). See references 16, 17, and 11.

[16] G. Källén, Arkiv Fysik **2**, 187, 371 (1951).

[17] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[18] This remark needs clarification. By "state" we do not mean here a physical state satisfying the Schrödinger equation and auxiliary condition, but we mean only some eigenfunction of the incident-photon occupation numbers and, therefore, of the free-photon energy in interaction representation. By the "value" of this free-photon energy we will then understand the corresponding eigenvalue. As the norm of the eigenfunction may be negative, this eigenvalue may be opposite in sign to the corresponding contribution to the expectation value in a physical state containing admixture of this eigenfunction.

as for A. Now, the rule of norms is a law of nature, important because it determines the expectation values of observables. Therefore, Gupta's rule of norms is acceptable only if it can be proved that, once it is valid for A, it will be valid for B automatically.

A somewhat complicated and not entirely complete proof of the covariance of this rule of norms was given by Bleuler.[2] He proved the following general theorem, which we shall purposely somewhat reformulate, in order to facilitate for us the completion of the proof we are asking for.[19]

Let A be, for instance, the x component of the potential four-vector operator. Let χ be an eigenfunction of A belonging to some given eigenvalue a. There is an operator S, such that $A'=S^{-1}AS$ is now the x' component of the potential four-vector operator in a second (primed) Lorentz frame. Then, obviously, $\chi'=S^{-1}\chi$ is the "corresponding" eigenfunction of A' belonging to that same eigenvalue a. "Covariance of norms" now requires that χ' shall have the same norm as χ had. This was shown by Bleuler to be so.

In order to complete the proof that the rule of norms remains valid for the primed Lorentz frame in the way we formulated it for the unprimed Lorentz frame, it is now sufficient to generalize Bleuler's theorem[19] to the case where A in the above is not just a component of the potential itself, but is some functional of it, such as an occupation number for a "scalar" photon. In this way one finds a proof, although a rather complicated one, of the covariance of Gupta's method. In the following, we shall find a much shorter and more direct proof of the covariance of Gupta's rule of norms. (See Chap. 4.)

Recently, Gupta himself has reformulated the definition of the metric in Gupta space in such way that its covariance becomes more or less evident. The general idea of this reformulated theory is approximately the following.[20]

We first postulate in one Lorentz frame that there shall be a state of nonvanishing norm, in which the incident-photon energy takes a minimum value.[15,18] It is then shown that this automatically leads to the validity of Gupta's rule of norms in this particular Lorentz frame. But it is also shown that this lowest energy state must then be the zero-energy zero-momentum state of the photon vacuum, and that all other states automatically have a time-like energy-momentum four-vector pointed into the positive energy direction. The latter property is invariant. Therefore, now also in any other Lorentz frame all energies are positive except the zero energy of the vacuum state. This, then, guarantees the automatic validity of Gupta's rule of norms in every Lorentz frame.

This new proof of Gupta is brief and elegant. However, it somewhat bypasses the question which we originally asked. For, while Gupta's new brief proof of covariance, of course, *implies* that the metric for B must satisfy the same rule of norms as for A, yet it does so without *explicitly calculating* the norm of a state which by B contains n scalar photons, in terms of the norms defined according to the rule of norms as postulated by A. If by a direct calculation of this type it could be demonstrated that the rule of norms then will hold automatically for B as well, this would be the most direct and therefore the most convincing proof of the covariance of Gupta's method.

The purpose of the present paper is to give this proof in a brief and simple form.

3. GUPTA'S REPRESENTATION OF EMISSION AND ABSORPTION MATRICES

We are here interested in the norms of the eigenfunctions of the occupation number operators N_μ used by Gupta in his rule of norms.[1] These N_μ are to be taken in interaction representation, not only in the definition of the Gupta norm in interaction representation, but also[21] in the definition of the Gupta norm in Heisenberg representation.[22]

These occupation-number operators can be introduced in a covariant way.[23] They form the diagonal components of mixed tensors,

$$N_\mu{}^\nu(K)=a_\mu{}^\dagger(K)\; a^\nu(K), \tag{1}$$

where K denotes a small domain on the positive-energy cone $k^0=+|\mathbf{k}|$ in momentum space. This so-called $+k$ cone[23] is assumed to be subdivided into many such domains; for each of these, there are four occupation-number operators, *viz.*,

$$N_0=-a_0{}^\dagger a_0,\quad N_j=+a_j{}^\dagger a_j\quad (j=1,2,3). \tag{2}$$

[19] In fact, Bleuler explicitly proved something slightly different from the theorem as we formulate it. For him, our χ' (called ψ by him) denoted a certain state, in which, for instance, the x component of the potential four-vector in the unprimed frame of reference has the expectation value $\bar{A}$, while the x' component in the primed frame of reference has the expectation value $\bar{A}'$. For him, our χ (his ψ') represents a state, in which then the x component in the unprimed Lorentz frame has the expectation value $\bar{A}'$. The different interpretation of Bleuler's formulas found in the text above makes it easier to pass from Bleuler's "proof of covariance of Gupta's method of quantization" to the more specific question of the covariance of the rule of norms, in which we are interested here.

[20] I thank Dr. Gupta for private communication of this reformulation of his theory. A full discussion of this treatment of the theory of the indefinite metric will appear in a forthcoming book *Quantum Theory of Fields* by S. N. Gupta [North Holland Publishing Company, Amsterdam (to be published)]. It uses the simplified formulation of quantum theory in Gupta space found in reference 4.

[21] This is most easily seen using Gupta's new formulation (see references 4 and 20). With the old formulation of Gupta's theory (see reference 1), it follows from the fact that the transformation between interaction and Heisenberg representation is unitary in the sense that its inverse is equal to its adjoint, not its Hermitian conjugate. Such unitary transformation of a state vector leaves its norm invariant.

[22] This fact will not complicate the use of Heisenberg representation in those problems, in which one can characterize the physical state of specifying the numbers of "incident" photons (see references 15–17).

[23] See the Appendix.

The operators $a_\mu(K)$ and $a_\mu^\dagger(K')$ are absorption and emission operators for the momentum regions K and K'. They are commutative for $K\neq K'$, while for $K=K'$ they satisfy the commutation relations,[23]

$$a_\lambda a_\mu^\dagger - a_\mu^\dagger a_\lambda = g_{\lambda\mu}(=\delta_{\lambda\mu}-2\delta_{\lambda 0}\delta_{\mu 0}). \tag{3}$$

Let χ be the normalized simultaneous eigenfunction of all occupation-number operators, which belongs to the eigenvalues $\eta_{K\mu}$:

$$N_\mu(K)\chi = n_{K\mu}\chi;\quad \chi^\dagger\chi = \pm 1. \tag{4a–b}$$

By $\chi(n_{K\mu}\pm 1)$ we shall denote the simultaneous eigenfunction belonging to this same set of eigenvalues except $(n_{K\mu}\pm 1)$ replacing $n_{K\mu}$; by $\chi(n_{K\mu}\pm 1,\ n_{K\nu}\mp 1)$ for $\mu\neq\nu$ we shall denote one belonging to the same eigenvalues as $\chi(n_{K\mu}\pm 1)$ except $(n_{K\nu}\mp 1)$ replacing $n_{K\nu}$; etc. Further, we shall omit the subscript or argument K wherever this does not cause confusion.

Gupta[24] then has found the following representation of the operators $a_\mu^\dagger$ and a_μ:

$$a_0\,\chi = n_0^{\frac{1}{2}}\,\chi(n_0-1), \tag{5a}$$

$$a_0^\dagger\,\chi = -[n_0+1]^{\frac{1}{2}}\,\chi(n_0+1), \tag{5b}$$

$$a_j\,\chi = n_j^{\frac{1}{2}}\,\chi(n_j-1),\quad (j=1,2,3), \tag{5c}$$

$$a_j^\dagger\,\chi = [n_j+1]^{\frac{1}{2}}\,\chi(n_j+1). \tag{5d}$$

Indeed, it is easily seen that Eqs. (5) ensure the validity of Eq. (3), and by (2) also the validity of (4a). In fact, the solution (5) of Eqs. (2)–(4) is unique but for the possible effect of a more arbitrary choice of the complex relative phase factors of the various eigenfunctions χ. Thus, (5) determines a fixed and convenient choice for these phases.

As for Eq. (4b): Eq. (5) also determines the signs of the norms of the eigenfunctions χ, if we choose the norm of the vacuum state to be $+1$. This is so because the meaning of the symbol $\dagger$ in Eqs. (5b) and (5d) involves that

$$\{\chi^\dagger a_\mu(K)\psi\}^* = \psi^\dagger a_\mu^\dagger(K)\chi, \tag{6}$$

if the asterisk (*) denotes ordinary complex conjugation. Now, introduce the notation η for the norm of χ, that is,

$$\eta = \chi^\dagger\chi,\quad \eta(n_\mu\pm 1) = \chi(n_\mu\pm 1)^\dagger\chi(n_\mu\pm 1),\quad \text{etc.} \tag{7}$$

Then, in Eq. (6) once take $\mu=0$ and $\psi=\chi(n_{K0}+1)$ and apply (5a–b), and once take $\mu=j(=1,2,3)$ and $\psi=\chi(n_{Kj}+1)$ and apply (5c–d). The two results thus obtained can be written by (7) as

$$\eta = -\eta(n_{K0}+1) = +\eta(n_{Kj}+1).$$

Thence, choosing $\eta=+1$ for the vacuum state, Gupta finally finds

$$\chi^\dagger\chi = \eta = \prod_K(-1)^{n_{K0}} = (-1)^{\Sigma_K n_{K0}} \tag{8}$$

for his rule of norms.[1] Finally, it is easily seen that the eigenfunctions χ satisfy the orthogonality relations[1,4]

$$\chi^\dagger\psi = \eta\prod_\mu\prod_K\delta(n_{K\mu},m_{K\mu}), \tag{9}$$

if ψ is an eigenfunction like χ, but for a completely independent set of eigenvalues $m_{K\mu}$, so that $N_\mu(K)\psi = m_{K\mu}\psi$. In (9), we wrote $\delta(n,m)$ for the Kronecker symbol δ_{nm}.

4. PROOF OF THE LORENTZ COVARIANCE OF GUPTA'S RULE OF NORMS

Suppose that the eigenfunctions χ of the occupation numbers $N_\mu(K)$ in our original (unprimed) coordinate system have been orthonormalized according to (8)–(9), and that the phases have been chosen according to Eqs. (5). This fixes the Gupta metric in the unprimed Lorentz system.

We are interested in the covariance of these formulas, so we want to show that in a new (primed) frame of reference the new eigenfunctions χ' of the new occupation-number operators $N_\mu'(K)$ can be chosen in such a way as to satisfy Eqs. (5′), (8′), and (9′); that is, Eqs. (5), (8), and (9) with primes.

It is sufficient to prove this covariance for infinitesimal Lorentz transformations only. For finite Lorentz transformations it then follows by integration.

For the sake of simplicity, we choose our x axis in the direction of the infinitesimal velocity v of the new (primed) Lorentz frame with respect to the old one. Let $b=v/c$, and $x^0=-x_0=ct$. Since the annihilation operators form a four-vector,[23] we then must have

$$\begin{aligned} a_0' &= a_0 + b\,a_1,\quad a_1' = a_1 + b\,a_0,\\ a_2' &= a_2,\quad a_3' = a_3\,. \end{aligned} \tag{10}$$

By (2) and (10), or by the tensor character of $N_{\mu\nu}=a_\mu^\dagger a_\nu$, it then follows, always for infinitesimal b, that

$$\begin{aligned} N_0' &= -a_0^\dagger a_0 - b(a_0^\dagger a_1 + a_1^\dagger a_0),\\ N_1' &= +a_1^\dagger a_1 + b(a_0^\dagger a_1 + a_1^\dagger a_0),\\ N_2' &= N_2,\quad N_3' = N_3, \end{aligned} \tag{11}$$

are the occupation-number operators in the primed Lorentz frame.[25]

The new simultaneous eigenfunctions χ' should now satisfy

$$N_\mu'(K)\,\chi' = n_{K\mu}\,\chi'\,. \tag{4a′}$$

We shall at once write down the solutions of (4a′) which fulfill our requirements. Expressed in terms of the old eigenfunctions χ of the operators $N_\mu(K)$, they are

$$\begin{aligned} \chi' = \chi &- b\textstyle\sum_k n_{k0}^{\frac{1}{2}}[n_{k1}+1]^{\frac{1}{2}}\,\chi(n_{k0}-1,\ n_{k1}+1)\\ &- b\textstyle\sum_k[n_{k0}+1]^{\frac{1}{2}}n_{k1}^{\frac{1}{2}}\,\chi(n_{k0}+1,\ n_{k1}-1), \end{aligned} \tag{12}$$

[24] See reference 1, Eqs. (16) and (20). Our $a_0=-a^0$ corresponds to Gupta's $-C_0$.

[25] Here, N_μ' and N_μ are abbreviations for $N_\mu'(K)$ and $N_\mu(K)$ with the same invariant argument K. A four-vector $\mathfrak{k}_\lambda$ inside the invariant region K, of course, changes its components to $\mathfrak{k}_\lambda'$, as it transforms like a four-vector.

where the dummy index k stands for K (therefore $\sum_k$ for an "integration" over the $+k$-cone) with the small letter k used merely for avoiding some primes in the following.

Next, it must be proved that the functions (12) satisfy our requirements indeed. First, we shall show that they satisfy the relations (5a′–d′), that is, the relations (5) with primes on the χ and the a, where a_μ' is given by (10).

Starting with (5a′), we first compare its right-hand member $n_{K0}^{\frac{1}{2}}\chi'(n_{K0}-1)$ with $a_0(K)\,\chi'$. In each term of (12), operation of the unprimed $a_0(K)$ will lower by 1 the argument n_{K0} of the featured unprimed eigenfunction, at the same time multiplying the term by the square root of the original number of $K0$ photons (=photons "scalar" with respect to the old frame of reference and with the proper wave vectors inside K). This yields a factor $n_{K0}^{\frac{1}{2}}$ for most terms of the expansions (12). The only exceptions are the terms with $k=K$ in the two sums. These terms become

$$-\,b\,n_{K0}^{\frac{1}{2}}[n_{K0}-1]^{\frac{1}{2}}[n_{K1}+1]^{\frac{1}{2}}\,\chi(n_{K0}-2,\,n_{K1}+1)\\ -\,b\,[n_{K0}+1]\,n_{K1}^{\frac{1}{2}}\,\chi(n_{K0},\,n_{K1}-1).$$

In $n_{K0}^{\frac{1}{2}}\chi'(n_{K0}-1)$, on the other hand, we find similar terms, except

$$-\,b\,n_{K0}^{\frac{1}{2}}[n_{K0}-1]^{\frac{1}{2}}[n_{K1}+1]^{\frac{1}{2}}\,\chi(n_{K0}-2,\,n_{K1}+1)\\ -\,b\,n_{K0}\,n_{K1}^{\frac{1}{2}}\,\chi(n_{K0},\,n_{K1}-1)$$

for the terms with $k=K$ in the two sums. Thence,

$$n_{K0}^{\frac{1}{2}}\chi'(n_{K0}-1)=a_0(K)\,\chi'+b\,n_{K1}^{\frac{1}{2}}\,\chi(n_{K0},\,n_{K1}-1). \tag{13}$$

The last term here is just $b\,a_1(K)\,\chi$. Since the difference between this and $b\,a_1(K)\,\chi'$ is quadratic in the infinitesimal b and therefore is to be neglected, Eq. (13) by Eq. (10) just gives the desired Eq. (5a′).

The other primed equations (5b′), (5c′), (5d′) can be verified similarly. The minus sign in Eq. (5b) causes no trouble at all, but combines in just the right way with the other signs in Eqs. (5d), (10), and (12).

Thus, the functions χ' given by (12) will for an observer in the primed frame of reference take the place which the functions χ take for an observer in the unprimed frame of reference. Just as (4a) by (2) followed from the Eqs. (5a–d), now (4a′) will follow from the validity of Eqs. (5a′–d′) by the relation (2′), or its equivalent (11).

This completes most of our task. The only thing yet to be verified is the automatic validity of the rule of norms in the primed Lorentz frame. That is, we should verify that $\chi'^\dagger\psi'$ satisfies Eqs. (8′)–(9′) as in Eq. (14) below, if $\chi'^\dagger$ is given by the adjoint (12†) of Eq. (12), while ψ' is given by an equation (12ψ) obtained from (12) by changing χ into ψ and $n_{K\mu}$ into $m_{K\mu}$.

The proof is easy. The expression (12) was written as a sum of three terms, say, $(12)_1+(12)_2+(12)_3$, so

$$\chi'^\dagger\psi'=[(12^\dagger)_1+(12^\dagger)_2+(12^\dagger)_3]\\ \cdot[(12\psi)_1+(12\psi)_2+(12\psi)_3].$$

Now, $[(12^\dagger)_2+(12^\dagger)_3][(12\psi)_2+(12\psi)_3]$ is infinitesimal of second order and can be omitted. Further,

$$(12^\dagger)_2(12\psi)_1+(12^\dagger)_1(12\psi)_3=\\ [\![-\,b\sum_k n_{k0}^{\frac{1}{2}}[n_{k1}+1]^{\frac{1}{2}}\{\chi^\dagger(n_{k0}-1,\,n_{k1}+1)\cdot\psi\}\\ -\,b\sum_k\{\chi^\dagger\cdot\psi(m_{k0}+1,\,m_{k1}-1)\}[m_{k0}+1]^{\frac{1}{2}}m_{k1}^{\frac{1}{2}}]\!]=0,$$

as by (9) the two expressions { } are nonvanishing for $n_{k0}=m_{k0}+1$, $n_{k1}+1=m_{k1}$ only, and then these expressions have opposite signs on account of (8). Similarly, $(12^\dagger)_3(12\psi)_1+(12^\dagger)_1(12\psi)_2=0$. Thus, we are left with $(12^\dagger)_1(12\psi)_1$ only; that is,

$$\chi'^\dagger\psi'=\chi^\dagger\psi=(-1)^{\Sigma_K n_{K0}}\prod_\mu\prod_K\delta(n_{K\mu},m_{K\mu}), \tag{14}$$

where we have finally used Eqs. (8)–(9) without primes.

Equation (14) then shows the validity of the rule of norms in the primed Lorentz frame, so that this completes our proof.

We thus find that the minus signs in Gupta's metric do not cause any difficulty in the demonstration of the covariance of his theory, but always combine with each other in such a way that this covariance is maintained.

APPENDIX

We shall show here the covariance of the occupation numbers for a photon field in interaction representation. In particular, we shall show that they form tensors together with the repolarization operators given by Eq. (1).

For an electromagnetic field in interaction representation, expand the positive-energy part $A_\mu^+(x)$ of the potential four-vector as a Fourier integral in four dimensions:

$$A_\mu^+(x)=(2\pi)^{-2}(2hc)^{\frac{1}{2}}C\int_{(k^0>0)}d^{(4)}k\\ \times\delta(k_\lambda k^\lambda)A_\mu(k)\exp(ik_\sigma x^\sigma). \tag{A1}$$

The factor $\delta(k_\lambda k^\lambda)$ here ensures vanishing of the Fourier components except on the half-cone $k^0=+|\mathbf{k}|$, which we shall call the "$+k$-cone." Similarly, we shall call $k^0=-|\mathbf{k}|$ the "$-k$-cone." Our Greek indices take the values 0, 1, 2, 3; never 4. The electrostatic potential is $A^0=-A_0$, and $x^0=-x_0=ct$. The numerical constants have been chosen for future convenience; $C=(4\pi)^{\frac{1}{2}}$ in the Gaussian system, and $C=1$ in the Heaviside-Lorentz system. By $\int_{(k^0>0)}d^{(4)}k$ we understand integration over any four-dimensional region which contains all of the $+k$-cone but nothing of the $-k$-cone. The total potential $A_\mu=A_\mu^++A_\mu^-=A_\mu^++(A_\mu^+)^\dagger$ is "self-adjoint" in the sense that its components have real expectation values.[1]

Four-vectors on the $+k$ cone we shall denote by $\mathfrak{k}$. Three parameters suffice to determine such a four-vector; for instance, the three components of its spatial part $\mathbf{k}$. Use of $\mathbf{k}$ for denoting $\mathfrak{k}$ does not destroy the covariance of our notation. While $\mathbf{k}$ is not invariant under Lorentz transformation, as space part of $\mathfrak{k}$ it is perfectly covariant, that is, its Lorentz transformations as the space part of a four-vector with

$$k^0=(k_x^2+k_y^2+k_z^2)^{\frac{1}{2}}$$

are well-defined (although not linear).

We define covariant integration over the $+k$-cone as follows. Let $K4$ denote an arbitrary four-dimensional volume in k space not intersecting the $-k$-cone, but intersecting the $+k$-cone by K. Let $F(\mathfrak{k})$ be a function of $\mathfrak{k}$ to be covariantly integrated over K. Then, let $F(k)$ be a function of the four-vector k, arbitrary except for its identity with $F(\mathfrak{k})$ on the $+k$-cone. Integration of $F(\mathfrak{k})$ over K shall then be defined Lorentz-covariantly by

$$\int_K F(\mathfrak{k})\, dV(\mathfrak{k})=2\int_{K4} F(k)\,\delta(k_\lambda k^\lambda)\, d^{(4)}k. \quad \text{(A2)}$$

Because of the delta-function, the shape of $K4$ outside the $+k$-cone and the values of $F(k)$ off the $+k$-cone do not matter. The factor 2 is for convenience: by

$$\delta(k_\lambda k^\lambda)=\delta(\mathbf{k}^2-k_0^2)=[\delta(|\mathbf{k}|-k^0)]/|2\mathbf{k}|$$

around the $+k$-cone, it enables us to write (A2) as

$$\int_K F(\mathfrak{k})\, dV(\mathfrak{k})=\int_{\mathbf{K}} F(\mathfrak{k})\, d^{(3)}\mathbf{k}/|\mathbf{k}|, \quad \text{(A3)}$$

where $\mathbf{K}$ is the projection of K on the $k_xk_yk_z$ hyperplane. Thence,

$$dV(\mathfrak{k})=d^{(3)}\mathbf{k}/|\mathbf{k}| \quad \text{(A4)}$$

can be regarded as the invariant "volume element on the $+k$-cone" corresponding to the definition (A2). (Never mind its dimensions!) The invariant "volume" of a finite domain K on the $+k$-cone is

$$V(K)=\int_K dV(\mathfrak{k})=\int_{\mathbf{K}} d^{(3)}\mathbf{k}/|\mathbf{k}|. \quad \text{(A5)}$$

We define the "German-$\mathfrak{d}$ function" on the $+k$-cone by

$$\int_K F(\mathfrak{k})\, \mathfrak{d}(\mathfrak{k}-\mathfrak{k}')\, dV(\mathfrak{k})=F(\mathfrak{k}')\,\delta(\mathfrak{k}',K), \quad \text{(A6)}$$

where

$$\mathfrak{d}(\mathfrak{k}',K)=\int_K \mathfrak{d}(\mathfrak{k}-\mathfrak{k}')\, dV(\mathfrak{k})=\left.\begin{cases}=1, \text{ if } \mathfrak{k}' \text{ inside } K,\\ =0, \text{ if } \mathfrak{k}' \text{ outside } K.\end{cases}\right\} \quad \text{(A7)}$$

From (A3) with (A6) we easily see that, in any Lorentz frame, the invariant German-$\mathfrak{d}$ function may be written as

$$\mathfrak{d}(\mathfrak{k}-\mathfrak{k}')=|\mathbf{k}|\;\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \quad \text{(A8)}$$

With this notation we may write (A1) and its adjoint as

$$\left.\begin{aligned} A_\nu^+(x) &= \tfrac{1}{4}\hbar^{\frac{1}{2}}c^{\frac{1}{2}}\pi^{-3/2}C\int_\infty A_\nu(\mathfrak{k})\exp(i\mathfrak{k}_\sigma x^\sigma)dV(\mathfrak{k}),\\ A_\mu^-(x) &= \tfrac{1}{4}\hbar^{\frac{1}{2}}c^{\frac{1}{2}}\pi^{-3/2}C\int_\infty A_\mu^\dagger(\mathfrak{k}')\\ &\qquad\times\exp(-i\mathfrak{k}_\sigma' x^\sigma)\, dV(\mathfrak{k}'). \end{aligned}\right\} \quad \text{(A9)}$$

Here, $\int_\infty$ denotes integration over the entire $+k$-cone. Using Schwinger's notation of the D^+ function,[26] which in our notation may be written as

$$D^+(x)=(16\pi^3 i)^{-1}\int_\infty \exp(i\mathfrak{k}_\sigma x^\sigma)\, dV(\mathfrak{k}), \quad \text{(A10)}$$

the commutation rules for $A_\mu=A_\mu^++A_\mu^-$ can be expressed by

$$\begin{aligned} &[A_\nu^+(x)\;;A_\mu^-(x')]=i\hbar cC^2 g_{\nu\mu}D^+(x-x'),\\ &[A_\nu^+(x)\;;A_\mu^+(x')]=[A_\nu^-(x)\;;A_\mu^-(x')]=0, \end{aligned} \quad \text{(A11)}$$

with $g_{00}=-1$, $g_{11}=g_{22}=g_{33}=+1$, and $[A\;;B]=AB-BA$. They can also be expressed covariantly, on account of Eqs. (A9), (A6), (A10), and $A_\nu=g_{\nu\lambda}A^\lambda$, by

$$\left.\begin{aligned} &[A^\lambda(\mathfrak{k})\;;A_\mu^\dagger(\mathfrak{k}')]=\delta_\mu^\lambda\;\mathfrak{d}(\mathfrak{k}-\mathfrak{k}'),\\ &[A^\lambda(\mathfrak{k})\;;A^\nu(\mathfrak{k}')]=[A_\mu^\dagger(\mathfrak{k})\;;A_\nu^\dagger(\mathfrak{k}')]=0. \end{aligned}\right\} \quad \text{(A12)}$$

Now consider the free-photon energy and momentum four-vector G^λ. Performing the necessary[27] subtraction of the meaningless infinite vacuum energy, we find[28]

$$G^\lambda=\int_\infty \hbar c\mathfrak{k}^\lambda\, A_\mu^\dagger(\mathfrak{k})A^\mu(\mathfrak{k})\, dV(\mathfrak{k}). \quad \text{(A13)}$$

This result suggests that

$$N_\mu(K)=\int_K A_\mu^\dagger(\mathfrak{k})A^\mu(\mathfrak{k})\, dV(\mathfrak{k}) \text{ (no sum over } \mu) \quad \text{(A14)}$$

should be regarded as the number of "incident" photons[11,15–17] of polarization μ and with energy-momentum four-vectors given by wave vectors $\mathfrak{k}$ inside the region K on the $+k$-cone. Evidently the $N_\mu(K)$ are the diagonal components of a mixed tensor

$$N_\mu{}^\nu(K)=\int_K A_\mu^\dagger(\mathfrak{k})A^\nu(\mathfrak{k})\, dV(\mathfrak{k}). \quad \text{(A15)}$$

The $N_\mu{}^\nu(K)$ for $\mu\neq\nu$ one might call the "repolarization operators," as they describe a change in polarization direction of a photon.

If K is an infinitesimal domain dK on the $+k$-cone, $N_\mu{}^\nu(K)$ may be denoted by $dN_\mu{}^\nu$:

$$dN_\mu{}^\nu=A_\mu^\dagger(\mathfrak{k})\; A^\nu(\mathfrak{k})\; dV(\mathfrak{k}). \quad \text{(A16)}$$

From (A13) and (A16) we obtain

$$G^\lambda=\int_\infty \hbar c\mathfrak{k}^\lambda\, dN_\mu{}^\mu. \quad \text{(A17)}$$

Our interpretation of $N_\mu(K)$ as an occupation number, and an interpretation of $A^\lambda(\mathfrak{k})$ and $A_\lambda^\dagger(\mathfrak{k})$ as

[26] J. Schwinger, Phys. Rev. **75**, 651 (1949).
[27] F. J. Belinfante, Phys. Rev. **76**, 461 (1949).
[28] J. L. Lopes, Anais acad. brasil. cienc. **23**, 39 (1951), Eq. (22).

absorption (=annihilation) and emission (=creation) operators for an incident photon of momentum $\hbar\mathbf{k}$, are justified by the following relations, which follow from Eqs. (A15), (A12), and (A7). First,

$$[A^\lambda(\mathfrak{k})\ ;N_\mu{}^\nu(K)]=\delta_\mu{}^\lambda\,\delta(\mathfrak{k},K)\,A^\nu(\mathfrak{k}),\qquad [A_\lambda{}^\dagger(\mathfrak{k})\ ;N_\mu{}^\nu(K)]=-\delta_\lambda{}^\nu\,\delta(\mathfrak{k},K)\,A_\mu{}^\dagger(\mathfrak{k}). \tag{A18}$$

Thence, if $N_\mu(K)\chi_n=n\chi_n$,

$$N_\mu(K)\{A^\lambda(\mathfrak{k})\chi_n\}=[n-\delta_\mu{}^\lambda\delta(\mathfrak{k},K)]\{A^\lambda(\mathfrak{k})\chi_n\},\qquad N_\mu(K)\{A_\lambda{}^\dagger(\mathfrak{k})\chi_n\}=[n+\delta_\lambda{}^\mu\delta(\mathfrak{k},K)]\{A_\lambda{}^\dagger(\mathfrak{k})\chi_n\}. \tag{A19}$$

Therefore, if we start from the eigenfunction χ_n belonging to the eigenvalue n of $N_\mu(K)$, application of the operator $A^\mu(\mathfrak{k})$ with $\mathfrak{k}$ inside K makes an eigenfunction belonging to $(n-1)$, and $A_\mu{}^\dagger$ makes an eigenfunction belonging to the eigenvalue $(n+1)$. By successive applications of these operators one can construct eigenfunctions corresponding to "stepladders" of eigenvalues $n\pm1$, $n\pm2$, $n\pm3$, $\cdots$. Now, using Eq. (A17) in Gupta's new definition of the indefinite metric,[29] we find that not only $dN_j=A_j{}^\dagger A_j dV(\mathfrak{k})$ with $j=1, 2, 3$, but also $dN_0=-A_0{}^\dagger A_0 dV(\mathfrak{k})$ cannot have bottomless stepladders of eigenvalues, so that obviously the stepladders for dN_μ downwards must break off. However, they can break off, say beneath a bottom step m, only if for all attempts to go down below m by applying another time $A^\mu(\mathfrak{k})$ with $\mathfrak{k}$ inside K one always finds $A^\mu(\mathfrak{k})\chi_m=0$. Then, however, by (A14), also $N_\mu(K)\chi_m=0$; that is, $m=0$ must be the bottom of every "stepladder" of eigenvalues. From here upwards, by successive applications of $A_\mu{}^\dagger(\mathfrak{k})$ with $\mathfrak{k}$ inside K, one then finds eigenfunctions belonging to the eigenvalues 1, 2, 3, $\cdots$. This fact, combined with the Eqs. (A19) furnishes the justifications we wanted.

Now, state vectors describing photon fields may be expanded in terms of the simultaneous eigenfunctions of the occupation numbers $N_\mu(K)$ for the various regions K on the $+k$-cone. For this purpose, the $+k$-cone is subdivided into a large number of very small regions K, each so small that we cannot measure differences of $\mathfrak{k}$ inside such a region. Therefore, the only absorption and emission operators that correspond to observably distinct changes of the state of the photon field, are obtained by bunching together all operators $A_\mu(\mathfrak{k})$ inside each such small finite region. Let us therefore define covariantly

$$\mathbf{A}^\mu(K)=\textstyle\int_K A^\mu(\mathfrak{k})\,dV(\mathfrak{k}),\qquad \mathbf{A}_\mu{}^\dagger(K)=\int_K A_\mu{}^\dagger(\mathfrak{k})\,dV(\mathfrak{k}). \tag{A20}$$

Now, if two regions K and K' would have a part in common, which we shall denote by $K\cap K'$, then the "volume" of this common region, given by (A5), can be written by (A7) as

$$V(K\cap K')=\textstyle\int_K \delta(\mathfrak{k},K')\,dV(\mathfrak{k}) =\int_K dV(\mathfrak{k})\int_{K'} dV(\mathfrak{k}')\,\mathfrak{d}(\mathfrak{k}-\mathfrak{k}'). \tag{A21}$$

From (A20), (A12), and (A21) we then find

$$[\mathbf{A}^\lambda(K)\ ;\mathbf{A}_\mu{}^\dagger(K')]=\delta_\mu{}^\lambda\,V(K\cap K'). \tag{A22}$$

Similarly, from (A20) and (A18),

$$[\mathbf{A}^\lambda(K)\ ;N_\mu{}^\nu(K')]=\delta_\mu{}^\lambda\,\mathbf{A}^\nu(K\cap K'),\qquad [\mathbf{A}_\lambda{}^\dagger(K)\ ;N_\mu{}^\nu(K')]=-\delta_\lambda{}^\nu\,\mathbf{A}_\mu{}^\dagger(K\cap K'). \tag{A23}$$

By putting $\nu=\mu$ (no sum over μ!) we obtain from (A23) commutation relations between $\mathbf{A}_\lambda$ and N_μ.

In practice, subdivisions of the $+k$-cone are made in such a way that the different regions K and K' either completely coincide, or are completely separate. Therefore we may introduce Kronecker symbols $\delta_{KK'}$, which are $=1$ and $=0$, respectively, for these two possibilities. We also introduce new four-vector operators a_λ and $a_\lambda{}^\dagger$, by

$$a_\lambda(K)=V(K)^{-\frac{1}{2}}\,\mathbf{A}_\lambda(K). \tag{A24}$$

Thus, the commutation relations (A22)–(A23) become

$$[a^\lambda(K)\ ;a_\mu{}^\dagger(K')]=\delta_\mu{}^\lambda\,\delta_{KK'}\,; \tag{A25}$$

$$[a^\lambda(K)\ ;N_\mu{}^\nu(K')]=\delta_\mu{}^\lambda\,\delta_{KK'}\,a^\nu(K),\qquad [a_\lambda{}^\dagger(K)\ ;N_\mu{}^\nu(K')]=-\delta_\lambda{}^\nu\,\delta_{KK'}\,a_\mu{}^\dagger(K). \tag{A26}$$

By a reasoning similar to that applied to Eqs. (A18) in and below Eqs. (A19), we conclude from Eqs. (A26) that $a^\mu(K)$ and $a_\mu{}^\dagger(K)$ are annihilation and creation operators for the occupation numbers $N_\mu(K)$. In order to find the matrix elements of the operators $a^\mu(K)$ between normalized eigenfunctions of the operators $N_\mu(K)$, however, we would need one more relation, *viz.*, the analog of Eq. (A16).

While the above equations are valid for any subdivision of the $+k$-cone into regions K, whatever large these are taken, we will not be able to express $N_\mu(K)$ in terms of $a_\mu{}^\dagger(K)$ and $a^\mu(K)$ for arbitrarily large domains K. One might think of by-passing this difficulty by defining new operators

$$\mathcal{N}_\mu{}^\nu(K)=a_\mu{}^\dagger(K)a^\nu(K)\,;\quad \mathcal{N}_\mu\equiv\mathcal{N}_\mu{}^\mu \text{ (no sum!)}. \tag{A27}$$

Truly enough, the operators $\mathcal{N}_\mu(K)$ have eigenvalues 0, 1, 2, $\cdots$, and the $a_\mu(K)$ and $a_\mu{}^\dagger(K)$ are absorption and emission operators for them. The catch, however, is that for arbitrarily large regions K the operators $\mathcal{N}_\mu(K)$ do not represent the true occupation numbers in which the physicist is interested. This interest stems from the possibility of expressing the energy and momentum in the form of Eq. (A17). From (A17) we see that dN_μ is the occupation number for an infinitesimal region dK. The physicist then wants to define the occupation number for finite K as the integral of dN_μ. By (A14), this integral is $N_\mu(K)$, not $\mathcal{N}_\mu(K)$.

[29] See the text above footnote 20.

In fact, while $N_\mu(K)$ by (A14) has the desirable property $N_\mu(K_1+K_2)=N_\mu(K_1)+N_\mu(K_2)$, the operators $\mathcal{N}_\mu(K)$ by (A27) with (A24) and (A20) do not even have this additive property, and therefore cannot be interpreted as the proper occupation number for any finite K which meaningfully may be cut up into smaller parts.

The way out of this difficulty is by taking the size of the regions K as small as we suggested above the Eqs. (A20). We might call these regions then "physically infinitesimal."[30] In this case, our various covariant definitions may be simplified. Equations (A15), (A20), and (A24) may then, by (A5), be re-written as

$$N_\mu{}^\nu(K) \approx A_\mu{}^\dagger(\mathfrak{k})\, A^\nu(\mathfrak{k})\, V(K), \qquad \text{(A15a)}$$

$$\mathbf{A}_\mu(K) \approx A_\mu(\mathfrak{k})\, V(K), \qquad \text{(A20a)}$$

$$a_\mu(K)=V(K)^{-\frac{1}{2}}\, \mathbf{A}_\mu(K) \approx V(K)^{\frac{1}{2}}\, A_\mu(\mathfrak{k}), \qquad \text{(A24a)}$$

where $\mathfrak{k}$ may have any value inside K, and where $\approx$ denotes experimental indistinguishability. From (A15a) and (A24a), then, we find for such small K

$$N_\mu{}^\nu(K) \approx a_\mu{}^\dagger(K)\, a^\nu(K)=\mathcal{N}_\mu{}^\nu(K). \qquad \text{(A28)}$$

Equations (A25), (A26), and (A28) then determine in the usual way[1] the matrix elements of the emission and absorption operators $a_\mu{}^\dagger(K)$ and $a_\mu(K)$ between states characterized by the occupation numbers $N_\mu(K) \approx \mathcal{N}_\mu(K)$. [Compare Eqs. (5a–d).]

Finally, replacing the integrals in (A9) by sums over the arbitrarily but invariantly chosen cells K on the $+k$-cone, we may write the potential four-vector $A_\mu(x)$ in terms of the matrices $a_\mu(K)$ and $a_\mu{}^\dagger(K)$, as

$$A_\mu(x)=\tfrac{1}{4}\hbar^{\frac{1}{2}}c^{\frac{1}{2}}\pi^{-3/2}C \textstyle\sum_K V(K)^{\frac{1}{2}} \times \{a_\mu(K)\exp(i\mathfrak{k}_\sigma x^\sigma)+a_\mu{}^\dagger(K)\exp(-i\mathfrak{k}_\sigma x^\sigma)\}. \qquad \text{(A29)}$$

This covariant expression contains as a special case the familiar expression

$$A_\mu(x)=\tfrac{1}{4}\hbar^{\frac{1}{2}}c^{\frac{1}{2}}c\mathcal{V}^{-\frac{1}{2}} \textstyle\sum_{\mathbf{k}} (2|\mathbf{k}|)^{-\frac{1}{2}} \times \{a_\mu(K)\exp(i\mathfrak{k}_\sigma x^\sigma)+a_\mu{}^\dagger(K)\exp(-i\mathfrak{k}_\sigma x^\sigma)\}, \qquad \text{(A30)}$$

in which one sums over a discrete set of $\mathbf{k}$ values forming a cubic lattice in such a way that there is one "allowed" $\mathbf{k}$ value for each cubic volume element $\mathbf{K}$ of size

$$d^{(3)}\mathbf{k}=(2\pi)^3\mathcal{V}^{-1} \qquad \text{(A31)}$$

in three-dimensional $\mathbf{k}$ space. Here, $\mathfrak{k}_\sigma$ has the components $\mathbf{k}$ and $k_0=-k^0=-|\mathbf{k}|$.

In order to see the equivalence of (A30) to (A29), we make for the latter a special subdivision of the $+k$-cone into cells K in such a way that the projection of each cell K onto the $k_xk_yk_z$ hyperplane will coincide with one cubic cell $\mathbf{K}$ of volume (A31) containing just one of the $\mathbf{k}$ values, over which we sum in (A30). But then, by (A4), the corresponding "invariant volume on the $+k$-cone" of such a cell K will be equal to

$$V(K)=d^{(3)}\mathbf{k}/|\mathbf{k}|=(2\pi)^3\mathcal{V}^{-1}|\mathbf{k}|^{-1}. \qquad \text{(A32)}$$

If this is inserted in Eq. (A29), that equation takes the familiar form (A30) indeed.

This shows that Eq. (A30) is nothing more than Eq. (A29) written up for some special subdivision of the $+k$-cone, in which the constant $\mathcal{V}$ by (A31) simply tells how finely we are subdividing $\mathbf{k}$ space into cubes. There is no need, therefore, for the conventional interpretation of $\mathcal{V}$ as "the volume of a cubic world."

Under Lorentz transformations, then, the $a^\nu(K)$ and the $a_\mu{}^\dagger(K)$ transform like four vectors, and the $N_\mu{}^\nu(K) \approx \mathcal{N}_\mu{}^\nu(K)$ for any "infinitesimal" region K on the $+k$-cone transform as a tensor. The region K itself is invariantly fixed on the $+k$-cone, and may be characterized by the four-vector(s) $\mathfrak{k}_\sigma$ which it contains. The fact that someone in a particular Lorentz frame may have characterized it by the projection $\mathbf{k}$ of $\mathfrak{k}$ onto $\mathbf{k}$ space—or may even have chosen this $\mathbf{k}$ as part of some cubic lattice, as in (A30)—does not effect the invariance of K or the covariance of the four-vector $\mathfrak{k}$. Also, it does not matter that in a different Lorentz frame the projection of $\mathfrak{k}$ onto the new $\mathbf{k}'$ space does not form part of any cubic lattice at all, since from (A29) it is clear that in subdividing the $+k$-cone into cells K it is *not necessary* to make the projections $\mathbf{k}$ form a cubic lattice, or even space them regularly as by (A31).

[30] Therefore, we could also denote them by dK. However, mathematically, these regions dK are finite. For, in the first place, there is no sense in making the subdivision into regions K much finer than what can be distinguished experimentally, as the state vector, expressed in terms of occupation numbers for these regions, should express our factual knowledge about the photon field, and not more than that. In the second place, we want them finite, as otherwise the definition (A24) would become meaningless.