Self-Energy Effects on Meson-Nucleon Scattering According to the Tamm-Dancoff Method*†

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The lowest-order Tamm-Dancoff equations for the meson-nucleon system are derived by the method of Cini, using the wave functions defined by Dyson. The contributions of the meson and nucleon self-energies to the kernel of the momentum-space integral equation are renormalized. Their effects are absorbed into the coupling G^2, making it momentum and energy dependent. The effective coupling turns out to exhibit an anomalous behavior, having a pole for a spacelike momentum, to which it is difficult to attach a sensible physical interpretation.

I. INTRODUCTION

CALCULATIONS of the meson-nucleon scattering phase shifts have recently been performed by the Tamm-Dancoff method.[1] This paper is essentially a continuation of one in which that undertaking is reported. The lowest-order Tamm-Dancoff equations were there derived and solved approximately by unmerical methods, with several of the terms in the kernel, namely, those which correspond to the nucleon, meson, and vacuum self-energies, and some which contribute to the interaction only in the $I=1/2, J=1/2$ state, simply dropped. It is the purpose of the present paper to ascertain the effects of the former omission on the scattering phase shifts; in other words, to find the renormalization corrections to the scattering in states in which either I or J, or both, are different from 1/2.

One of the omitted terms, the vacuum self-energy, may be eliminated by a simple redefinition of the Tamm-Dancoff amplitudes, as was shown by Dyson.[2] In addition, this redefinition makes possible a consistent relativistic renormalization of mass and charge.[3]

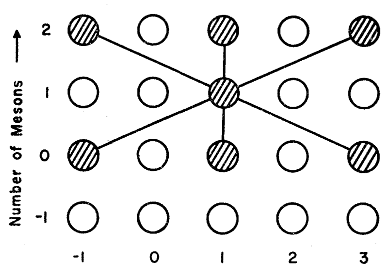

FIG. 1. Intermediate states of the meson-nucleon scattering system which are allowed in the Tamm-Dancoff approximation. Vertically is plotted the number of mesons present in the real state in excess of those present in the vacuum, horizontally the number of nucleons.

In order to perform a reliably unambiguous separation of the divergent parts of the self-energy integrals, it is essential to write them in a covariant form. For this purpose we use the method of Cini,[4] applied to the DTD theory.

In Part III we reduce the equations derived in Part II to a form similar to that obtained formerly.[1] They turn out to be, in fact, identical, except for some of the self-energy terms, and one of the terms that occurs only in the $I=1/2$, $J=1/2$ state. Thus the self-energy corrections which will be derived in Part IV may be applied to the phase shifts calculated previously.[1] The results are discussed in Part V.

II. THE CINI METHOD APPLIED TO THE DTD EQUATIONS

We define a two-particle amplitude,

$$\Phi_{\rho\alpha}(x,y;t)=(\Psi_0^*(t)\psi_\rho(x)\phi_\alpha(y)\Psi(t)), \tag{1}$$

where $\Psi_0(t)$ is the interacting vacuum state in the interaction representation, $\Psi(t)$ is the real state vector in the interaction representation, and $\psi_\rho(x)$ and $\phi_\alpha(y)$ are interaction representation operators for the nucleon and meson fields, respectively. x and y need not be on the spacelike surface t. $\Psi(t)$ and $\Psi_0(t)$ satisfy the Schrödinger equation,

$$id\Psi(t)/dt=H_I(t)\Psi(t). \tag{2}$$

It is a consequence of Eqs. (1) and (2) that

$$\Phi_{\rho\alpha}(x,y;t)-\Phi_{\rho\alpha}(x,y;-\infty)=-\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt'' \times(\Psi_0^*(t'')[H_I(t''),[H_I(t'),\psi_\rho(x)\phi_\alpha(y)]]\Psi(t'')), \tag{3}$$

if one requires that the initial state $\Psi(-\infty)$ be a state with one meson present.

We choose Eq. (3) as the fundamental equation for our system, because it will yield results in a simple way which are equivalent to the lowest-order Tamm-Dancoff approximation. The Tamm-Dancoff approximation as applied to Eq. (3) may be stated in the following manner. One arranges the operators occurring

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[1] Bethe, Dyson, *et al.*, Phys. Rev. (to be published); M. Fubini, Nuovo cimento **10**, 564 (1953).

[2] F. J. Dyson, Phys. Rev. **90**, 994 (1953). We shall call Dyson's modification of the Tamm-Dancoff theory, the DTD theory.

[3] F. J. Dyson, Phys. Rev. **91**, 421, 1543 (1953).

[4] M. Cini, Nuovo cimento **10**, 526 (1953).

in the integrand in the normal order (absorption operators to the right of emission operators), yielding a sum of terms of the general form of $\Phi(x, y; t)$, possibly containing more than two field operators. We then drop all the terms except those having exactly the form of $\Phi(x, y; t)$. That this procedure is equivalent to the usual Tamm-Dancoff approximation may be seen in the following way. An array of states can be constructed (Fig. 1) which helps the enumeration of the nonzero matrix elements of the interaction Hamiltonian H_I. Since H_I contains two nucleon field operators and one meson operator, the nonzero elements are only those between states which differ by zero or two nucleons, and by one meson. The lines in Fig. 1 represent matrix elements which connect the "ground state" (1 meson, 1 nucleon in state Ψ in excess of those in Ψ_0) to other states. It is clear that our approximation is equivalent to the Tamm-Dancoff approximation which restricts the allowed states of the system to the cross-hatched circles in Fig. 1.

When the above procedure has been carried out, we shall be left with an integral equation for $\Phi_{\rho\alpha}(x, y; t)$ similar in appearance to the lowest-order Bethe-Salpeter equation for the meson-nucleon system, with the important difference that our field operators are in the interaction representation, and their time dependence is known.

Inserting

$$H_I(x_0)=\int d^3x H_I(x),$$

where

$$H_I(x)=iG\bar{\psi}(x)\gamma_5\tau_\alpha\phi_\alpha(x)\psi(x), \tag{4}$$

into Eq. (3), we obtain

$$\begin{aligned}\Phi_{\rho\alpha}(x, y; t)-\Phi_{\rho\alpha}(x, y; -\infty)=G^2\int_{-\infty}^{t}dz\int_{-\infty}^{z_0}dw \\ \times(\Psi_0^*(w_0)[\bar{\psi}(w)\gamma_5\tau_\beta\phi_\beta(w)\psi(w), \\ [\bar{\psi}(z)\gamma_5\tau_\delta\phi_\delta(z)\psi(z), \psi_\rho(x)\phi_\alpha(y)]]\Psi(w_0)),\end{aligned} \tag{5}$$

where

$$\int_{-\infty}^{t} dz \text{ means } \int_{-\infty}^{t} dz_0\int d^3\mathbf{z}.$$

The double commutator in Eq. (5) may, without much difficulty, be put into normal order; if one contains only those terms which correspond to a one-meson, one-nucleon amplitude, it becomes

$$\begin{aligned}&-3iS(x-z)\gamma_5[1+\epsilon(z-w)P][\langle\phi(w)\phi(z)\rangle_0 \\ &\qquad\times\langle\bar{\psi}(w)\psi(z)\rangle_0]\gamma_5\phi_\alpha(y)\psi(w) \qquad (6a)\\ &-2i\Delta(y-z)(1-P)\,\mathrm{Sp}[\gamma_5\langle\bar{\psi}(w)\psi(z)\rangle_0\gamma_5 \\ &\qquad\times\langle\psi(w)\bar{\psi}(z)\rangle_0]\phi_\alpha(w)\psi_\rho(x) \qquad (6b)\\ &+[\langle\phi(z)\phi(y)\rangle_0\langle\bar{\psi}(z)\psi(x)\rangle_0+\langle\phi(y)\phi(z)\rangle_0\langle\psi(x)\bar{\psi}(z)\rangle_0] \\ &\qquad\times\gamma_5(1+\epsilon(z-w)P)\langle\bar{\psi}(w)\psi(z)\rangle_0\gamma_5\tau_\alpha\tau_\beta\phi_\beta(w)\psi(w) \qquad (6c)\\ &+i\Delta(y-z)[\langle\psi(w)\psi(z)\rangle_0\gamma_5\langle\psi(w)\bar{\psi}(z)\rangle_0 \\ &\qquad-\langle\psi(x)\bar{\psi}(w)\rangle_0\gamma_5\langle\bar{\psi}(z)\psi(w)\rangle_0]\gamma_5\tau_\beta\tau_\alpha\phi_\beta(w)\psi(z) \qquad (6d)\\ &-iS(x-z)[\langle\phi(w)\phi(y)\rangle_0\gamma_5\langle\bar{\psi}(w)\psi(z)\rangle_0 \\ &\qquad+\langle\phi(y)\phi(w)\rangle_0\gamma_5\langle\psi(z)\bar{\psi}(w)\rangle_0]\gamma_5\tau_\beta\tau_\alpha\phi_\beta(z)\psi(w). \qquad (6e)\end{aligned}$$

By application of the usual rules for drawing the Feynman diagram corresponding to a given S-matrix element, the terms (6a)–(6e) may be seen to correspond to the time-ordered graphs drawn in Fig. 2. It will be noticed that the graphs (a), (b), and (c) occur alone, without their negative-energy intermediate state analogs, and that the vacuum self-energy fails to appear at all, in contrast to what one finds with the old Tamm-Dancoff method.[2] The reason that these terms do not appear is that the structure of the commutator in Eq. (5) is such that any Feynman diagram representing parts of it must have a line connecting z with x or y, or both.

If we next insert the propagation functions defined by[5]

$$\begin{aligned}P\langle\phi(x)\phi(y)\rangle_0&=\tfrac{1}{2}\Delta_F(x-y),\\ \epsilon(x-y)P\langle\psi(x)\bar{\psi}(y)\rangle_0&=-\tfrac{1}{2}S_F(x-y),\\ \langle\psi(z)\bar{\psi}(y)\rangle_0&=-iS^+(z-y)\\ &=-iS_A^+(z-y)\quad z_0<y_0\\ &=+iS_R^+(z-y)\quad z_0>y_0,\\ \langle\bar{\psi}(y)\psi(z)\rangle_0&=-iS_A^-(z-y)\quad z_0<y_0\\ &=+iS_R^-(z-y)\quad z_0>y_0,\\ \langle\phi(z)\phi(y)\rangle_0&=+i\Delta_A^+(z-y)\quad z_0<y_0\\ &=-i\Delta_R^+(z-y)\quad z_0>y_0,\\ \langle\phi(y)\phi(z)\rangle_0&=-i\Delta_A^-(z-y)\quad z_0<y_0\\ &=+i\Delta_R^-(z-y)\quad z_0>y_0\end{aligned} \tag{7}$$

for the vacuum expectation values in (6d) and (6e) which depend on field operators at w and z, we may extend the integration over w in (5) to all space-time, because (6a), (6b), and (6c) are already zero when $z_0<w_0$. The result of this substitution is that (5)

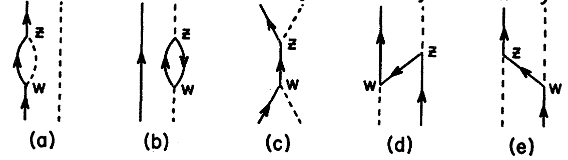

FIG. 2. Feynman diagrams corresponding to the terms of Eq. (6). Time increases vertically.

[5] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

becomes

$$\Phi_\alpha(x,y;t)-\Phi_\alpha(x,y;-\infty)=iG^2\int_{-\infty}^{t}dz\int_{-\infty}^{\infty}dw$$
$$\times\{[3S(x-z)\gamma_5\Delta^-(z-w)S^-(z-w)$$
$$+\tfrac{3}{4}S(x-z)\gamma_5\Delta_F(z-w)S_F(z-w)\gamma_5]\Phi_\alpha(w,y;w_0)$$
$$+\Delta(y-z)\,\mathrm{Sp}[2\gamma_5S^-(z-w)\gamma_5S^+(w-z)$$
$$+\tfrac{1}{2}\gamma_5S_F(z-w)\gamma_5S_F(w-z)]\Phi_\alpha(x,w;w_0)$$
$$+\tfrac{1}{2}i[-\Delta^-(y-z)S^-(x-z)+\Delta^+(y-z)S^+(x-z)]\gamma_5$$
$$\times[2iS^-(z-w)+S_F(z-w)]\gamma_5\tau_\alpha\tau_\beta\Phi_\beta(w,w;w_0)$$
$$+\Delta(y-z)[-S^-(x-w)\gamma_5S_A{}^+(w-z)$$
$$+S^+(x-w)\gamma_5S_A{}^-(w-z)]\gamma_5\tau_\beta\tau_\alpha\Phi_\beta(z,w;w_0)$$
$$+S(x-z)[-\Delta^-(y-w)\gamma_5S_R{}^-(z-w)$$
$$+\Delta^+(y-w)\gamma_5S_R{}^+(z-w)]\gamma_5\tau_\beta\tau_\alpha\Phi_\beta(w,z,w_0)\}.\quad(8)$$

III. REDUCTION TO THE NONCOVARIANT FORM

It will be noticed from the form of $\Phi(x,y;t)$ that it is nonzero in four different physical situations.[3] Specifically, decomposing ψ and ϕ into their positive and negative energy parts, one may write

$$\Phi(x,y;t)=\Phi^{++}+\Phi^{+-}+\Phi^{-+}+\Phi^{--}(x,y;t),\quad(9)$$

where, for example,

$$\Phi_{\rho\alpha}{}^{-+}(x,y;t)=(\Psi_0{}^*(t)\phi_\alpha{}^-(x)\psi_\rho{}^+(y)\Psi(t)).$$

This decomposition may be done explicitly here because the time dependence of the interaction representation operators ϕ and ψ is known. A nonzero Φ^{++} corresponds to having a meson and a nucleon in the state Ψ in excess of those in Ψ_0; Φ^{-+} to one more nucleon and one less meson, Φ^{+-} to one more meson and one less nucleon, Φ^{--} to one less meson and one less nucleon in Ψ than in Ψ_0. We will now operate from the left on Eq. (9) with a positive energy projection operator for both nucleons and mesons, yielding a somewhat simplified equation:

$$\Phi_\alpha{}^{++}(x,y;t)-\Phi_\alpha{}^{++}(x,y;-\infty)=iG^2\int_{-\infty}^{t}dz\int d\xi$$
$$\times\{[3S^+(x-z)\gamma_5\Delta^-(\xi)S^-(\xi)\gamma_5$$
$$+\tfrac{3}{4}S^+(x-z)\Delta_F(\xi)\gamma_5S_F(\xi)\gamma_5][\Phi_\alpha{}^{++}+\Phi_\alpha{}^{-+}]$$
$$\times(z-\xi,y;z_0-\xi_0)+\Delta^+(y-z)\,\mathrm{Sp}\{2\gamma_5S^-(\xi)\gamma_5S^+(-\xi)$$
$$-\tfrac{1}{2}\gamma_5S_F(\xi)\gamma_5S_F(-\xi)\}[\Phi_\alpha{}^{++}+\Phi_\alpha{}^{+-}](x,z-\xi;z_0-\xi_0)$$
$$+\tfrac{1}{2}i\Delta^+(y-z)S^+(x-z)\gamma_5[2iS^-(\xi)+S_F(\xi)]$$
$$\times\gamma_5\tau_\alpha\tau_\beta\Phi_\beta(z-\xi,z-\xi;z_0-\xi_0)+\Delta^+(y-z)S^+$$
$$\times(x-z+\xi)\gamma_5S_A{}^-(-\xi)\gamma_5\tau_\beta\tau_\alpha\Phi_\beta(z,z-\xi;z_0-\xi_0)$$
$$+S^+(x-z)\Delta^+(y-z+\xi)\gamma_5S_R{}^+(\xi)$$
$$\times\gamma_5\tau_\beta\tau_\alpha\Phi_\beta(z-\xi,z;z_0-\xi_0)\}.\quad(10)$$

This is exactly the equation which would have been obtained if we had started with Φ^{++} in Eq. (3). We will next drop all terms on the right-hand side of Eq. (10) which contain either meson or nucleon negative-energy amplitudes. This is consistent with the Tamm-Dancoff approximation; these components of the wave function are not among those connected on Fig. 1 to the ground state.

The terms in Eq. (10) containing $\Delta^-(\xi)S^-(\xi)$, $S^-(\xi)S^+(-\xi)$, and $S^-(\xi)$ will vanish in the integration over ξ_0. After dropping the negative-energy amplitudes, we take the Fourier transform of Eq. (10). The expansion of Φ^{++} is found to be

$$\Phi_\alpha{}^{++}(x,y;t)=\int d^3\mathbf{q}\int d^3\mathbf{r}(\omega_\mathbf{q}E_\mathbf{r})^{-\frac{1}{2}}$$
$$\times e^{i(r\cdot x+q\cdot y)}u(\mathbf{r})a(\mathbf{r}u,\mathbf{q}\alpha;t),\quad(11)$$

where a sum is implied over positive-energy nucleon spinors $u(\mathbf{r})$ (normalized so that $\bar{u}u=u^*\gamma_4u=1$), $r\cdot x=\mathbf{r}\cdot\mathbf{x}-r_0x_0$, $E_\mathbf{r}=(M^2+\mathbf{r}^2)$, $\omega_\mathbf{q}=(\mu^2+\mathbf{q}^2)$ (M and μ are the masses of the nucleon and meson), and

$$a(\mathbf{r}u,\mathbf{q}\alpha;t)=(\Psi_0{}^*(t)b_{\mathbf{r}u}a_{\mathbf{q}\alpha}\Psi(t)).\quad(12)$$

Annihilation operators for the nucleon and meson, respectively, are $b_{\mathbf{r}u}$ and $a_{\mathbf{q}\alpha}$. By inserting the time dependence of the state vectors,

$$\Psi(t)=e^{-i(E-H_0)t}\Psi(0),\qquad \Psi_0(t)=e^{-i(E_0-H_0)t}\Psi_0(0),\quad(13)$$

where H_0 is the noninteraction Hamiltonian, E is the total energy of the system, and E_0 is the vacuum state energy, Eq. (12) becomes ($\epsilon=E-E_0$):

$$a(\mathbf{r}u,\mathbf{q}\alpha;t)=e^{-i(\epsilon-E_\mathbf{r}-\omega_\mathbf{q})t}a(\mathbf{r}u,\mathbf{q}\alpha),\quad(14)$$

where $a(\mathbf{r}u,\mathbf{q}\alpha)=(\Psi_0{}^*(0)b_{\mathbf{r}u}a_{\mathbf{q}\alpha}\Psi(0))$. With this notation, Eq. (10) written in momentum space in the center-of-mass system becomes ($\mathbf{p}$=momentum of nucleon; $\Delta=\epsilon-E_\mathbf{p}-\omega_\mathbf{p}$):

$$a(\mathbf{p}u,-\mathbf{p}\alpha)=-\tfrac{3}{4}iG^2\frac{M}{E_\mathbf{p}}\int d\xi(\bar{u}(\mathbf{p})\gamma_5S_F(\xi)\gamma_5u(\mathbf{p}))\Delta_F(\xi)$$
$$\times e^{-ip\cdot\xi+i(\Delta+E_\mathbf{p})\xi_0}a(\mathbf{p}u,-\mathbf{p}\alpha)-\tfrac{1}{4}iG^2\frac{1}{\omega_\mathbf{p}}\int d\xi$$
$$\times\mathrm{Sp}[\gamma_5S_F(\xi)\gamma_5S_F(-\xi)]e^{ip\cdot\xi+i(\Delta+\omega_\mathbf{p})\xi_0}a(\mathbf{p}u,-\mathbf{p}\alpha)$$
$$+\frac{M^2G^2}{16\pi^3}\int\frac{d^3\mathbf{k}}{(E_\mathbf{p}\omega_\mathbf{p}E_\mathbf{k}\omega_\mathbf{k})^{\frac{1}{2}}}\left\{\left[\frac{(\bar{u}(\mathbf{p})\gamma_5\Lambda_+(0)\gamma_5v(\mathbf{k}))}{M(M-\epsilon)}\right.\right.$$
$$\left.+\frac{(\bar{u}(\mathbf{p})\gamma_5\Lambda_-(0)\gamma_5v(\mathbf{k}))}{M(M+\epsilon)}\right]\tau_\alpha\tau_\beta$$
$$+\left[\frac{(\bar{u}(\mathbf{p})\gamma_5\Lambda_-(-\mathbf{p}-\mathbf{k})\gamma_5v(\mathbf{k}))}{E_{\mathbf{p}+\mathbf{k}}(E_\mathbf{p}+E_{\mathbf{p}+\mathbf{k}}+E_\mathbf{k}-\epsilon)}\right.$$
$$\left.\left.+\frac{(\bar{u}(\mathbf{p})\gamma_5\Lambda_+(\mathbf{p}+\mathbf{k})\gamma_5v(\mathbf{k}))}{E_{\mathbf{p}+\mathbf{k}}(E_{\mathbf{p}+\mathbf{k}}+\omega_\mathbf{k}+\omega_\mathbf{q}-\epsilon)}\right]\tau_\beta\tau_\alpha\right\}a(\mathbf{k}v,-\mathbf{k}\beta).\quad(15)$$

Here, for convenience in the next section, we have left the self-energy parts in terms of invariant functions. Except for these terms, and for the terms involving the energy denominator $1/(M+\epsilon)$, which contributes only to the states for which $I=J=1/2$, this equation is identical to that derived by the old Tamm-Dancoff method.[1] We may therefore evaluate these self-energy parts, and use them to find corrections to the phase shifts previously calculated, except for the $I=J=1/2$ states.

IV. RENORMALIZATION

Let $\Omega(\xi)=\gamma_5 S_F(\xi)\gamma_5\Delta_F(\xi)$, and

$$\Sigma(\xi)=\mathrm{Spur}\{\gamma_5 S_F(\xi)\gamma_5 S_F(-\xi)\}. \tag{16}$$

There are the well-known nucleon and meson self-energy propagation factors from second-order perturbation theory, whose strong singularity at $\xi=0$ causes the divergence of their momentum-space integrals. We shall follow the usual relativistic subtraction procedures to extract their finite parts. First take the 4-dimensional Fourier transform of Ω and Σ,

$$\Omega(p)=\int e^{-ip\cdot\xi}\Omega(\xi)d^4\xi,$$
$$\Sigma(\kappa)=\int e^{-i\kappa\cdot\xi}\Sigma(\xi)d^4\xi. \tag{17}$$

These are now relativistically invariant functions of the 4-vector momenta p and κ. More specifically, $\Omega(p)$ is a function of $p\cdot\gamma$ only, since $p\cdot\gamma$ is the simplest invariant matrix function of p. Similarly $\Sigma(\kappa)$ is a function of κ^2 only. These facts may be easily verified by explicit calculation.

In the usual way,[6] we set the observable (renormalized) parts of the self-energy operators equal to the following expressions:

$$\Omega_R(p\cdot\gamma)=\Omega(p\cdot\gamma)-\Omega(iM)-(p\cdot\gamma-iM)\left[\frac{\partial\Omega}{\partial(p\cdot\gamma)}\right]_{p\cdot\gamma=iM}, \tag{18}$$

$$\Sigma_R(\kappa^2)=\Sigma(\kappa^2)-\Sigma(-\mu^2)-(\kappa^2+\mu^2)\left[\frac{\partial\Sigma}{\partial\kappa^2}\right]_{\kappa^2=-\mu^2}. \tag{19}$$

Nucleon Self-Energy Renormalization

The subtractions indicated in (18) may be carried out by following exactly the procedure that Karplus and Kroll[7] used for the electron self-energy,

$$\Omega(p\cdot\gamma)=\int e^{-ip\cdot\xi}\gamma_5 S_F(\xi)\gamma_5\Delta_F(\xi)d^4\xi$$
$$=-\frac{4i}{(2\pi)^4}\int d^4s\,\gamma_5\frac{s\cdot\gamma+p\cdot\gamma+iM}{(p+s)^2+M^2}\gamma_5\frac{1}{s^2+\mu^2}. \tag{20}$$

[6] P. T. Matthews, Phil. Mag. **41**, 185 (1950).
[7] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

We apply the formula

$$\frac{1}{ab}=\int_0^1\frac{du}{[au+b(1-u)]^2} \tag{21}$$

to (20) to obtain

$$\Omega(p\cdot\gamma)=\frac{4i}{(2\pi)^4}\int d^4s\int_0^1 du\times\frac{s\cdot\gamma+p\cdot\gamma-iM}{\{[(p+s)^2+M^2]u+[s^2+\mu^2](1-u)\}^2}. \tag{22}$$

The denominator of the integrand of (22) may be written $[(s+pu)^2+\Lambda^2]^2$, where $\Lambda^2=p^2(u-u^2)+M^2u+\mu^2(1-u)$. We expand the integrand in a power series in pu; keeping only the first two terms, because the higher ones may be transformed into vanishing surface integrals. This casts (22) into the form

$$\Omega(p\cdot\gamma)=\frac{4i}{(2\pi)^4}\int d^4s\int_0^1 du\left\{\frac{s\cdot\gamma+p\cdot\gamma(1-u)-iM}{[s^2+\Lambda^2]^2}+pu\cdot\frac{\partial}{\partial s}\frac{s\cdot\gamma}{[s^2-\Lambda^2]^2}\right\}. \tag{23}$$

Upon carrying out the differentiation in (23), and making use of the facts that

$$\int f(t^2)t\cdot\gamma d^4t=0,$$
$$\int f(t^2)(a\cdot t)(b\cdot t)d^4t=\tfrac{1}{4}(a\cdot b)\int f(t^2)t^2d^4t, \tag{24}$$

we find the following form for Ω:

$$\Omega(p\cdot\gamma)=\frac{4i}{(2\pi)^4}\int d^4t\int_0^1 du\left(\frac{p\cdot\gamma-iM}{(t^2+\Lambda^2)^2}-\frac{ut^2}{(t^2+\Lambda^2)^3}\right). \tag{25}$$

The subtraction procedures (18) may be applied to this equation, and, making use of[8]

$$\int d^4t\left(\frac{1}{(t^2+\Lambda^2)^2}-\frac{1}{(t^2+\Lambda_0^2)^2}\right)=-i\pi^2\log\left|\frac{\Lambda^2}{\Lambda_0^2}\right|$$
$$=-i\pi^2\int_0^1 dv\frac{\Lambda^2-\Lambda_0^2}{\Lambda_0^2+(\Lambda^2-\Lambda_0^2)v}, \tag{26}$$

[8] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

one obtains

$$\Omega_R(\rho\cdot\gamma)=\frac{1}{(2\pi)^2}(\rho\cdot\gamma-iM)^2\int_0^1 du\int_0^1 dv \times\frac{u-u^2}{\Lambda_0^2+(\rho^2+M^2)(u-u^2)v} \times\left[(\rho\cdot\gamma+iM)\left(1-u-\frac{2M^2u^2(1-u)v}{\Lambda_0^2}\right)-iMu\right], \quad (27)$$

where $\Lambda_0^2=M^2u^2+\mu^2(1-u)$.

It is worth-while noting that, although it would greatly simplify the approximate evaluation of (27) to put $\mu^2=0$, it may not be done, because Ω_R is not an analytic function of μ^2 at $\mu^2=0$.

Meson Self-Energy Renormalization

$$\Sigma(\kappa)=\int e^{-i\kappa\cdot\xi}\,\mathrm{Sp}\{\gamma_5 S_F(\xi)\gamma_5 S_F(-\xi)\}d^4\xi = \frac{1}{4\pi^4}\int d^4p\,\mathrm{Sp}\left\{\gamma_5\frac{p\cdot\gamma+iM}{p^2+M^2}\gamma_5\frac{(p-\kappa)\cdot\gamma+iM}{(p-\kappa)^2+M^2}\right\}. \quad (28)$$

By evaluating the spur in (28) one may write

$$\Sigma(\kappa)=-\frac{1}{8\pi^4}\int\frac{d^4p}{(p-\kappa)^2+M^2}-\frac{1}{8\pi^4}\int\frac{d^4p}{p^2+M^2} +\frac{\kappa^2}{8\pi^4}\int\frac{d^4p}{[(p-\kappa)^2+M^2][p^2+M^2]}. \quad (29)$$

The second term of (29) clearly gives no contribution to Σ_R, since it will drop out in the first subtraction in (19). This is also true of the first term, as may be seen by expanding the integrand, as was done in (23). We find

$$\int\frac{d^4p}{(p-\kappa)^2+M^2}=\int\frac{d^4p}{p^2+M^2}-\kappa^2M^2\int\frac{d^4p}{(p^2+M^2)^3},$$

which will completely drop out in the subtraction (19). Therefore the observable parts of Σ are all contained in the third term of (29), Σ_3.

Application of (21) to Σ_3 yields

$$\Sigma_3=\frac{\kappa^2}{8\pi^4}\int_0^1 du\int d^4s\frac{1}{[s^2+u(1-u)\kappa^2+M^2]^2}, \quad (30)$$

after a change of variable $s=p-u\kappa$. We define

$$J(\lambda)=\frac{\kappa^2}{8\pi^4}\int_0^1 du\int d^4s\left(\frac{1}{[s^2+u(1-u)\kappa^2+M^2]^2} -\frac{1}{[s^2+u(1-u)\kappa^2+\lambda^2]^2}\right), \quad (31)$$

where λ is a very large mass. By using (26), (31) becomes

$$J(\lambda)=-\frac{i}{8\pi^2}\kappa^2(M^2-\lambda^2)\int_0^1 du\int_0^1 dw \times\frac{1}{\lambda^2+\kappa^2(1-u)u+(M^2-\lambda^2)w}. \quad (32)$$

The subtractions of (19) may be conveniently performed on this expression. When one does them, and lets $\lambda^2\to\infty$, (32) is transformed to

$$\Sigma_R(\kappa^2)=\frac{1}{8\pi^2}(\kappa^2+\mu^2)^2\int_0^1 du\int_0^1 dw\frac{u(1-u)}{M^2-u(1-u)\mu^2} \times\left(\frac{\kappa^2u(1-u)w}{u(1-u)w(\kappa^2+\mu^2)+M^2-u(1-u)\mu^2}-1\right), \quad (33)$$

which may be quite accurately approximated by letting $\mu^2=0$ under the integral sign. Equation (33) then reduces to

$$\Sigma_R=-\frac{i}{8\pi^2}(\kappa^2+\mu^2)^2\int_0^1 du\frac{1}{\kappa^2}\log\left|1+u(1-u)\frac{\kappa^2}{M^2}\right|. \quad (34)$$

V. CONCLUSION

The integrals in (27), (34) are elementary, and may be carried out without difficulty. The four-vector arguments ρ and κ of Ω and Σ are replaced by the quantities which occur in the integral Eq. (15), namely $(\mathbf{p}, \Delta+E_\mathrm{p})$ and $(-\mathbf{p}, \Delta+\omega_\mathrm{p})$, respectively. Since Ω is surrounded in (15) by $\bar{u}(\mathbf{p})$ and $u(\mathbf{p})$ and $(\mathbf{p}\cdot\gamma+i\gamma_4E_\mathrm{p}-iM)u(\mathbf{p})=0$, the Dirac matrices $(\rho\cdot\gamma-iM)^2$ and $(\rho\cdot\gamma-iM)^2\times(\rho\cdot\gamma+iM)$, which occur in (27), become $-\Delta^2$ and $-i\Delta E_\mathrm{p}(\Delta+2E_\mathrm{p})/M$.

By collecting terms which multiply $a(\mathbf{p}u, -\mathbf{p}\alpha)$, Eq. (15) may be rewritten in the form

$$a(\mathbf{p}u,-\mathbf{p}\alpha)=(F^2/4\pi)\times(\text{interaction terms}), \quad (35)$$

where the "interaction terms" are all those on the right side of (15), except the first two (self-energy terms), divided by $G^2/4\pi$; and

$$\frac{F^2}{4\pi}=\frac{G^2}{4\pi}\left(1+\frac{3iG^2}{4}\frac{M}{E_\mathrm{p}}\Omega+\frac{iG^2}{4\omega_\mathrm{p}}\Sigma\right)^{-1}. \quad (36)$$

Equation (35), therefore, absorbs the self-energy effects into the coupling "constant" $F^2/4\pi$, which now depends on $|\mathbf{p}|$ and ϵ.

The asymptotic expressions for Ω and Σ for $|\mathbf{p}|/M\to\infty$ may be easily derived. When they are inserted into (36) it becomes

$$\frac{F^2}{4\pi}=\frac{G^2}{4\pi}\left(1-\frac{3G^2}{32\pi^2}\log\left|\frac{\mathbf{p}}{M}\right|-\frac{G^2}{32\pi^2}\frac{\epsilon}{|p|}\log\left|\frac{\mathbf{p}}{M}\right|\right)^{-1}. \quad (37)$$

The first term within the bracket arises from the nucleon self-energy, and dominates for large $\mathbf{p}$. $F^2/4\pi$ is clearly negative for $\mathbf{p}$ large enough.[9] For $\Delta=0$ it is by definition positive and equal to $G^2/4\pi$. Therefore, for some $\mathbf{p}=\mathbf{k}$ for which $\Delta<0$ $F^2/4\pi$ has a pole. A pole in $F^2/4\pi$ implies a pole in the wave function $a(\mathbf{p}u, -\mathbf{p}\alpha)$ or, in other words, an incoming or outgoing wave of momentum $\mathbf{k}$ in the coordinate-space wave function. It would be possible to interpret the pole as an inelastic scattering if $2|\mathbf{k}|<\epsilon$, but a rough numerical calculation shows this condition not to be fulfilled for laboratory energies (values of ϵ from M to $2M$). If $2|\mathbf{k}|>\epsilon$ an interpretation as inelastic scattering is not possible, because there is too much momentum for the fixed amount of energy in the system, and at least one of the scattered particles would have a spacelike energy-momentum four vector, or an imaginary mass. No reconciliation oft his nonsensical prediction with reality has yet been made, and at present it must be considered as raising serious doubt about the consistency of the Tamm-Dancoff approximation.

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[9] R. H. Dalitz (private communication) has pointed out an error made by the author in the evaluation of these terms which radically altered their behavior. His information has prevented the possible publication of qualitatively wrong results, and is greatly appreciated.

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Mesonic Corrections to the Quadrupole Moment of the Deuteron*

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Mesonic corrections to the quadrupole moment of the deuteron are calculated by means of the method of Tamm and Dancoff. Only the two-nucleon and two-nucleon—one-meson amplitudes are included. The first is associated with a phenomenological wave function for the deuteron, the second yielding a correction to the quadrupole moment. A correction exists even in neutral scalar theory, and for pseudoscalar (pseudoscalar) symmetric theory the result in adiabatic approximation to second order is $\Delta Q=3.1$ percent or $\Delta Q=-0.7$ percent, depending on choice of wave functions (assuming $g^2/4\pi=10$). Contributions from multiple meson amplitudes are examined, and for a hard-core deuteron function they are shown to contribute only slightly.

I. INTRODUCTION

A PHENOMENOLOGICAL theory of the deuteron consists of choosing an arbitrary potential which is to act between two nucleons and only depends explicitly on nucleon variables, and then calculating the binding energy of the deuteron, the quadrupole moment, the n-p triplet scattering length, and the n-p triplet effective range. These quantities are then compared with experiment, and if they all agree, within experimental error, then the potential is an acceptable one.

The phenomenological theory is not complete in that it ignores the coordinates of the mesons associated with the two-nucleon system. It does not, however, completely ignore the mesons, in so far as they contribute to the potential acting between nucleons. The question arises as to how valid the phenomenological theory is, or, equivalently, how much of a correction will the inclusion of meson coordinates make in which potentials are considered acceptable.

It is not known how to obtain from meson theory a quantitatively correct answer to this question. However, as we shall see, the only one of the above mentioned quantities we need calculate with meson theory is the quadrupole moment of the deuteron. Because this is an outside quantity one should be able to obtain a very reasonable estimate of the magnitude of the effect, and in view of the rather large estimates which appear in the literature,[1,2] it was felt that a re-examination of the problem was necessary.

The results of a field-theoretic calculation of the two-nucleon system may be expressed in terms of a wave functional in Fock space. The method of Tamm[3] and Dancoff,[4] in lowest approximation, consists of equating to zero all amplitudes of the functional other than the two-nucleon (2,0), and two-nucleon—one-meson (2,1) amplitudes. We shall make this lowest approximation first, and examine its validity in Sec. IV.

We may associate the (2,0) amplitude with a phenomenological wave function. It satisfies a Schroedinger equation in which the potential term arises from the amplitudes involving mesons, but does not contain meson coordinates. Furthermore, the binding energy of the deuteron, and the n-p scattering length, and the n-p effective range may be obtained from the (2,0) amplitude alone. The quadrupole moment differs in that it

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