

frequency, but at least one pair of spectral components will coalesce to a single line along this locus, making it readily identifiable experimentally.

We note finally the methods for determining η from experimental Zeeman data. One set of measurements for this purpose is the location of $\theta_0(\phi)$ of (30). The principal coordinate system for the field gradient under investigation is determined from the symmetry that the Zeeman splittings show as functions of the orientation of the magnetic field and from the fact that $\theta_0(\phi)$ has its maximum and minimum values in the xz and the yz planes, respectively. η is found from the formula:

$$\eta = \frac{3[\sin^2\theta_0(0^\circ) - \sin^2\theta_0(90^\circ)]}{\sin^2\theta_0(0^\circ) + \sin^2\theta_0(90^\circ)}. \quad (31)$$

As we have seen, this method of determining η can be adapted for use with magnetic fields of any magnitude. In this procedure the only precision measurements are those of the direction of the magnetic field; the spectrometer serves only as an indicator.

The other method that our theory provides for determining η requires the measurement of the amounts of splitting when the magnetic field is parallel to one of the principal axes of the field gradient in question. The differences in frequency between the Zeeman components and the unsplit quadrupole resonance, expressed in energy units, are found from (19):

	α	β	
$H\ z$	2ξ	$4\rho\xi$	
$H\ x$	$2(1-\eta)\rho\xi$	2ξ	(32)
$H\ y$	2ξ	$2(1+\eta)\rho\xi$	

η is therefore found directly from a measurement of the splitting of the proper pair of Zeeman components along one of the principal axes. The magnetic field can be calibrated by measuring the splitting for one of the other pairs.

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Particle Derivation of Nuclear Rotation Properties Associated with a Surface Wave

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The dynamical conception of nuclear rotation in terms of a surface wave on a droplet of irrotational fluid has achieved some success in spite of the great differences between nucleons and the particles of a normal classical fluid. As a justification for the simplifying assumption of irrotational fluid flow, the collective rotational energy is here derived from a suitable set of nucleon wave functions in the approximation in which there is a rotating distortion, slow compared with the internal nucleon motions. The wave functions are those of a three-dimensional harmonic oscillator that is made anisotropic by having the force constant along one axis different from those along the other two in a rotating cartesian coordinate system. For the case of steady rotation about a fixed axis, the perturbation problem with first-order wave functions leads to a second-order rotational energy which agrees with the droplet-model result. The observed level spacings appear to lie between this result and that of a rigid rotator, and the discrepancy is probably to be attributed to higher orders. The result is also derived by another method without introducing a steady rotation.

INTRODUCTION

THE spectacular success of the (jj) coupling shell model,¹ in accounting particularly for the magic numbers and the succession of ground-state angular momenta J of the moderately heavy nuclei, leaves little doubt that the extrashellular nucleons have an independence of motion quite different from that of the molecules in a classical fluid. These particular successes were first achieved on the assumptions that the extrashellular nucleons move in a spherically symmetric potential, have large spin-orbit coupling, and interact

with one another besides. The equally impressive success of the introduction of a nonspherical core,^{2,3} in accounting for the large quadrupole moments and especially the ratios of certain low excitation energies,⁴ gives the impression that the degree of freedom permitting distortion of the core is important to the external features of most nuclei without radically modifying the internal coupling of the nucleons as

² J. Rainwater, *Phys. Rev.* **79**, 432 (1950).

³ A. Bohr, *Phys. Rev.* **81**, 134, 331 (1951); *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* **26**, No. 14 (1952). A. Bohr and B. R. Mottelson, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* **27**, No. 16 (1953); D. L. Hill and J. A. Wheeler, *Phys. Rev.* **89**, 1102 (1953).

⁴ F. Asaro and I. Perlman, *Phys. Rev.* **91**, 763 (1953).

¹ M. G. Mayer, *Phys. Rev.* **75**, 1969 (1949); **78**, 16, 22 (1950). Haxel, Jensen, and Suess, *Phys. Rev.* **75**, 1766 (1949); *Naturwiss.* **36**, 155 (1949).

calculated in the original (jj) coupling scheme which neglected this collective degree of freedom. The distortion is assumed to arise from the pressures exerted from within on the surface of a bag of liquid representing the core by the nucleons not belonging to closed shells. It is true that many of the results originally calculated by treating the specific nuclear interactions between the nucleons not belonging to closed shells can be derived by the "strong coupling" assumption³ that they interact only through their several interactions with a common surface, but it seems likely that their direct interaction is not so unimportant and that both types of interaction contribute to the observed results. In any case, separability into products of nucleon wave functions and strong spin-orbit coupling are needed to justify the individual-nucleon quantum numbers j , and if one chooses to think intuitively of a classical fluid he must consider the mean free path to be long compared with the size of the vessel.

One then has no continuum at all and the classical equations for a continuous fluid with small density fluctuations do not apply. Quantization does introduce a probability density that has one property of a classical fluid in that it obeys the equation of continuity, but it does not follow that it has other properties of usual classical fluids. In particular, one cannot assume low viscosity, for the concept of viscosity loses its meaning with such a long mean free path. There is thus no logical basis for assuming irrotational flow. One may feel that after quantization, in the limit for very slow distortion of the nucleon wave functions, there is no way for the distortion to excite the nucleons, and thus no mechanism for dissipation of energy, which may be intuitively associated with vanishing viscosity, but this conception merely suggests that one should treat quantitatively the departure from the limit of vanishingly slow distortion, as in the following calculation. It is thus unsatisfactory tacitly to assume that the equations of classical fluid and in particular of irrotational flow apply to nuclear matter.

The basic question that is raised by the classical analogy is how much mass motion is to be associated with a given motion of the boundary. A surface wave traveling around a liquid droplet has both angular momentum and kinetic energy, the amount depending classically on the depth of penetration of the wave motion as is calculable by assuming incompressible irrotational flow. The motion of the surface is transmitted to the internal volume elements by the sequence of pressures between the volume elements. A nucleus is perhaps more nearly analogous to a bag of rarified gas with a mean free path somewhat greater than the linear dimensions. The effect of a motion of an element of surface is transmitted mainly by rectilinear flight of the recoiling particles to various remote volume elements, and it has not been shown that the result is the same as with transmission by a sequence of pres-

ures. But this analogy, too, is inadequate because nucleon wavelengths are comparable to nuclear size.

So long as the nucleus or its core is given only a static deformation, the kinematic assumption of irrotational motion is unimportant and the liquid drop model provides a plausible and very useful tool. The energy of surface tension is properly associated with the smaller number of statistically distributed neighbors with which an element of nucleon density near the surface can interact (the interaction being fairly uniformly modified by saturation requirements for various nuclei), and the kinetic energy per unit volume remains constant for the nucleons moving within fixed boundaries, as in an incompressible fluid. This is about all that is used of the droplet model in Weizsäcker's "semiempirical" treatment⁵ of nuclear energies, in the Bohr-Wheeler treatment⁶ of stability of heavy nuclei against fission, and in Rainwater's discussion² of the large quadrupole moments.

The kinematic deformation of the nucleus becomes important in more detailed considerations of the fission process, and the dynamical problem associated with the rate of deformation is recognized by Hill and Wheeler,³ who, in the discussion of their Figs. 7 and 8, show that in the limit of a large nucleus, wherein a typical nucleon wave function has many nodes, the mass motion may be treated in terms of a quantum analog of a classical velocity potential and that the classical hydrodynamic result for irrotational flow may plausibly be taken over as an approximation. The dynamics of the deformation of the core are likewise important in Bohr's treatment of the circulating surface waves and in Bohr and Mottelson's discussion of the coupling of nucleon motions to these waves,³ and for simplicity of discussion they assume the equivalence to classical irrotational fluid flow.

The oscillator model provides a somewhat more systematic attempt to approximate the mechanical system than does the droplet model, and is similarly capable of describing a deformation in shape, but with more detailed accounting for the behavior of the nucleons. It introduces a zeroth-order potential V_0 idealized as a three-dimensional harmonic oscillator potential (not necessarily spherically symmetric) for each nucleon, and considers the perturbation to be the difference between the sum of the two-body interactions and this V_0 . The oscillator model thus provides an alternative to the droplet treatment and by demonstrating some of the qualitative phenomena accompanying distortion provides an interesting critique of the droplet procedure.

In a very interesting comment on the oscillator model, Gallone and Salvetti⁶ show that if one fills a shell (without spin) so that the sum of the three oscillator quantum numbers is less than or equal to k for as many nucleons as possible, the nucleus is stiffer against de-

⁵ N. Bohr and J. A. Wheeler, *Phys. Rev.* **56**, 426 (1939).

⁶ S. Gallone and C. Salvetti, *Nuovo cimento* **10**, 145 (1953).

formation than the droplet model supposes, but that a fairly large static deformation favors transition of some of the nucleons to states ($k+1$), thus softening the nucleus and making the energy of static deformation vary about as it does in the droplet model. To emphasize the essentially static nature of this excitation of a "higher" shell, let us consider a case in which shell k is almost filled with protons and shell $k+1$ partly filled with neutrons, and that the attractive interaction energy is minimized when both the protons and the neutrons tend to concentrate near the equatorial plane. Then if we consider adding one or two more protons, it may be found to give lower energy to add them in the shell $k+1$ where they may concentrate near the equatorial plane, rather than in the shell k where, in filling the shell, they must concentrate near the poles. This "spilling over" from shell k to $k+1$ thus facilitates the spontaneous distortion of the nucleus, and may contribute to a large quadrupole moment. For an undistorted nucleus consisting of closed shells only, a rather large distortion would have to be imposed to make the spilling over energetically favorable, and it is not apt to happen spontaneously. The energies of the single-nucleon states between which the "spilling over" takes place consist of both potential and kinetic energy, but contribute to the energy as a function of a slow distortion which may be considered a *potential* energy in the same way that the electronic energy of a molecule as a function of internuclear distance is considered to be a potential energy in molecular vibration problems. Collective nuclear rotation, as well as vibration, may be considered to involve a dynamic and temporary transition between single-nucleon states which, being dependent on the rate of change of the shape of the system, contributes to an equivalent *kinetic* energy of the system. For simplicity we confine the treatment to cases where the static "spilling over" does not take place.

ROTATING DISTORTION OF A CLOSED SHELL

The interest in the rotating distortion arises from the physical situation in which outside of closed shells we have, let us say for simplicity, two nucleons of the same fairly large l (which is approximately equal to j) but oppositely directed so they move almost in a plane with approximately zero resultant angular momentum of their individual orbital motions. They distort the bag of liquid representing the closed shells into an oblate ellipsoid and we consider now the collective degree of freedom corresponding to rotation of this ellipsoid about an axis in the equatorial plane. If we treat nuclear matter as a classical irrotational liquid, this is not a rigid rotation of the nucleus, but rather (aside from the two nucleons not in closed shells) the motion of a surface wave around it, and the rate at which the matter below the surface is swept along is proportional to the amplitude of the wave.

This classical-fluid model gives the result³ that the

nucleus has low rotational states with $J=0, 2, 4, 6 \dots$ and with rotational energy proportional to $J(J+1)$, the constant of proportionality depending on the nuclear size and deformation. In several nuclei two even excited states are found with excitation energies in the ratio 20/6 (and even a third with ratio to the first 42/6) with remarkable fidelity.⁴ The effective moment of inertia is small, making the level spacing large, when the wave has a small amplitude requiring a small mass motion for a given angular speed. The observed spacings of rotational states are in general an order of magnitude larger than would be given by a rigid rotation of the nucleus. This empirical success of the classical-fluid model raises the question why a nucleus which may for many purposes be approximated by a suitable product of single-particle wave functions (or combinations thereof) behaves dynamically so nearly like a simple classical fluid, particularly in its rotational properties.

Once one ignores the coordinates of the individual nucleons within the closed shells and assumes they behave collectively as an incompressible, irrotational fluid, the problem is greatly simplified. There is a definite equation of motion for the fluid from which one can proceed to quantization. This provides a useful tool for treating³ the more detailed problems of coupling several nucleons to the same nuclear axis. In investigating the validity of the assumption of a simple fluid and the concept of an effective moment of inertia when a wave travels around the nuclear surface, it will suffice to examine the problem not in all its generality but in a special case which retains the essential feature of the circulating wave.

In the special case here treated we find it convenient to introduce a preferred axis, in such a way as to establish the projection M_J of the angular momentum along this axis as a constant of the motion, rather than the total angular momentum J . In this " M representation" the rotational levels associated with a surface wave in the droplet model would be proportional to M_J^2 , rather than to $J(J+1)$, and we here derive a similar result from a collective distortion of the nucleon wave functions.

In a hydrodynamic treatment of this special case we could start with an ellipsoidal container of frictionless and incompressible fluid at rest (rather than the usual free droplet in space), then start it rotating with a constant angular speed Ω and calculate the subsequent motion within the fluid, in which matter is always flowing beneath the surface from recent to future positions of the bulge at the largest diameter. We could then adjust Ω to give a desired value of the angular momentum, such as one allowed by quantization, and could calculate the corresponding kinetic energy of rotation. In this case [apart from the uncertainty of direction associated with the distinction between M_J^2 and $J(J+1)$] the uniformly rotating container becomes rather nonessential because there is no transfer of angular momentum between it and the fluid. In the

oscillator-model treatment, our introduction of a rotating ellipsoidal zero-order potential is analogous to this procedure.

PERTURBATION TREATMENT OF ROTATION

In any treatment of the nuclear rotation problem one must assume in some way that the collective behavior of the nucleons designates a preferred rotating coordinate system, and we assume in particular that the preferred set of axes is that in which the potential energy arising from all of the interactions of the nucleons is best approximated by a sum of single-nucleon energies, so that in it an applied zeroth-order potential best approximates a self-consistent field.

In our first treatment of the problem we avoid introducing an extra coordinate by assuming that the rotating coordinate system rotates about the x axis with a fixed angular velocity Ω , so that $y' = y \cos \varphi + z \sin \varphi$, $z' = -y \sin \varphi + z \cos \varphi$ with $\varphi = \Omega t$. In terms of the primed coordinates x'_i of the rotating system the Lagrangian is

$$\mathcal{L}(\dot{x}'_i, x'_i) = (M/2) \sum (\dot{x}'_i + [\Omega \times r]_i)^2 - V(x'_i), \quad (1)$$

which defines the momenta,

$$p'_i = \partial \mathcal{L} / \partial \dot{x}'_i = M \dot{x}'_i + [\Omega \times r]_i, \quad (2)$$

whence the Hamiltonian relative to the laboratory but expressed in the primed coordinates is

$$\begin{aligned} \mathcal{H} &= \sum p'_i \dot{x}'_i - \mathcal{L} = (M/2) \sum \{ (\dot{x}'_i)^2 - [\Omega \times r]^2 \} + V \\ &= \sum (p'^2_i / 2M - p'_i [\Omega \times r]_i) + V = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \end{aligned} \quad (3)$$

with

$$\begin{aligned} \mathcal{H}^{(0)} &= -(\hbar^2 / 2M) \sum \partial^2 / \partial x'^2_i + V(x'_i), \\ \mathcal{H}^{(1)} &= i\hbar \Omega \sum (y'_i \partial / \partial z'_i - z'_i \partial / \partial y'_i) = -\hbar \Omega L_x. \end{aligned} \quad (4)$$

Thus the Hamiltonian in the rotating system differs from that which we would have in a stationary system, $\mathcal{H}^{(0)}$, only by the term $\mathcal{H}^{(1)}$, as is well known. When Ω is small compared to the internal frequencies arising from $\mathcal{H}^{(0)}$, the term $\mathcal{H}^{(1)}$ may be treated as a perturbation. This is closely related to the adiabatic approximation which one might use if $\mathcal{H}^{(1)}$ were to be considered a slowly time-dependent perturbation in the stationary system with an expansion in slowly time-dependent wave functions determined by $\mathcal{H}^{(0)}$. Instead it is simpler, and it is customary in molecular problem,⁷ to make use of the circumstance that the time does not appear in \mathcal{H} as expressed Eq. (3) in the rotating system, and to assume a stationary solution $\psi(x'_i) \exp[(i/\hbar)Et]$ to the Schrödinger equation, $[\mathcal{H} - (\hbar/i)\partial/\partial t]\psi(x'_i, t) = 0$. One thus obtains a time-independent wave equation which may be put in perturbation-theory form:

$$\begin{aligned} &\{\mathcal{H}^{(0)}(x'_i, \partial/\partial x'_i) + \mathcal{H}^{(1)}(x'_i, \partial/\partial x'_i) \\ &- E^{(0)} - E^{(1)} - E^{(2)}\} \{U_0(x'_i) + \sum' a_n U_n(x'_i)\} = 0. \end{aligned} \quad (5)$$

⁷ J. H. Van Vleck, Phys. Rev. **31**, 591 (1928); G. C. Wick, Phys. Rev. **73**, 51 (1947).

ROTATING DISTORTION IN THE OSCILLATOR MODEL

In the "central model" treatment of nucleon interactions it has been found convenient to introduce in zeroth order a spherically symmetric three-dimensional harmonic oscillator potential to define a set of wave functions amenable to analytic integration, and it has been considered a good approximation because this potential possesses a relatively flat bottom corresponding to the presumed roughly constant nuclear density and potential energy. The wave function is separable in the three cartesian coordinates and in each the equation $[-(\hbar^2/2M)\partial^2/\partial x^2 + (K/2)x^2]u_n = E_n u_n$ is put in the form $(\partial^2/\partial \xi^2 + \lambda_n - \xi^2)u_n(\xi) = 0$ by use of the convenient "oscillator coordinate" $\xi = \beta x$, with $\beta = (KM/\hbar^2)^{1/4} = (K/\hbar\omega)^{1/4} = (M\omega/\hbar)^{1/4}$, $\omega = (K/M)^{1/2}$, and $E_n = (n + \frac{1}{2})\hbar\omega$.

We may introduce wave functions to describe a rotating distortion by relinquishing the spherical symmetry and introducing the zeroth-order rotating spherical potential,

$$\begin{aligned} V &= (K/2) [(x^2 + y'^2)/b^2 + z'^2/a^2] \\ &= (\hbar\omega/2) [(\xi^2 + \eta'^2)/b + \zeta'^2/a] \\ &= (\hbar\omega/2) [(\xi^2 + \eta'^2)\alpha^2 + \zeta'^2/a^4]. \end{aligned} \quad (6)$$

Here the equipotential surfaces are ellipsoids of revolution $(x^2 + y'^2)/b^2 + z'^2/a^2 = R^2$, distorted from the spheres of radius R without change of volume by keeping $ab^2 = 1$, and the usual "oscillator coordinates" have been introduced appropriate to each dimension by putting $\xi = \beta_1 x$, $\eta' = \beta_1 y'$, $\zeta' = \beta_3 z'$, $\beta_1 = \beta/b^{1/2} = \alpha\beta$ and $\beta_3 = \alpha^{-2}\beta$, with $\beta = (KM/\hbar^2)^{1/4} = (K/\hbar\omega)^{1/4} = (M\omega/\hbar)^{1/4}$, $\alpha = (a/b)^{1/6}$, where, of course, M is the nucleon mass and $\omega = (K/M)^{1/2}$. The deviation of α from unity measures the distortion of the ellipsoid from a sphere.

In the artificial problem to be compared with the classical fluid in a rotating container, we may simply ignore any further potentials and consider the terms of the Hamiltonian as given in Eq. (4). In a nucleus we may consider it plausible that a similar result is accomplished by introducing into $\mathcal{H}^{(1)}$ an additional term $V^{(1)}(x'_i) = V(x'_i - x'_f) - V(x'_i)$, that is, the difference between the potential representing the internucleon interactions and the oscillator potential, and then assuming that the average value of $V^{(1)}$ either vanishes (self-consistent field assumption) or is unaffected by the rotation, so we may neglect it.

The zeroth-order wave function for a single nucleon, though anisotropic, is still separable in the three primed coordinates,

$$\begin{aligned} u_{lmn}(\xi, \eta', \zeta') &= u_l(\xi) u_m(\eta') u_n(\zeta') \\ &= H_l(\xi) H_m(\eta') H_n(\zeta') \exp[-(\rho'^2)/2], \end{aligned} \quad (7)$$

where $\rho'^2 = \xi^2 + \eta'^2 + \zeta'^2$, and from these the zero-order wave functions of the system are built up by taking the usual (antisymmetric sum of) products:

$$U = \Pi_i u_{li}(\xi_i) u_{mi}(\eta'_i) u_{ni}(\zeta'_i). \quad (8)$$

These are solutions of the zeroth-order term of Eq. (5), $(\mathcal{H}^{(0)} - E_n^{(0)})U_n = 0$. The perturbation has no diagonal element for the ground state, so $E^{(1)} = 0$, and the rotational energy is given by

$$E^{(2)} = \sum |\mathcal{H}^{(1)}_{0i}|^2 / (E_i^{(0)} - E_0^{(0)}).$$

In terms of the oscillator coordinates the angular momentum operator L_x appearing in (4) is

$$L_x = i \sum (\alpha^3 \zeta'_i \partial / \partial \eta'_i - \alpha^{-3} \eta'_i \partial / \partial \zeta'_i). \quad (9)$$

The normalized Hermite polynomials appearing in Eq. (7) obey the recursion relations,

$$\begin{aligned} 2^{\frac{1}{2}} \xi H_n(\xi) &= (n+1)^{\frac{1}{2}} H_{n+1} + n^{\frac{1}{2}} H_{n-1}, \\ \partial H_n(\xi) / \partial \xi &= (2n)^{\frac{1}{2}} H_{n-1}, \end{aligned} \quad (10)$$

and from them we find

$$\begin{aligned} \partial u_n(\xi) / \partial \xi &= (\partial / \partial \xi) H_n \exp(-\xi^2/2) \\ &= -2^{\frac{1}{2}} [(n+1)^{\frac{1}{2}} u_{n+1} - n^{\frac{1}{2}} u_{n-1}]. \end{aligned} \quad (11)$$

Thus between the ground state and an excited state in which any one of the single-nucleon states appearing in the product Eq. (8), say l, m, n , is replaced by the excited nucleon state $l, m+1, n+1$, we have the matrix element,

$$(L_x)_{l, m+1, n+1, lmn} = -(i/2)(\alpha^3 - \alpha^{-3})(m+1)^{\frac{1}{2}}(n+1)^{\frac{1}{2}}. \quad (12)$$

Each of these states has an excitation energy $E_i^{(0)} - E_0^{(0)} = 2\hbar\omega$. Since $\mathcal{H}^{(1)}$ is a sum of single-nucleon terms, there are no exchange terms arising from antisymmetrizing (8), and there are no further matrix elements in this order. The energy of rotation is thus

$$\begin{aligned} E^{(2)} &= (\hbar\Omega^2/2\omega) \sum |(L_x)_{i0}|^2 \\ &= (\hbar\Omega^2/8\omega)(\alpha^3 - \alpha^{-3})^2 \sum' (m+1)(n+1). \end{aligned} \quad (13)$$

The prime on the summation sign is to remind us that it is to be carried out only over those single-nucleon excitations that do not lead to vanishing wave functions because of the antisymmetry, that is, in the closed-shell cases we shall consider, over the proton states or neutron states of the last two filled shells. In the sense that the last filled shells with the highest nucleon energy have density distributions concentrated most predominately toward the outside of the nucleus, we see here that the rotation energy is to some extent a surface phenomenon comparable to the energy of a surface wave.

The angular momentum about the x axis evaluated for the first-order ground-state wave function $\psi_0 = U_0 + \sum a_i U_i$ with $a_i = \mathcal{H}^{(1)}_{i0} / (E_0^{(0)} - E_i^{(0)})$ is

$$\begin{aligned} \langle L_x \rangle &= \sum [\bar{a}_i (L_x)_{i0} + (L_x)_{0i} a_i] \\ &= (\Omega/\omega) \sum |(L_x)_{i0}|^2 \\ &= (\Omega/4\omega)(\alpha^3 - \alpha^{-3})^2 \sum' (m+1)(n+1). \end{aligned} \quad (14)$$

By eliminating Ω we may express the rotational energy

in terms of this angular momentum:

$$\begin{aligned} E^{(2)} &= 2\hbar\omega \langle L_x \rangle^2 / [(\alpha^3 - \alpha^{-3})^2 \sum' (m+1)(n+1)] \\ &= \hbar^2 \langle L_x \rangle^2 / 2(\mathcal{J}_x)_{\text{calc}}. \end{aligned} \quad (15)$$

Here $(\mathcal{J}_x)_{\text{calc}}$ is introduced as an effective moment of inertia calculated from the properties of the nucleons. It is at once apparent that this result has general features anticipated from familiarity with the droplet model, the rotational energy decreasing with increasing distortion as α departs from unity, and the energy decreasing with increasing atomic number A as ω decreases and $\sum' (m+1)(n+1)$ grows larger.

In the droplet model the effective moment of inertia is, in the notation of Bohr,³

$$(\mathcal{J}_x)_{\text{droplet model}} = 3B\alpha_0^2 = (9/8\pi)AMR_0^2\alpha_0^2, \quad (16)$$

where R_0 is the radius of a spherical nucleus, $R_0 = A^{\frac{1}{3}}r_0$, and where α_0 is the parameter defining the deviation of the spheroidal surface of the droplet from sphericity,

$$R(\theta) = R_0[1 + (\frac{5}{16}\pi)^{\frac{1}{2}}(3\cos^2\theta - 1)\alpha_0] \quad (17)$$

[Bohr's Eqs. (1), (12), (13) with $\gamma=0$ and with $Y_2 = (\frac{5}{16}\pi)^{\frac{1}{2}}(3\cos^2\theta - 1)$]. In order to establish the numerical correspondence between (15) and (16) it is necessary to relate α_0 to α in spite of the nebulous nature of the nuclear surface in the more realistic oscillator model. Instead of a definite nuclear surface we have a fairly sudden decay of density induced by the exponential factor from Eq. (7) in competition with the more slowly-varying factors arising from the sum of squares of Hermite polynomials, a sum which is isotropic in ξ, η', ζ' space for closed shells. For closed shells an equi-density surface may be defined by putting $\rho'^2 = C$. At this surface the exponential factor in the density is reduced below its maximum value (at the center) by e^{-C} and some arbitrary value of C may be taken to define an equivalent nuclear surface, as is discussed further below. In the coordinate system of the principal axes we have the identity,

$$r^2 = x^2 + y'^2 + z'^2 = \rho'^2/\beta^2\alpha^2 + (1 - \alpha^{-6})z'^2,$$

with $z' = r \cos\theta$, and this equi-density surface is then

$$r = (C^{\frac{1}{2}}/\beta)[1 + \frac{1}{6}(1 - \alpha^{-6})(3\cos^2\theta - 1) + \dots], \quad (18)$$

where higher powers of the small parameter $(1 - \alpha^{-6})$ have been omitted. By comparing Eq. (17) and Eq. (18) we have to this order,

$$1 - \alpha^{-6} = 3(5/4\pi)^{\frac{1}{2}}\alpha_0 = 2\Delta R/R_0, \quad \alpha = 1 + (5/16\pi)^{\frac{1}{2}}\alpha_0. \quad (19)$$

Here ΔR is the difference between major and minor semiaxes of the equidensity ellipsoid of average radius R_0 .

The size of the nucleus is represented in Eq. (15) by the parameter ω and this must be determined in such a way as to facilitate the comparison with the droplet model. In the comparison with experiment the comparison with quadrupole moments is important,

and they may be made the same in the two models by selecting ω to make the mean square radius the same in both models. With the wave functions (7) for a closed shell, in which $\sum l = \sum m = \sum n$, we have

$$\sum (r^2)_{Av} = (\xi^2 + \eta^2 + \zeta^2)_{Av} / \beta^2 = (3\hbar/M\omega) \sum (m + \frac{1}{2}), \quad (20)$$

where the sum is to be carried out over the states of all nucleons. In a uniform distribution of A nucleons in a sphere of radius R_0 the corresponding result is

$$\sum (r^2)_{Av} = \frac{3}{5} A R_0^2, \quad (21)$$

and by equating these we have from Eq. (15) the effective moment of inertia as calculated in the oscillator model expressed in terms of the radius R_0 of a sphere in which a uniform distribution would have the same mean square radius:

$$(\mathcal{I}_x)_{\text{calc}} = \frac{1}{5} M A R_0^2 (\Delta R/R_0)^2 \times [\sum' (m+1)(n+1) / \sum (m + \frac{1}{2})], \quad (22)$$

which differs from the droplet-model result³ only by half the ratio of the two sums in the square bracket, since $(\mathcal{I}_x)_{\text{droplet model}} = \frac{2}{5} M A (\Delta R)^2$.

Superficially these two sums appear quite different, the first being obviously bilinear in the quantum numbers but summed over only the highest two shells of nucleons, and the second apparently linear but acquiring a corresponding quadratic nature through the fact that the summation extends over all filled shells. On evaluating them for the various closed shells one finds in fact the remarkable equality,

$$\sum' (m+1)(n+1) = 2 \sum (m + \frac{1}{2}), \quad (23)$$

which leaves

$$(\mathcal{I}_x)_{\text{calc}} = (\mathcal{I}_x)_{\text{droplet model}}. \quad (24)$$

The closed shells considered here where we neglect spin-orbit coupling are, of course, not closed shells of the "magic numbers," but they provide a valid indication of the trends of these sums through the periodic table. In Table I they are listed both for summation over all nucleons and for summation over protons as is of interest in connection with other data on isotope shifts and on electric quadrupole moments eQ_0 , with

$$Q_0 = 2(\hbar/M\omega)(\alpha^4 - \alpha^{-2}) \sum_{\text{protons}} (m + \frac{1}{2}) \approx \frac{4}{5} A R_0^2 (\Delta R/R_0) \sum_{\text{protons}} (m + \frac{1}{2}) / \sum (m + \frac{1}{2}). \quad (25)$$

For the cases in which there is a neutron excess, the ratio of sums in Eq. (25) is 7 or 8 percent less than Z/A .

The equality (24) indicates that the assumption of irrotational fluid flow in the droplet model corresponds to the result of calculation of the second-order energy $E^{(2)}$ with nucleon wave functions.

A criterion for the validity of the perturbation approximation may be obtained by requiring that with the perturbation the probability density of ψ is still

TABLE I. Sums over the closed shells of the oscillator model. Here k_p is the total quantum number $l+m+n$ for the last proton shell and k_n likewise for neutrons.

k_p	k_n	Z	N	A	$\Sigma(m + \frac{1}{2})$	$\Sigma_{\text{protons}}(m + \frac{1}{2})$
0	0	2	2	4	2	1
1	1	8	8	16	12	6
2	2	20	20	40	210	20
3	3	40	40	80	100	50
3	4	40	70	110	155	50
4	4	70	70	140	210	105
4	5	70	112	182	301	105
5	5	112	112	224	392	196
5	6	112	168	280	532	196

predominantly associated with U_0 , that is, that

$$\begin{aligned} \sum |a_{i0}|^2 &= E^{(2)} / 2\hbar\omega \\ &= (\Omega/4\omega)^2 (\alpha^3 - \alpha^{-3})^2 \sum' (m+1)(n+1) \\ &= (\Omega/4\omega) \langle L_x \rangle \ll 1, \end{aligned} \quad (26)$$

from Eq. (13) and (14). Since $\langle L_x \rangle$ or J for the states of interest is of order of magnitude 4, we see that this criterion is practically identically identical with the adiabatic condition $\Lambda/\omega \ll 1$. From Eq. (14) the implied contradiction is apparent that Ω/ω is not small and the approximation does not apply in the limit for small values of the deformation parameter $(\alpha^3 - \alpha^{-3})$ which has been considered small in our final numerical evaluation [beginning with Eq. (18)]. However, the values of $\sum' (m+1)(n+1)$ listed in Table I [see Eq. (23)] are sufficiently large that Ω/ω is considerably less than unity and the adiabatic approximation is valid for the fairly small J 's and substantially finite nuclear deformations of most practical interest.

COMPARISON WITH EXPERIMENT

The principal success of such a collective description of nuclear behavior is the agreement with the *ratios* of level spacings in individual nuclei, which depends only on a valid separation of the collective motion from the individual-nucleon motion so as to give an effective moment of inertia, presumably arising from a nuclear distortion, independent of the rotational quantum number. These ratios do not depend on whether the collective motion is more nearly irrotational or rigid and thus do not test the validity of the droplet-model assumptions. The quantitative comparison of the level spacings with experimental values is complicated by the necessity of taking an indication of the deformation from the interpretation of some other experimental data. This has been done for the droplet model by others,³ especially by Ford⁸ who plots against neutron number N the distortions as inferred from rotational levels, from isotope shifts, and from quadrupole moments. There are so many uncertainties involved that it is the study of the trend rather than of individual cases that seems most significant, and the trend is that the moments of inertia effective in the rotation are at least

⁸ K. W. Ford, Phys. Rev. **90**, 29 (1953); **95**, 1250 (1954).

about four times as large as calculated on the assumption of irrotational fluid flow, that is, the observed level spacings are about one-fourth as large as calculated by the droplet model. This means that the nucleus seems to rotate more nearly as a rigid body than called for by the first-order wave functions giving rise to the second-order energy $E^{(2)}$. One may plausibly speculate that the configuration interaction of a higher-order calculation would correspond to a transition part way toward a "rigid" alpha model, taking advantage of the inherent stability of a clustering in space into alpha groups.⁹

Configuration interaction results from $V^{(1)}$, that is, from the introduction of two-body forces in place of the oversimplified oscillator potential, and the observed proportionality⁴ of rotational energy to $J(J+1)$ seems to indicate that, if this be the significant mechanism, the amount of configuration interaction is not much affected by the "Coriolis term" $\mathcal{H}^{(1)}$ of Eq. (4).

COMPARISON WITH ANOTHER METHOD

In the above treatment we have assumed that a simple approximation to a self-consistent field is best in a certain coordinate system rotating with a constant Ω . Instead one may, before setting up the Hamiltonian, introduce as a coordinate system free to rotate the one defined by the instantaneous principal axes of the system of A nucleons, and treat the orientation angles as coordinates in addition to the position coordinates x'_i of the nucleons in these axes. This means that the system is over-determined, having $3A+3$ coordinates if a general rotation is permitted or $3A+1$ in the special case here discussed with rotation about the x axis only. A similar difficulty is encountered with the center of mass in all shell-model treatments, and especially for large A it may plausibly be hoped that the extra coordinate (s) may cause little trouble, being somehow absorbed in the many coordinates lost in the "core."

We thus consider again only the deviation from cylindrical symmetry about the x axis, and define the orientation angle of the principal axes in the yz plane through the vanishing of the product of inertia,

$$\sum_i y'_i z'_i = \sum_i [(z_i^2 - y_i^2) \cos \varphi \sin \varphi + y_i z_i (\cos^2 - \sin^2 \varphi)] = 0. \quad (27)$$

This is equivalent to Eq. (II) of a very recent publication of Bohr,¹⁰ in which he also discusses in somewhat more general terms the approximate equivalence of a particle approach to irrotational flow [Eq. (XIII)]. If we postulate a wave function, $\psi(x'_i, \varphi)$, dependent on φ and on the internal coordinates x'_i , we may as he sug-

gests write the Hamiltonian in the separated form,

$$\mathcal{H}(x_i, \partial/\partial x_i) = \mathcal{H}_{\text{int}}(x'_i, \partial/\partial x'_i) + \mathcal{H}_{\text{rot}}(x'_i, \partial/\partial \varphi) + \mathcal{H}_{\text{coup1}}(x'_i, \partial/\partial x', \partial/\partial \varphi), \quad (28)$$

with

$$\mathcal{H}_{\text{rot}} = -(\hbar^2/2M) \sum_i [(\partial \varphi / \partial y_i)^2 + (\partial \varphi / \partial z_i)^2] (\partial^2 / \partial \varphi^2). \quad (29)$$

Taking the y_i and z_i derivatives of (27), with $y'_i = y_i'(y_\alpha, z_\alpha, \varphi(y_\alpha, z_\alpha))$ and $\partial y'/\partial \varphi = z'$, $\partial z'/\partial \varphi = -y'$, we have

$$\begin{aligned} \partial \varphi / \partial y_i &= (y'_i \sin \varphi - z'_i \cos \varphi) / [\sum_j (z_j'^2 - y_j'^2)], \\ \partial \varphi / \partial z_i &= (y'_i \cos \varphi + z'_i \sin \varphi) / [\sum_j (y_j'^2 - z_j'^2)]. \end{aligned} \quad (30)$$

If in Eq. (28) we put $\mathcal{H}_{\text{int}} = \mathcal{H}^{(0)}$, $\mathcal{H}_{\text{rot}} = \mathcal{H}^{(1)}$ and neglect $\mathcal{H}_{\text{coup1}}$ (which contains several terms and does not contribute to $E^{(1)}$) in a perturbation equation of the form Eq. (5), we have the zeroth-order wave functions,

$$\psi(x'_i, \varphi) = U(x'_i) R_{Lx}(\varphi), \quad (31)$$

where U is again an antisymmetrized product (8), and obtain

$$\begin{aligned} E^{(1)} &= \langle 0 | \mathcal{H}_{\text{rot}} | 0 \rangle = (\hbar^2/2M) L_x^2 \\ &\times \int \bar{U} \{ \sum_i (y_i'^2 + z_i'^2) / [\sum_j (y_j'^2 - z_j'^2)]^2 \} U d\tau_{3A}', \end{aligned} \quad (32)$$

where the integration extends over the $3A$ coordinates of A nucleons. This is closely related to Eq. (IX) of Bohr.¹⁰ Although the summation in the numerator is a sum of single-nucleon functions, the entire curly bracket in the integrand is not, and it is not strictly permissible to neglect exchange terms arising from the antisymmetry nor to carry out any simple evaluation of the integral. The complicating denominator entered Eq. (28) in such a way as to indicate that the greater the gross distortion from spherical symmetry, the smaller the effect of a given small displacement Δy_i of a single particle on the orientation φ of the principal axes. A simple and plausible procedure is then to neglect the fluctuations of this gross distortion about its mean, by evaluating the summations in the numerator and denominator separately with the wave function U which amounts to averaging over the fluctuations in a somewhat indefinite way so the result may be approximately valid even though the zero-point oscillations of shape may in some cases be almost as large as the mean distortion. This gives

$$\begin{aligned} \langle 0 | \mathcal{H}_{\text{rot}} | 0 \rangle &= \hbar^2 L_x^2 / 2 (g_x)_0 \\ &\approx \hbar L_x^2 (\alpha^2 + \alpha^{-4}) / [2(\alpha^3 - \alpha^{-3})^2 \sum (m + \frac{1}{2})], \end{aligned} \quad (33)$$

for the case of completely filled shells for which $\sum m_i = \sum n_i$, the summations being carried out over all nucleons. In the approximation in which $\alpha^2 + \alpha^{-4} = 2$ and in the light of Eq. (23), the two expressions Eq. (15) and Eq. (33) for the rotational energy are then identical.

⁹ E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938); D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); D. M. Dennison, Phys. Rev. **96**, 378 (1954).

¹⁰ A. Bohr, *Rotational States of Atomic Nuclei* (Munksgaards Forlag, Copenhagen, 1954). The unspecified q_α are presumably fewer than our x'_i to keep the right number of coordinates.